Writing problems

**Problem 1:** In order to derive Simpson’s rule, use Lagrange formula to prove that: if $p_2(x)$ is the unique polynomial of degree $\leq 2$ which interpolates the function $f(x)$ at three equispaced points $x_0$, $x_1$ and $x_2$, then

$$\int_{x_0}^{x_2} p_2(x) \, dx = \frac{h}{3} (f(x_0) + 4f(x_1) + f(x_2))$$

Remark: You are asked to derive this result in a more rigorous way: First use Lagrange’s formula and then integrate the polynomial that you obtained.

**Problem 2:** Problem 10, page 195.

**Problem 3:** Problem 11, page 195.

**Problem 4:** Problem 13, page 195.

**Problem 5:** Problem 4 (a)(b), page 204.

**Problem 6:** Problem 1 (a), page 211.

**Problem 7:** (Gaussian Quadrature v.s Simpson’s rule) Let $f(x) = \frac{1}{x+1.5}$

(a) Compute $\int_{-1}^{1} f(x) \, dx$ by hand.

(b) Approximate $\int_{-1}^{1} f(x) \, dx$ using Gaussian Quadrature with 3 nodes.

(c) Approximate $\int_{-1}^{1} f(x) \, dx$ using Simpson rule with 3 equispaced nodes.

(d) Compare the errors of the above two approximation and which method works better?
**Problem 8:** In the Romberg algorithm, $R(n, 1)$ denotes an estimate of $\int_a^b f(x)\,dx$ with subintervals of size $h = (b - a)/2^{n-1}$. If it were known that
\[ \int_a^b f(x)\,dx = R(n, 1) + a_3 h^3 + a_6 h^6 + \cdots \]
how would we have to modify the Romberg algorithm?

**Programming problems:**

**Problem 9:** Let $f(x) = e^x$. We want to compute $\int_0^1 e^x\,dx$. You can first compute the exact integral by hand.

(a) In class, we know that the error of the trapezoidal rule is given by
\[ E_T^n = \left| \int_a^b f(x)\,dx - I_T^n(f) \right| \leq \frac{(b - a)M}{12} h^2, \tag{1} \]
where $h = \frac{b-a}{n}$ and $M = \max_{x \in [0,1]} |f''(x)|$. Suppose you want to use the trapezoidal rule to compute $\int_0^1 e^x\,dx$. Use estimation (1) to find how many node points are necessary in order to be sure that the error will be less than $10^{-8}$.

(b) Use the trapezoidal rule to compute $\int_0^1 e^x\,dx$ with the number of node points from part a). What is the actual error?

(c) Repeat a) and b) but this time with Simpson’s rule. (The error of Simpson’s rule is given as follows:
\[ E_S^n = \left| \int_a^b f(x)\,dx - I_S^n(f) \right| \leq \frac{(b - a)M}{180} h^4, \]
where $h = \frac{b-a}{n}$ and $M = \max_{x \in [0,1]} |f^{(4)}(x)|$.)

(d) Write a program which compute $E_T^n$ and $E_S^n$ for $n = 10, 110, 210, 310, \ldots, 510$. Plot $\log E_T^n$ versus $\log n$ and $\log E_S^n$ versus $\log n$. The points should line on lines. What are the equations of these lines? (you can click “tools → basic fitting” menu in your plot window and do a linear fitting).

(e) Why are the slopes of above lines $-2$ and $-4$ respectively?

**Problem 10:** Design and carry out an experiment using the Romberg algorithm.
(a) Try on a "good" function (that possesses many continuous derivatives on the intervals), for example $\int_{0}^{1} \frac{1}{1+x} \, dx$

(b) Try on a "bad" function, for example $\int_{0}^{1} \sqrt{x} \, dx$.

(c) For a sequence of $n$, Check the ratio

$$\frac{R(n, m) - R(n - 1, m)}{R(n + 1, m) - R(n, m)} \approx 4^{m+1}$$

for both functions to test whether the algorithm is working.