Homework 5
Due on Thursday, Nov 10

Writing problems

Problem 1: Use a calculator to solve Problem 9, Section 8.1, page 495

Problem 2: Find the least square polynomials approximation of degree 1, 2, 3 to \( f(x) \) on the indicated interval:

\[
\begin{align*}
f(x) &= \frac{1}{x}, [1, 3] \\
f(x) &= e^x, [0, 2]
\end{align*}
\]

Problem 3:
1) Using the Gram-Schmidt process (Theorem 8.7, page 504) to construct orthogonal set of polynomial functions \( \phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x) \) for the interval \([0, 2]\) (Note, set the weight function \( w(x) \equiv 1 \)).

2) Find the least square approximation polynomial of degree 3 for the function \( f(x) = e^x \) on \([0, 2]\) using the basis functions \( \phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x) \). Compare the result with what you obtained in Problem 2.

Problem 4: Let \( \Pi_n \) be the set of all polynomials of degree at most \( n \). Suppose that \( \{\phi_0, \phi_1, \cdots, \phi_n\} \) is any linearly independent set in \( \Pi_n \). Show that for any element \( Q(x) \in \Pi_n \), there exists unique constants \( c_0, \cdots, c_n \), such that

\[
Q(x) = \sum_{k=0}^{n} c_k \phi_k(x)
\]

(Hint: Use the concept of "linear vector space").

Problem 5: Show that if \( \{\phi_0(x), \cdots, \phi_n(x)\} \) is an orthogonal set of functions on \([a, b]\), then \( \{\phi_0(x), \cdots, \phi_n(x)\} \) is a linearly independent set.

Problem 6: Assume that \( f \in C^4[a, b] \) and find a formula to approximate \( f'(x_0) \) with the four points \((x_0, f(x_0)), (x_0 - h, f(x_0 - h)), (x_0 - 2h, f(x_0 - 2h)), (x_0 + h, f(x_0 + h))\) with the error of \( O(h^3) \). Assume \( x_0, x_0 - h, x_0 - 2h, x_0 + h \in [a, b] \).
Problem 7: Values for $f(x) = xe^x$ are given in the following table. Since $f'(x) = (x + 1)e^x$, we have $f'(2.0) = 22.167168$. Approximate $f'(2.0)$ using the following formulas, and give their errors respectively.

(a) Forward difference formula.
(b) Backward difference formula.
(c) Central difference formula.
(d) The three-point formula $\frac{-3f(x_0)+4f(x_0+h)-f(x_0+2h)}{2h}$.
(e) The five-point formula $\frac{f(x_0-2h)-8f(x_0-h)+8f(x_0+h)-f(x_0+2h)}{12h}$.

<table>
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<tr>
<th>$x$</th>
<th>$f(x)$</th>
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<tbody>
<tr>
<td>1.8</td>
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<td>2.2</td>
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Problem 8: Derive the five point formula via Richardson extrapolation and the error term by using $f(x_0 - 2h), f(x_0 - h), f(x_0), f(x_0 + h), f(x_0 + 2h)$.

Problem 9: Derive the second-derivative $f''(x_0)$ formula via Taylor series by using $f(x_0 - h), f(x_0), f(x_0 + h)$. (Hint: represent $f(x_0 - h)$ and $f(x_0 + h)$ at the point $x_0$ by Taylor’s expansion)

Programming problems

Problem 10: Solve Problem 5, Section 8.1, page 495. In your report, you are required to

- specify the coefficients matrix $A$,
- solve the normal equation $A^T A x = A^T b$ in MATLAB(SCILAB) using matrix inversion (for example: ”inv(A'*A)*A'*b” in MATLAB(or equivalent command in SCILAB)),
- write out the least square approximation functions (polynomials or other forms).
• Plot the graph of the least square approximation function by using "fplot" and the data ("hold on, plot(xi,yi)" where xi,yi are the data vector).

**Problem 11:** Let \( f(x) = \cos(x) \) and \( x_0 = 0 \). Carry out a numerical experiment to compare the accuracy of the "derivative" formulas (a)-(e) in Problem 7 with the exact derivative \( f'(x_0) \) computed directly. Take a sequence of values for \( h = 4^{-n} \) with \( 0 \leq n \leq 12 \).