Homework 3
Due on Thursday, Oct 13

Writing problems

Problem 1: (Remedy of Newton’s method)
1. We have seen in class that, if $f(x)$ has a root of multiplicity $m \geq 2$, Newton’s method converges only linearly. One of remedies is to define

$$p_{n+1} = p_n - \frac{f(p_n)f'(p_n)}{(f'(p_n))^2 - f(p_n)f''(p_n)}.$$  \hspace{1cm} (1)

Prove that this method converges quadratically.

2. We have another modified version of Newton’s method:

$$p_{n+1} = p_n - m \frac{f(p_n)}{f'(p_n)}.$$  \hspace{1cm} (2)

Show that, if it converges, then it converges quadratically.

Problem 2: (An application of Newton’s method) Newton’s method is the commonly used method for calculating square roots on a computer.

(a) What equation would you solve in order to find $\sqrt{a}$?

(b) Show that in this case, Newton’s method reduces to the following iteration:

$$p_{n+1} = \frac{1}{2} \left( p_n + \frac{a}{p_n} \right).$$ \hspace{1cm} (1)

(c) In the lecture, we showed that Newton’s method converges when $p_0$ is close enough to $p$. However, for this special problem, we can choose any $p_0$ to get a convergent Newton’s method. Show that (1) always converges to $\sqrt{a}$ for any initial guess $p_0 > 0$.

Problem 3: Show that the following sequences converges linearly to $p = 0$. How large must $n$ be before we have $|p_n - p| \leq 5 \times 10^{-2}$?

(a) $p_n = \frac{1}{n^2}$, $n \geq 1$

(b) $p_n = \frac{1}{n^2}$, $n \geq 1$
**Problem 4:** (Generalization of Newton’s method) Using Taylor’s expansion of two variables, derive Newton’s method for solving the following system of nonlinear equations

\[
\begin{align*}
  f_1(x, y) &= 0 \\
  f_2(x, y) &= 0.
\end{align*}
\]

(Assuming the differentiability of the functions \(f_1\) and \(f_2\) if needed, and assume that the Jacobian matrix \(
\begin{pmatrix}
  \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
  \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}
\end{pmatrix}
\) is invertible in a small neighborhood of a root of the system).

**Problem 5:** Find the unique polynomial of form \(P(x) = a_0 + a_1x + a_2x^2\) which passes through the three points \((0, 1)\), \((-2, 4)\) and \((2, 5)\)

(a) by solving the linear equation:

\[
X \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}
\]

where \(X\) is the associated Vandermonde matrix.

(b) by Lagrange formula.

**Problem 6:** Let \(P_3(x)\) be the interpolating polynomial for the data \((0, 0)\), \((0.5, y)\), \((1, 3)\) and \((2, 2)\). Find \(y\) if the coefficient of \(x^3\) in \(P_3(x)\) is 6.

**Problem 7:** (Comparison of algorithms) Find a solution of

\[x^3 = x^2 + x + 1\]

using

(a) Bisection method. Precise the initial interval you choose and justify your choice.

(b) Newton’s method. Precise the initial guess you choose and justify your choice.

(c) Secant method. Precise the initial guess you choose and justify your choice.
In each case, use the stopping criterion $|p_n - p_{n-1}| \leq 10^{-6}$, and plot the graph of $p_n$ versus $n$.

(Hint: Before to do the iterations, plot the function with matlab("fplot" and "grid on") so that you can see roughly where is the zero.)

**Problem 8:** Let $f(x) = e^x - x - 1$. We have shown in class that 0 is a root of $f(x)$ of multiplicity 2.

(a) How many iterations are needed with Newton’s method to find this zero with accuracy $10^{-10}$ (start with $p_0 = 1$, and set the stopping criteria as $|p_n - 0| \leq 10^{-10}$)

(b) For the same accuracy and initialization, how many iterations are needed with the modified versions of Newton’s method (1) and (2) that we have seen in Problem 1?

(c) Plot the error log($|p_n - 0|$) versus iterations number (until 1000) for Newton’s method, the modified Newton’s methods (1) and (2). Which method has the fastest convergence for this example?

**Problem 9:** Use the generalized Newton’s method in Problem to solve the nonlinear equations:

$$\begin{cases} x^2 + y^2 = 1 \\ x^2 - y = 0. \end{cases}$$

with $x_0 = 1, y_0 = 0.5$ with stopping criteria $\sqrt{(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2} \leq 10^{-5}$. 