Representation Dimension and Tilting

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joint work with Dieter Happel

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Representation dimension

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$k$: algebraically closed field
$\text{mod}\Lambda$: category of fin. dim. left $\Lambda$-modules
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$\text{rep.dim}\Lambda$: smallest global dimension of $\text{End}_{\Lambda} M$, $M \in \text{mod}\Lambda$ a generator-cogenerator [introduced by Auslander in 1971]
**Representation dimension**

**Alternative characterization**

$M$ a generator-cogenerator such that for all $X \in \text{mod}\Lambda$ there is

\[ 0 \rightarrow M^d \rightarrow \cdots \rightarrow M^1 \rightarrow M^0 \rightarrow X \rightarrow 0 \]

exact with $M^i \in \text{add} M$ and

\[ 0 \rightarrow \text{Hom}_\Lambda(M, M^d) \rightarrow \cdots \rightarrow \text{Hom}_\Lambda(M, M^0) \rightarrow \text{Hom}_\Lambda(M, X) \rightarrow 0 \]

is exact. We call such an $M$ an **Auslander generator**.
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- $\text{rep.dim} \Lambda = 2$ iff $\lambda$ is representation finite.
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[Igusa, Todorov, 2005:] If $\text{rep.dim} \Lambda \leq 3$ then finitistic dimension conjecture holds for $\Lambda$. 

[Iyama 2003:] $\text{rep.dim} \Lambda$ is finite (conjectured by Auslander).

[Rouquier 2006:] For all $n \geq 2$ there is $\Lambda_n$ with $\text{rep.dim} \Lambda_n = n$. 
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Some tame blocks of finite groups [Holm 2002]
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Behaviour of rep.dim under stable equivalence [Guo 2005]

Behaviour of rep.dim under tilting [Xi 2006], [Chen-Hu 2010]
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Tilting modules

Λ \mathcal{T} \in \text{mod}\Lambda \text{ called tilting module if}

1. \text{pd}_\Lambda \mathcal{T} \leq 1,

2. \text{Ext}_\Lambda^1(\mathcal{T}, \mathcal{T}) = 0 and

3. there is 0 \rightarrow \Lambda \Lambda \rightarrow T^0 \rightarrow T^1 \rightarrow 0 \text{ exact with } T^0, T^1 \in \text{add}\mathcal{T}.

[Brenner-Butler 1980]: Let \Gamma = \text{End}_\Lambda \mathcal{T}. Then \mathcal{T} \in \text{mod}\Lambda \text{ induces torsion pairs } (\mathcal{T}((\mathcal{T})), F(\mathcal{T})) \text{ on } \text{mod}\Lambda \text{ and } (X((\mathcal{T})), Y(\mathcal{T})) \text{ on } \text{mod}\Gamma, \text{ where}

1. T((T)) = \{X \in \text{mod}\Lambda | \text{Ext}_\Lambda^1(\mathcal{T}, X) = 0\}

2. F(T) = \{X \in \text{mod}\Lambda | \text{Hom}_\Lambda(\mathcal{T}, X) = 0\}

3. X(T) = \{X \in \text{mod}\Gamma | T \otimes \Gamma X = 0\}

4. Y(T) = \{X \in \text{mod}\Gamma | \text{Tor}_\Gamma^1(\mathcal{T}, X) = 0\}.
Tilting modules

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[Brenner-Butler 1980]: Let $\Gamma = \text{End}_\Lambda T$. Then $\Lambda T \in \text{mod}\Lambda$ induces torsion pairs $(\mathcal{T}(T), \mathcal{F}(T))$ on $\text{mod}\Lambda$ and $(\mathcal{X}(T), \mathcal{Y}(T))$ on $\text{mod}\Gamma$, where

1. $\mathcal{T}(T) = \{X \in \text{mod}\Lambda \mid \text{Ext}_\Lambda^1(T, X) = 0\}$
2. $\mathcal{F}(T) = \{X \in \text{mod}\Lambda \mid \text{Hom}_\Lambda(T, X) = 0\}$
3. $\mathcal{X}(T) = \{X \in \text{mod}\Gamma \mid T \otimes_\Gamma X = 0\}$
4. $\mathcal{Y}(T) = \{X \in \text{mod}\Gamma \mid \text{Tor}_1^\Gamma(T, X) = 0\}$. 
Splitting tilting modules

The restrictions

\[ \text{Hom}_\Lambda(\mathcal{T}, -) : \mathcal{T}(\mathcal{T}) \to \mathcal{Y}(\mathcal{T}) \]

\[ \text{Ext}^1_\Lambda(\mathcal{T}, -) : \mathcal{F}(\mathcal{T}) \to \mathcal{X}(\mathcal{T}) \]

are equivalences of categories.

\[ \Lambda \mathcal{T} \] is called a splitting tilting module if each indecomposable \( \Gamma \)-module \( X \) either \( X \in \mathcal{X}(\mathcal{T}) \) or \( X \in \mathcal{Y}(\mathcal{T}) \).

[Hoshino 1983:] A tilting module \( \Lambda \mathcal{T} \) is splitting if and only if \( \text{inj.dim} \ X \leq 1 \) for all \( X \in \mathcal{F}(\mathcal{T}) \).
**Splitting tilting modules**

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**Approximations**

\[ C \subseteq \text{mod}\Lambda \text{ a full subcategory, } X \in \text{mod}\Lambda. \text{ A map } f : X \to F_X \]

called **left \( C \)-approximation** of \( X \) if \( F_X \in C \) and for all \( g : X \to C \in C \) there is \( h : F_X \to C \) making

\[
\begin{array}{ccc}
X & \xrightarrow{f} & F_X \\
C & \xleftarrow{g} & C
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\[ \text{commutative.} \]
**Approximations**

$\mathcal{C} \subseteq \text{mod}\Lambda$ a full subcategory, $X \in \text{mod}\Lambda$. A map $f : X \to F_X$ called **left $\mathcal{C}$-approximation** of $X$ if $F_X \in \mathcal{C}$ and for all $g : X \to C \in \mathcal{C}$ there is $h : F_X \to C$ making

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$\mathcal{C} \subseteq \text{mod}\Lambda$ called **functorially finite** if each $X \in \text{mod}\Lambda$ admits a left and a right $\mathcal{C}$-approximation.
Approximations
A result we’ll need

$\mathcal{C} \subseteq \text{mod}\Lambda$ called \textbf{functorially finite} if each $X \in \text{mod}\Lambda$ admits a left and a right $\mathcal{C}$-approximation.

[Happel-Ringel 1982], [Smalø 1984]: The categories $\mathcal{T}(T)$ and $\mathcal{F}(T)$ are functorially finite.
Piecewise hereditary algebras

Λ called **piecewise hereditary** if $D^b(Λ)$ is equivalent to $D^b(\mathcal{H})$ for some hereditary abelian category $\mathcal{H}$.

[Happel 2001]: Two possibilities for $\mathcal{H}$:

• $\mathcal{H} = \text{mod} \, H$ for some hereditary $k$-algebra $H$,
• $\mathcal{H} = \text{coh} \, X$ for a weighted projective line $X$.

We call $\text{mod} \, H$ resp. $\text{coh} \, X$ the **type** of $\Lambda$.

$\Lambda$ piecewise hereditary of type $\text{mod} \, H$ is called an **iterated tilted** algebra.
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Examples:

• hereditary algebras
• tilted algebras
• iterated tilted algebras
• quasitilted algebras (i.e. $\text{End}_\mathcal{H} T$ for some tilting object $T$ in a hereditary category $\mathcal{H}$)

Auslander’s philosophy: These algebras are homologically easy. They should have small rep. dim.
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- hereditary algebras have \( \text{rep.dim} \) at most 3 [Auslander]
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Characterization of piecewise hereditary algebras

[Happel-Richard-Schofield 1988] [Happel-Reiten-Smalø 1996]

**Theorem** Let \( \Lambda \) be piecewise hereditary.

1. If \( \Lambda \) is of type \( \text{mod} H \), then
   \[ \exists \] algebras \( \Lambda_i \) and splitting tilting modules \( \Lambda_i T_i \), \( 0 \leq i \leq m \) such that
   \[ \Lambda_0 = H, \; \Lambda_{i+1} = \text{End}_{\Lambda_i} T_i \text{ and } \Lambda_m = \Lambda. \]

2. If \( \Lambda \) is of type \( \text{coh} X \), then
   \[ \exists \] quasitilted algebra \( \Gamma \) and a sequence of algebras \( \Lambda_i \), \( 0 \leq i \leq m \) and splitting tilting or cotilting modules \( \Lambda_i T_i \), \( 0 \leq i \leq m \) such that
   \[ \Lambda_0 = \Gamma, \; \Lambda_{i+1} = \text{End}_{\Lambda_i} T_i \text{ and } \Lambda_m = \Lambda. \]
Main theorem

Both types: we know $\text{rep.dim}\Lambda_0$. If we knew how rep. dim. changes under taking endomorphism rings of splitting tilting/cotilting modules, we knew rep. dim of piecewise hereditary algebras.
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**Theorem:** $\Lambda$ f.d. $k$-algebra with $\text{rep.dim.}\Lambda \leq 3$, $\Lambda T$ a splitting tilting/cotilting module, $\Gamma = \text{End}_\Lambda T$.
Then $\text{rep.dim}\Lambda \leq 3$.

**Corollary:** $\Lambda$ piecewise hereditary $\Rightarrow \text{rep.dim}\Lambda \leq 3$. 
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Sketch of proof

- May assume that $\Lambda$ and $\Gamma$ are representation infinite.
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- What to take as Auslander generator: $\Lambda M$ an Auslander generator of $\text{mod}\Lambda$, $\Lambda M \to \Lambda E_M$ minimal left $\mathcal{T}(T)$-approximation of $\Lambda M$.

$$\Gamma N = \text{Hom}_\Lambda(T, T \oplus E_M) \oplus \text{Ext}_\Lambda^1(T, \tau T)$$

Easy: $\Gamma N$ is a generator-cogenerator for $\text{mod}\Gamma$. 


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Easy: $\Gamma N$ is a generator-cogenerator for $\text{mod}\Gamma$.

- $\Lambda T$ splitting, hence every indecomposable $\Gamma X$ lies in $\mathcal{X}(T)$ or $\mathcal{Y}(T)$. In either case we explicitly construct

$$0 \longrightarrow \Gamma N^1 \longrightarrow \Gamma N^0 \overset{\pi}{\longrightarrow} \Gamma X \longrightarrow 0$$

such that $\pi$ is an $\text{add}_\Gamma N$-approximation. Then $\Gamma N$ is an Auslander-generator and $\text{rep.dim}\Gamma \leq 3$. 