Harmonic metrics on unipotent bundles over quasi-compact Kähler manifolds

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Abstract

In this note, we propose an approach to the study of the analogue for unipotent harmonic bundles of Schmid’s Nilpotent Orbit Theorem. Using this approach, we construct harmonic metrics on unipotent bundles over quasi-compact Kähler manifolds with carefully controlled asymptotics near the compactifying divisor; such a metric is unique up to some isometry. Such an asymptotic behavior is canonical in some sense.

1 Introduction

In this paper, we construct a harmonic metric on a unipotent bundle over a quasi-compact Kähler manifold. A quasi-compact Kähler manifold here is a Kähler manifold that can be compactified to a complex variety by adding a divisor that consists of smooth hypersurfaces with at most normal crossings as singularities. The key point of our construction is the prescribed and carefully controlled asymptotic behavior of the harmonic metric when approaching that compactifying divisor. This control is canonical in some sense. In fact, we have shown existence results for harmonic metrics on unipotent bundles already in our earlier papers [10], but the methods employed there do not yield the control near the compactifying divisor. Therefore, here we shall develop a new and more subtle method.

We now introduce some notation. Let $\overline{X}$ be an $n$-dimensional compact Kähler manifold, $D$ a normal crossing divisor; $X = \overline{X} \setminus D$ then is the quasi-compact Kähler manifold that we are interested in. Let

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¶Supported partially by SFB/TR 45 Periods, Moduli Spaces and Arithmetic of Algebraic Varieties of the DFG.
\( \rho : \pi_1(X) \to SL(r, \mathbb{Z}) \subset SL(r, \mathbb{C}) \) be a linear representation. Equivalently, we have a flat vector bundle \( L_{\rho} \) over \( X \).

Take \( p \in D \) and a small neighborhood in \( X \) of \( p \), say \( \Delta \). By our normal crossing assumption, we can assume that \( \Delta \cap X = (\Delta^*)^k \times \Delta^{n-k} \), where \( \Delta^* \) is the punctured disk. Let \( (z_1, z_2, \cdots, z_n) \) be the corresponding local complex coordinate at \( p \) covering \( \Delta \), \( z_i = r_i e^{\sqrt{-1} \theta_i} (1 \leq i \leq n) \). Let \( \gamma_1, \gamma_2, \cdots, \gamma_k \) be the generators of \( \pi_1((\Delta^*)^k \times \Delta^{n-k}) \), where \( \gamma_i \) corresponds to the \( i \)-th \( \Delta^* \) \((1 \leq i \leq k)\). We restrict \( \rho \) to \( \pi_1((\Delta^*)^k \times \Delta^{n-k}) \) (note that the induced map from \( \pi_1((\Delta^*)^k \times \Delta^{n-k}) \) into \( \pi_1(X) \) is not necessarily injective), and denote \( \rho(\gamma_i) \) still by \( \gamma_i \). We note that \( \gamma_1, \gamma_2, \cdots, \gamma_k \) commute, which will be important for our constructions below.

In this note, we always assume that each \( \gamma_i \) is unipotent; so \( N_i = \log \gamma_i \) is nilpotent and the \( N_1, \cdots, N_k \) commute. We call \( L_{\rho} \) a unipotent bundle. If \( L_{\rho} \) is endowed with a harmonic metric \( h \) with tame growth condition (Simpson’s terminology [22]) near the divisor, we call \( (L_{\rho}, h) \) a unipotent harmonic bundle. For the notion of general harmonic bundles and its history and related definitions, we refer the reader to [23].

The notion of a unipotent harmonic bundle is a natural generalization of a variation of Hodge structures introduced by P. Griffiths [7, 22]. In the study of variations of Hodge structures, it is very important to understand the asymptotic behavior near the divisor of the Hodge metric [3, 15]; likewise, for the study of a unipotent harmonic bundle, we also need to have a good understanding of the asymptotic behavior of the endowed tame harmonic metric. In this aspect, Mochizuki [18], along the way initiated by Cattani-Kaplan-Schmid [2], provides a systematic description for the asymptotic behavior of tame harmonic metrics on a unipotent bundle. In this paper, we try to propose a slightly different approach to the study of asymptotic behavior of a unipotent harmonic bundle; our approach is perhaps able to be considered as an analogue of Schmid’s Nilpotent Orbit Theorem [20].

In order to formulate the idea, we first go back to Schmid’s earlier paper [20]; there, in the case of one variable, an equivalent description of the asymptotic behavior of the period mapping (equivalently, the Hodge metric) is developed in terms of an equivariant geodesic embedding of the upper half plane into the period domain—a homogeneous (complex) manifold; although this was not explicitly stated, it clearly is implied by the nilpotent orbit theorem and \( SL_2 \)-orbit theorem for one variable. The existence and uniqueness of such a geodesic embedding are obtained by using Jacobson-Morosov’s theorem and a result of Kostant [16]. We hope to generalize this description to the case of several variables; consequently, we also need to extend Jacobson-Morosov’s theorem and the result of Kostant in some sense. However, we will not directly use such a generalization to obtain general information about
the asymptotic behavior of a unipotent harmonic bundle; instead we first use this idea to construct an initial metric of finite energy on a unipotent bundle and then deform this initial metric to a harmonic metric of finite energy without changing the behavior near the divisor. So, our tame harmonic metric is a more special one, namely one with trivial filtered (or parabolic) structure in the sense of local systems [22, 18]; a general tame harmonic metric may be of infinite energy [12].

The idea of using a geodesic embedding of the upper half plane as the asymptotic behavior of a unipotent harmonic bundle in the case of one variable has already been successfully used in our previous work [14]—the study of cohomologies for unipotent harmonic bundles over a noncompact curve, more precisely, the study of the $L^2$-Poincaré lemma and the $L^2$-Dolbeault lemma for unipotent harmonic bundles.

In order to apply Schmid’s description of a geodesic embedding of the upper half plane as the asymptotic behavior of a Hodge metric to harmonic metrics in the higher dimensional case, one needs to suitably extend Jacobson-Morosov’s theorem and the result of Kostant; we feel that such an extension is an appropriate (algebraic) substitute in the harmonic bundles theory of Cattani-Kaplan-Schmid’s $\text{SL}_2$-orbit theorem for several variables in the Hodge theory. In §2, we work out this extension; mainly, we use the notions of parabolic subgroups (algebras) and the corresponding horospherical decomposition. Our argument is slightly geometric; we think that there should also exist a purely Lie-theoretic proof.

In Schmid’s description, one of the key points is how to get a related geodesic embedding by using one single nilpotent element; this is achieved by finding a semi-simple element by using Jacobson-Morosov’s theorem, then Kostant’s result implies uniqueness in a certain sense. For several commuting nilpotent elements $N_1, \ldots, N_k$ in the present setting, correspondingly we hope to get a group of semi-simple elements, $Y_1, \ldots, Y_k$, which are commutative and such that each pair $(N_i, Y_i)$ can be extended to an $\mathfrak{sl}_2$-triple. (We here remark that in general these triples are not commutative.) The commutativity of $Y_1, \ldots, Y_k$ implies that they are contained in a maximal abelian subspace; this motivates us to use the theory of parabolic subgroups (subalgebras). We also remark that these semisimple elements are unique after fixing a maximal abelian subspace.

After getting the semi-simple elements $Y_1, \ldots, Y_k$, in §3, we are able to construct initial metrics of finite energy. These metrics are not yet harmonic, but for our local construction, it is indeed harmonic when restricted to a punctured disk transversal to the divisor; this is enough for us to prove the asymptotic behavior of the obtained harmonic metric.

Let $\mathcal{P}_\gamma$ be the set of positive definite hermitian symmetric matrices
of order $r$ with determinant 1. $SL(r, \mathbb{C})$ acts transitively on $\mathcal{P}_r$ by

$$g \circ H =: g H \bar{g}, \quad H \in \mathcal{P}_r, g \in SL(r, \mathbb{C}).$$

Obviously, the action has the isotropy subgroup $SU(r)$ at the identity $I_r$. Thus $\mathcal{P}_r$ can be identified with the symmetric space of noncompact type $SL(r, \mathbb{C})/SU(r)$, and can be uniquely endowed with an invariant metric up to some constant. Our local construction at $p$ then takes the following form

$$H_0(z_1, z_2, \cdots, z_n) = \exp(\frac{1}{2\pi} \sum_{i=1}^{k} \theta_i N_i) \circ \exp(\sum_{i=1}^{k} \frac{1}{2} \log |\log r_i|) Y_i). \quad (1)$$

Such a construction is compatible for all $z_i$-directions; namely, for each $z_i$-direction, it gives an equivariant geodesic embedding of the upper half plane.

In §4, we then deform the initial metric of finite energy to a harmonic metric with the same asymptotic behavior. In order to make this deformation successful, one needs to impose a geometric condition on the representation $\rho$, namely semi-simplicity; such a condition first appeared in [6] and then [4]. Under this condition, we can deform the initial metric to a harmonic one of finite energy. The harmonic metric is pluriharmonic by using Siu’s Bochner technique for harmonic map theory into a Riemannian manifold due to Sampson [19]. The essential difficulty is to prove that the harmonic metric has the same asymptotic behavior as the initial metric; to this end, restricting the harmonic metric and the above local construction of the initial metric to a punctured disk transversal to the divisor so that both of them are harmonic, we can then show that the distance function between both metrics is actually bounded on the punctured disk, which accordingly implies that the harmonic metric and the initial metric have the same asymptotic behavior.

We can now state our main results.

**Theorem 1** Let $\overline{X}$ be an $n$-dimensional compact Kähler manifold, $D$ a normal crossing divisor; set $X = \overline{X} \setminus D$. Let $\rho : \pi_1(X) \to SL(r, \mathbb{Z}) \subset SL(r, \mathbb{C})$ be a linear semisimple representation that is unipotent near the divisor, and $L_\rho$ the corresponding unipotent bundle. Then, up to a certain isometry in $\mathcal{P}_r$, there exists uniquely a harmonic metric on $L_\rho$ with the same asymptotic behavior as $H_0$ (see (1)) near the divisor.

Our proof in §4 also gives the following corollary where we do not assume that $\rho$ is semisimple, which is perhaps able to be considered as an analogue of Schmid’s Nilpotent Orbit Theorem. Based on this, we consider the asymptotic behavior (1) as a canonical one.
Theorem 2 Any harmonic metric $H$ on $L_\rho$ of finite energy must have the same asymptotic behavior as the local construction (1) near the divisor; namely the distance function $\text{dist}_p(H, H_0)$ is bounded near the divisor.

Remark: The point in Theorem 2 that is unsatisfactory is that we can not yet give the decay estimate of the distance function $\text{dist}_p(H, H_0)$ near the divisor presently.

The research of harmonic metrics on noncompact manifolds was initiated by Simpson [22] in the complex one dimensional case under a more algebraic geometric background; there, he also suggest that one should study the analogues for harmonic bundles of the nilpotent orbit theorem and the $SL_2$-orbit theorem in the Hodge theory; this paper can be considered as an attempt to it. The general construction of harmonic metrics on a quasi-compact Kähler manifold was later considered in [10, 11] in a more general setting — equivariant harmonic maps. [10] also obtained a harmonic metric of finite energy; however, due to their construction, the behavior at infinity of the metric is not controlled; in particular, there is no norm estimate for flat sections when translated into the case of harmonic bundles. Hopefully, our construction will give a new understanding of asymptotic behavior of variation of Hodge structures or period mapping [2].

This work was begun in February-April of 2004 when the second author was visiting CUHK and HKU in Hong Kong and continued in April-June of 2007 when he was visiting the Universities of Mainz and Essen; its main part was finished when he visited the Max-Planck Institute for Mathematics in the Sciences, Leipzig in January-February and July-August of 2009. He thanks the above institutes for hospitality and good working conditions; especially, he wants to thank Professors Ngaiming Mok, Hélène Esnault and Eckart Viehweg for their kind invitation.

2 Construction of semisimple elements

In this section, we first recall Morosov-Jacobson’s theorem, the theorem of Kostant and the horospherical decomposition associated to a parabolic subgroup; then we present the construction of semisimple elements $Y_1, \cdots, Y_k$.

2.1 Some Lie-theoretic and geometric preliminaries

2.1.1. Morosov-Jacobson’s theorem (cf. e.g. [16]): Let $G$ be a noncompact real simple Lie group, $\mathfrak{g}$ the corresponding Lie algebra. As-
sume $N$ is a nilpotent element in $\mathfrak{g}$. Then, one can extend $N$ to an $\mathfrak{sl}(2, \mathbb{R})$-embedding into $\mathfrak{g}$: $\{N, Y, N^-\} \subset \mathfrak{g}$ satisfying

$$
[N, Y] = 2N, \quad [N, Y^-] = -2N^-, \quad [N, N^-] = Y.
$$

Geometrically, this means the following: Let $X$ be the corresponding Riemannian symmetric space of $G$. Then through any fixed point, there exists a geodesic embedding into $X$ of the upper half plane whose horocycles are generated by $N$ and whose geodesics perpendicular to the horocycles are generated by the corresponding semi-simple element (or say the orbits of the corresponding one-parameter group).

2.1.2. Kostant’s theorem (cf. [16]): Let $\mathfrak{g}$ and $N$ be as in Morosov-Jacobson’s theorem. Then,

1) the elements in $\text{Im}(adN) \cap \text{Ker}(adN)$ are nilpotent;
2) if $\{N, Y, N^-\}$ is an $\mathfrak{sl}_2$-embedding extended by $N$ in $\mathfrak{g}$ as in Morosov-Jacobson’s theorem, satisfying

$$
[N, Y] = 2N, \quad [N, Y^-] = -2N^-, \quad [N, N^-] = Y,
$$

then $Y$ is unique up to nilpotent elements in $\text{Im}(adN) \cap \text{Ker}(adN)$.

Consequently, we have: Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition and $\mathfrak{a}$ a maximal abelian subalgebra in $\mathfrak{p}$. If $Y$ lies in $\mathfrak{a}$, then, such a $Y$ is unique. Namely, for a fixed maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$, $Y \in \mathfrak{a}$ is unique.

Geometrically, this means that, through a fixed point in $X$, the geodesic embedding in $X$ of the upper half plane corresponding to $N$ in Morosov-Jacobson’s theorem is unique.

2.1.3. Parabolic subgroups (subalgebras) and the corresponding horospherical decompositions (cf. e.g. Borel and Ji’s book [1])

Let $\mathfrak{g}$ be a real semi-simple Lie algebra of noncompact type, $G$ the corresponding noncompact semisimple Lie group, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ Cartan decomposition, and $\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{p}$. Relative to $\mathfrak{a}$, one has the restricted root system, denoted by $\Phi$; choose a simple root system, denoted by $\Delta$; denote the corresponding positive root system by $\Phi^+$. Let $I$ be a subset of $\Delta$. One can construct a corresponding parabolic subalgebra $\mathfrak{p}_I$ (resp. $P_I$) of $\mathfrak{g}$ (resp. $G$), called the standard parabolic subalgebra (subgroup) relative to $I$. Any parabolic subalgebra (subgroup) is conjugate to such a one under $G$ and also under $K$, the maximal compact subgroup of $G$ corresponding to $\mathfrak{k}$; moreover, for any two distinct subsets $I, I'$ of $\Delta$, $P_I, P_{I'}$ are not conjugate under $G$.

Let $\mathfrak{m}$ be the centralizer in $\mathfrak{k}$ of $\mathfrak{a}$, $\mathfrak{a}_I = \cap_{\alpha \in I} \text{Ker} \alpha \subset \mathfrak{a}$, $\mathfrak{a}'_I$ the orthogonal complement in $\mathfrak{a}$ of $\mathfrak{a}_I$; let $\Phi_I$ be the set of roots generated
by $I$. Set

$$n_I = \sum_{\alpha \in \Phi^-} g^{\alpha}, \quad m_I = m \oplus a^I \oplus \sum_{\alpha \in \Phi^I} g^{\alpha}.$$ 

Then, $p_I = n_I \oplus a_I \oplus m_I$ is the desired parabolic subalgebra, its corresponding subgroup of $G$ denoted by $P_I$.

Corresponding to the above decomposition of $p_I$, one also has the decomposition of $P_I$ in the level of groups: Let $N_I, A_I, M_I$ be the Lie subgroups of $G$ having the Lie algebras $n_I, a_I, m_I$ respectively, then

$$P_I = N_I A_I M_I \cong N_I \times A_I \times M_I,$$

where by $\cong$ we mean an analytic diffeomorphism, i.e. the map

$$(n, a, m) \rightarrow nam \in P_I \quad (n \in N_I, a \in A_I, m \in M_I)$$

is an analytic diffeomorphism. This is the so-called Langlands decomposition of $P_I$. When $I$ is empty, $P_I = P_\emptyset$ is a minimal parabolic subgroup of $G$, the corresponding decomposition is $P_\emptyset = N_\emptyset AM$, here $A = \exp a, M = \exp m$.

Let $K$ be the maximal compact subgroup with the Lie algebra $\mathfrak{k}$ of $G$. The Iwasawa decomposition $G = N_\emptyset A K$ tells us that $G = P_I K$ for any subset $I$ of $\Delta$. So, $P_I$ acts transitively on the symmetric space $X = G/K$. Thus, the Langlands decomposition for $P_I$ induces a decomposition of $X$ associated to $P_I$, called the horospherical decomposition

$$X \cong N_I A_I M_I \cong N_I \times A_I \times M_I,$$

where $X_I = M_I/(M_I \cap K)$, called the boundary symmetric space associated to $P_I$; by $\cong$ we again mean an analytic diffeomorphism, i.e. the map

$$(n, a, m(M_I \cap K)) \rightarrow namK \in X \quad (n \in N_I, a \in A_I, m \in M_I)$$

is an analytic diffeomorphism.

Let $P$ be a parabolic subgroup of $G$. As mentioned before, it is conjugate to a unique standard parabolic subgroup $P_I$ under $K$. Choose $k \in K$ such that under $k$, $P$ is conjugate to $P_I$, denoted by $P =^k P_I$. Define

$$N_P =^k N_I, \quad A_P =^k A_I, \quad M_P =^k M_I.$$ 

Though the choice of $k$ is not unique, the subgroups $N_P, A_P, M_P$ are well-defined. We call $N_P, A_P$ (resp. the corresponding Lie algebras $\mathfrak{n}_P, \mathfrak{a}_P$) the unipotent radical, the split component of $P$ (resp. $\mathfrak{p}$) respectively. Thus, we can translate the Langlands decomposition of $P_I$ into that of $P$, namely

$$P = N_P A_P M_P \cong N_P \times A_P \times M_P.$$
Consequently, we also have the horospherical decomposition of $X = G/K$ associated to $P$

$$X \cong N_P \times A_P \times X_P,$$

where $X_P = M_P/(M_P \cap K)$, called the boundary symmetric space associated with $P$.

### 2.2 The construction of semisimple elements

Now, we return to the setting of §1. As mentioned there, $N_1, N, \cdots, N_k$ are some commutative nilpotent matrices. By Engel’s theorem, we can assume that all of them are upper triangular with the entries of the diagonal being zero, and hence that $\gamma_1, \gamma_2, \cdots, \gamma_k$ are upper triangular with the entries of the diagonal being 1.

Choose a maximal parabolic subalgebra (resp. subgroup) $p$ (resp. $P$) of $\mathfrak{sl}(r, \mathbb{R})$ (resp. $SL(r, \mathbb{R})$) the unipotent radical of which contains $N_1, N, \cdots, N_k$ (resp. $\gamma_1, \gamma_2, \cdots, \gamma_k$); furthermore, we can choose $p$ (resp. $P$) so that its split component is contained in the set of diagonal matrices of $\mathfrak{sl}(r, \mathbb{R})$ (resp. $SL(r, \mathbb{R})$).

We remark that the set of diagonal matrices in $\mathfrak{sl}(r, \mathbb{R})$ is a maximal abelian subspace contained in the noncompact part of a Cartan decomposition of $\mathfrak{sl}(r, \mathbb{R})$; and that the use of the Engel’s theorem shows that we consider the set of diagonal matrices in $\mathfrak{sl}(r, \mathbb{R})$ as such a maximal abelian subspace. We also remark that such a choice of parabolic subalgebras (resp. subgroups) is not unique, even for a fixed maximal abelian subspace.

Let $P = N_PA_PM_P$ ($p = n_P \oplus a_P \oplus m_P$, $N_P = \exp n_P$, $A_P = \exp a_P$) be the Langlands decomposition of $P$, correspondingly, we have the horospherical decomposition

$$SL(r, \mathbb{R})/SO(r) := X = N_PA_PX_P,$$

where $X_P = M_P/(M_P \cap SO(r))$, the boundary symmetric space associated with $P$. From the previous choice for $p$ (resp. $P$), we know that $a_P$ (resp. $A_P$), as a set of matrices, is contained in the set of diagonal matrices.

For $N_i \in n_P$, $i = 1, \cdots, k$, by the Morosov-Jacobson’s theorem, we can extend it to an embedding into $\mathfrak{sl}(r, \mathbb{R})$ of $\mathfrak{sl}(2, \mathbb{R})$, say $\{N_i, Y_i, N_i^-\} \subset \mathfrak{sl}(r, \mathbb{R})$, satisfying

$$[N_i, Y_i] = 2N_i, \quad [N_i, Y_i^-] = -2N_i^-, \quad [N_i, N_i^-] = Y_i.$$

**Lemma 1** For this embedding, one can choose $Y_i$ such that it lies in $a_P$, and hence such a semisimple element is unique.
In the following, we will show that this can actually be done by using (the geometric interpretations of) the Morosov-Jacobson theorem and the Kostant theorem, and the horospherical decomposition associated to \( P \). It should be an interesting question whether one can give a purely Lie-theoretic proof of this result.

Using the Killing form, we can easily show that the factors \( n_P, a_P, m_P \) of \( p \) are orthogonal to each other, so the orbits of \( N_P \) and \( A_P \) in the horospherical decomposition are orthogonal and also orthogonal to the boundary symmetric space \( X_P \); in addition, we also can consider \( X_P \) as the set of fixed points at infinity of \( N_P \) and \( A_P \). Fix a point \( x_0 \in X \). By \( \infty_0 \), we denote the (unique) intersection point of the orbit of \( x_0 \) under \( \exp(tN_\text{i}) \), \( t \in \mathbb{R} \) (denoted by \( \exp(tN_\text{i}) \circ x_0 \)) with \( X_P \). One has then a unique geodesic in \( X \) connecting \( x_0 \) and \( \infty_0 \), denoted by \( \sigma_0 \). Clearly, the orbit \( \exp(tN_\text{i}) \circ \sigma_0, t \in \mathbb{R} \) is a geodesic embedding into \( X \). We need to show that the orbit \( \exp(tN_\text{i}) \circ \sigma_0, t \in \mathbb{R} \) is a geodesic embedding of the upper half plane and hence the orbit of any point in it under \( \exp(tN_\text{i}) \) is its horocycle. This is a consequence of the Morosov-Jacobson theorem and the Kostant theorem. By the Morosov-Jacobson theorem, through \( x_0 \), we have a geodesic embedding of the upper half plane into \( X \); the Kostant theorem implies that such an embedding is unique. On the other hand, \( \exp(tN_\text{i}) \circ x_0 \) is contained in this embedding, so the intersection point of this embedding with \( X_P \) is also \( \infty_0 \). That is to say, this embedding also contains the geodesic \( \sigma_0 \). So, this embedding is just \( \exp(tN_\text{i}) \circ \sigma_0, t \in \mathbb{R} \). Furthermore, by the horospherical decomposition associated to \( P \), we can choose a semisimple element \( Y_i \in a_P \), the orbits in this embedding of the one-parameter group of which are geodesics perpendicular to the horocycles. Such a \( Y_i \) is just the desired one. This completes the proof of the Lemma.

Remarks 1) From the construction, \( Y_1, \cdots, Y_k \in a_P \) seem to depend on the choice of the parabolic subalgebra \( p \). But, by Kostant’s theorem, we know that \( Y_1, \cdots, Y_k \) only depend on the choice of a maximal abelian subspace in the noncompact part of a Cartan decomposition; namely they are unique up to some conjugations when we require that all of them be contained in a maximal abelian subspace. So, when we fix a maximal abelian subspace \( a \), we can uniquely get semisimple elements \( Y_1, \cdots, Y_k \in a \) such that \( [N_i, Y_i] = 2N_i, i = 1, \cdots, k \). 2) The above construction for semisimple elements works for any semisimple Lie algebra, not only for \( \mathfrak{sl}(r, \mathbb{R}) \); so, this may provide a new way to understand variations of Hodge structures and their degeneration, different from Cattani-Kaplan-Schmid’s theory [2].
3 The construction of initial metrics (maps) and their asymptotic behavior

As in the Introduction, let $\overline{X}$ be a compact Kähler manifold, $D$ a normal crossing divisor, $X = \overline{X} \setminus D$. Taking $p \in D$ and a small neighborhood $U$ at $p$, then $X \cap U$ is of the form $(\Delta^*)^k \times \Delta^{n-k}$. Let $(z_1, z_2, \cdots , z_n)$ be a local complex coordinate on $U$ with

$$(\Delta^*)^k \times \Delta^{n-k} = \{(z_1, z_2, \cdots , z_n) : z_1 \neq 0, z_2 \neq 0, \cdots , z_k \neq 0\}.$$ 

On $(\Delta^*)^k \times \Delta^{n-k}$, one has the following product metric

$$ds^2_P = \frac{\sqrt{-1}}{2} \left[ \sum_{i=1}^{k} \frac{dz_i \wedge d\bar{z}_i}{|z_i|^2(\log |z_i|)^2} + \sum_{i=k+1}^{n} dz_i \wedge d\bar{z}_i \right].$$

In general, one has the following

**Proposition 1** There exists a complete, finite volume Kähler metric on $X$ which is quasi-isometric to the metric of the above form near any point in the divisor $D$.

**Proof.** cf. [5].

Due to the above proposition, when we consider local constructions and various estimates near the divisor in this section, if not involving the derivatives of the Kähler metric (in fact we indeed do not need estimates involving the derivatives), we always use the above local product metric $ds^2_P$.

3.1 The construction of initial metrics of finite energy

Using the previous constructions for semisimple elements $Y_1, \cdots , Y_n$, we can construct an initial map from the universal covering of $(\Delta^*)^k \times \Delta^{n-k}$ into $SL(r, \mathbb{C})/U(r)$, which is $\rho$-equivariant; equivalently, we can also consider such a map as a metric on the corresponding flat bundle $L_\rho|_{(\Delta^*)^k \times \Delta^{n-k}}$.

To this end, let us first give some preliminaries. Let $\mathcal{P}_r$ be the set of positive definite hermitian symmetric matrices of order $r$ with determinant 1. $SL(r, \mathbb{C})$ acts transitively on $\mathcal{P}_r$ by

$$g \circ H = : gH \bar{g}, \ H \in \mathcal{P}_r, g \in SL(r, \mathbb{C}).$$

Obviously, the action has the isotropy subgroup $SU(r)$ at the identity $I_r$. Thus $\mathcal{P}_r$ can be identified with the coset space $SL(r, \mathbb{C})/SU(r)$, and
can be uniquely endowed with an invariant metric\(^1\) up to some constants. In particular, under such a metric, the geodesics through the identity \(I_r\) are of the form \(\exp(tA)\), \(t \in \mathbb{R}\), \(A\) being a hermitian symmetric matrix of trace-free.

### 3.1.1 Local construction

Let \((z_1, z_2, \ldots, z_n)\) be the above complex coordinates on \((\Delta^*)^k \times \Delta^{n-k}\), \(z_i = r_i e^{\sqrt{-1} \theta_i}\), \(0 < r_i < 1\), \(-\infty < \theta_i < \infty\). Set

\[
H_0(z_1, z_2, \ldots, z_n) = \exp\left(\frac{1}{2\pi} \sum_{i=1}^{k} \theta_i N_i\right) \circ \exp\left(\sum_{i=1}^{k} \left(1 \frac{1}{2} \log |\log r_i|Y_i\right)\right),
\]

which is independent of \(z_{k+1}, \ldots, z_n\). Clearly, it is \(\rho\)-equivariant. Similar to [13], one can show that under the product metric \(ds^2_{P}\), \(H_0\) has finite energy \(^2\).

Geometrically, the finiteness of the energy can be explained as follows. For any fixed \(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n, i \leq k\), \(H_0\) can be considered as a geodesic (and hence harmonic) isometric embedding

\[
\exp\left(\frac{1}{2\pi} \theta_i N_i\right) \circ \exp\left(\sum_{j \neq i} \left(1 \frac{1}{2} \log |\log r_i|Y_i\right)\right),
\]

of a neighborhood of a point at infinity of the upper half plane, say \(\{w_i \in \mathbb{C} | \Im w_i > \alpha > 0\}\) with \(w_i = -\sqrt{-1} \log z_i\), into \(P_r\), which is equivariant with respect to \(\gamma_i\), as in the 1-dimensional case [14]. This point is also important in the proof in §4; namely, our local constructions are harmonic on punctured disks transversal to the divisor.

Since the metric \(ds^2_{P}\) is a product metric on \((\Delta^*)^k \times \Delta^{n-k}\), for the estimate of the energy density (and energy) of \(H_0\), we can consider each \(\partial_z H_0, i \leq k\) separately. Again since the map

\[
p : \{w_i = x_i + \sqrt{-1} y_i \in \mathbb{C} | y_i > \alpha > 0\}, \quad \frac{dw_i \wedge d\overline{w_i}}{|Im w_i|^2} \rightarrow (\Delta^*, \frac{dz_i \wedge d\overline{z}_i}{|z_i|^2(\log |z_i|)^2})
\]

where \(p(w_i) = z_i = e^{\sqrt{-1} w_i}\), is a Riemannian covering, so restricting to a fundamental domain of \(p\), say \(\{x_i + \sqrt{-1} y_i \in \mathbb{C} | y_i > \alpha > 0, 0 \leq x_i < 1\}\),

---

\(^1\)In terms of matrices, such an invariant metric can be defined as follows. At the identity \(I_r\), the tangent elements just are hermitian symmetric matrices of trace-free; let \(A, B\) be such matrices, then the Riemannian inner product \(\langle A, B \rangle_{p_r}\) is defined by \(\text{tr}(AB)\). In general, let \(H \in P_r\), \(A, B\) two tangent elements at \(H\), then the Riemannian inner product \(\langle A, B \rangle_{p_r}\) is defined by \(\text{tr}(H^{-1}AH^{-1}B)\).

\(^2\)We here remark that, in Proposition 1 of [13], the estimate of \(|dh|^2\) should be read as \(\eta^2 |dh|^2 \leq C\) instead of \(\eta^2 |dh|^2 \leq C|\log r|^{2n}\).
we can write $H_0$ as

$$H_0 = \exp\left(\frac{1}{2\pi} \sum_{i=1}^{k} x_i N_i\right) \circ \exp\left(\sum_{i=1}^{k} \left(\frac{1}{2} \log y_i\right) Y_i\right);$$

and estimating $\partial_{z_i} H_0$ is equivalent to estimating $\partial_{w_i} H_0$. Since for fixed $w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_n$, $H_0$ is a geodesic isometric embedding, we have

$$|\partial_{w_i} H_0| = \text{const}.$$ 

On the other hand, the domain $\{x_i + \sqrt{1-y} y_i \in \mathbb{C} \mid y_i > \alpha > 0, 0 \leq x_i < 1\}$ has finite volume, so the energy of $H_0$ is finite.

### 3.1.2 Patching local constructions together on a tube neighborhood of the divisor

Using the above local construction for $H_0$ and a finite partition of unity on $X$, we can construct a smooth metric on $L_\rho$ which takes the $H_0$ as asymptotic behavior near the divisor $D$, still denoted by $H_0$. So, using the above complete Kähler metric on $X$, such a metric, as a $\rho$-equivariant map, has finite energy. However, although our local constructions are harmonic on punctured disks transversal to the divisor, the metric $H_0$, after patched together, is not necessarily harmonic on such a punctured disk. Here, for convenience later on, we do a special patching; we choose an appropriate partition of unity to patch these local constructions together so that the metric $H_0$ is harmonic on some open subsets of the divisor at infinity when restricted to a certain small punctured disk transversal to the divisor with the puncture contained in the divisor. This can be done as follows.

For simplicity, we assume in the following that $\dim_\mathbb{C} X = 3$, $D = D_1 + D_2$ and $D_1 \cap D_2 \neq \emptyset$; so $\dim_\mathbb{C} D_1 \cap D_2 = 1$. The discussion for the general case is completely similar.

First, we patch together local constructions near the intersection $D_1 \cap D_2$. Take two enough small tube neighborhoods $U_1^{12} \subset U_1^{12}$ of $D_1 \cap D_2$ with a holomorphic projection $\pi^{12} : U_1^{12} \rightarrow D_1 \cap D_2$. We take $U_1^{12}$ enough small so that for any point $p$ of $D_1 \cap D_2$ there exists a neighborhood $U$ in $D_1 \cap D_2$, such that $(\pi^{12})^{-1}(U)$ can be covered by a local complex coordinate $(z_1, z_2, z_3)$ at $p$ of $X$ with $D_1 = \{z_1 = 0\}$ and $D_2 = \{z_2 = 0\}$. Obviously, the fibres of $\pi^{12}$ are the product of two disks.

Take a finite open covering $\{U_\alpha\}$ of $D_1 \cap D_2$ satisfying that each $U_\alpha$ has the property of the above $U$ and that for each $U_\alpha$ one can take a smaller open subset $U'_\alpha \subset U_\alpha$ so that $U'_\alpha \cap U_\beta' = \emptyset$. Corresponding to $\{U'_\alpha \subset U_\alpha\}$, we can choose a partition of unity $\{\phi_\alpha\}$ with $\phi_\alpha|U'_\alpha \equiv 1$; consequently, $\{\phi_\alpha \circ \pi^{12}\}$ is a partition of unity on $U_1^{12}$ with $\phi_\alpha \circ \pi^{12}|_{(\pi^{12})^{-1}(U'_\alpha)} \equiv 1$. Using this partition of unity, we can patch smoothly all local constructions together along $D_1 \cap D_2$ to get a metric of $L_\rho$ on $U_1^{12}$. We remark that from the above patching process, we can see
that the obtained metric is harmonic on \((\pi^{12})^{-1}(U'_\alpha)\) when restricted to punctured disks transversal to \(D_1\) or \(D_2\).

Using the same way, we can patch all local constructions together along \(D_1 - U^{12}_i\) and \(D_2 - U^{12}_i\). For convenience of notations in the next section, we here give some details. Take two enough small tube neighborhoods \(V'_i \subset V_i\) of \(D_i - U^{12}_i\), \(i = 1, 2\); for our purposes, w.l.o.g we may assume that \(V'_i \cap U^{12}_i\) is empty. Then there is a holomorphic projection \(\pi^i : V'_i \to D_i - U^{12}_i\). We also assume that \(V'_i\) is small enough so that for any interior point \(p\) of \(D_i - U^{12}_i\) there exists a neighborhood \(V_i\) of \(D_i - U^{12}_i\), \((\pi^1)^{-1}(V)\) can be covered by a local complex coordinate \((z_1, z_2, z_3)\) at \(p\) of \(X\) with \(D_i = \{z_i = 0\}\). Take a finite open covering \(\{V_\alpha\}\) of \(D_i - U^{12}_i\) satisfying that each \(V_\alpha\) has the property of the above \(V\) and that for each \(V_\alpha\) one can take a smaller open subset \(V'_\alpha \subset V\) so that \(V'_\alpha \cap V'_\beta = \emptyset\). Corresponding to \(\{V'_\alpha \subset V_\alpha\}\), we can choose a partition of unity \(\{\psi_\alpha\}\) with \(\psi_\alpha|_{V'_\alpha} \equiv 1\); consequently, \(\psi_\alpha \circ \pi^i\) is a partition of unity on \(V'_i\) with \(\psi_\alpha \circ \pi^i(\pi^i)^{-1}(V'_\alpha) \equiv 1\). Using this partition of unity, we can patch smoothly all local constructions together along \(D_i - U^{12}_i\) to get a metric of \(L_\rho\) on \(V'_i\).

Finally, we can patch smoothly the above three metrics together so that the metrics on \(U^{12}_i, V^{12}_i - U^{12}_i, V^{12}_2 - U^{12}_i\) are preserved. Thus, near the divisor, we get a smooth metric which is harmonic on certain open subsets, say \((\pi^{12})^{-1}(U'_\alpha) \cap U^{12}_i\) and \((\pi^i)^{-1}(V'_\alpha)\), when restricted to a punctured disk transversal to the divisor. We can then extend the metric smoothly to all of \(X\) to get a metric of \(L_\rho\), still denoted by \(H_0\).

### 3.2 The norm estimate under \(H_0\) of a flat section of \(L_\rho\)

Here, we want to observe what the asymptotic behavior of the norm of a flat section of \(L_\rho\) under the metric \(H_0\) is near the divisor. To this end, we continue to restrict ourselves to \((\Delta^*)^k \times \Delta^{n-k}\). Since the \(\theta_i\)-directions have nothing to do with asymptotic behavior of the norm (actually, we can consider \(\exp\left(\frac{1}{\pi\varepsilon} \sum_{i=1}^k \theta_i N_i\right)\) as an isometry on \(\mathcal{P}_\varepsilon\)), we only need to observe \(\exp\left(\sum_{i=1}^k \frac{1}{2} \log |\log r_i| Y_i\right)\). By the previous construction for semisimple elements \(Y_i\), they can be diagonalized simultaneously under a suitable basis of \(L_\rho\); assuming this and expanding
\[
\exp(\sum_{i=1}^{k} \left( \frac{1}{2} \log \left| \log r_i \right| \right) Y_i),
\]
one has the following form
\[
\begin{pmatrix}
\prod_{i=1}^{k} | \log r_i |^{a_i^1} & 0 & \cdots & 0 & 0 \\
0 & \prod_{i=1}^{k} | \log r_i |^{a_i^2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \prod_{i=1}^{k} | \log r_i |^{a_i^{r-1}} & 0 \\
0 & 0 & \cdots & 0 & \prod_{i=1}^{k} | \log r_i |^{a_i^r}
\end{pmatrix},
\]
where
\[
Y_i = \begin{pmatrix}
a_i^1 \\
0 \\
\vdots \\
0 \\
0 \\
a_i^{r-1} \\
0 \\
0
\end{pmatrix}.
\]
This gives explicitly the asymptotic behavior of the norm of a flat section under the metric \( H_0 \).

**Remark:** In Cattani-Kaplan-Schmid’s theory \([2]\), the norm estimates of flat sections under the Hodge metric depend on the order of \( N_1, \ldots, N_n \) and use the notion of weight filtration, and consequently one needs to restrict to the corresponding sectors of \((\Delta^*)^k\). Our estimate can be considered as a supplement.

### 3.3 The behavior of the differential of \( H_0 \)

Finally, we also need to understand the asymptotic behavior near the divisor of the differential of \( H_0 \). This is very different from the estimates of the norm, where one does not need to consider the \( \theta_i \)-directions.

First, we do an explicit computation for the differential of \( H_0 \). We still restrict to \((\Delta^*)^k \times \Delta^{n-k}\),
\[
H_0 = \exp \left( \frac{1}{2\pi} \sum_{i=1}^{k} \theta_i N_i \right) \circ \exp \left( \sum_{i=1}^{k} \left( \frac{1}{2} \log | \log r_i | \right) Y_i \right).
\]

We can consider \( \exp(\frac{1}{2\pi} \sum_{i=1}^{k} \theta_i N_i) \) as an isometry on \( P_r \); so for the \( r := (r_1, \cdots, r_n) \)-direction, we have
\[
d_r H_0 = \left( \exp \left( \frac{1}{2\pi} \sum_{i=1}^{k} \theta_i N_i \right) \right) \left( \sum_{i=1}^{k} \frac{dr_i}{2r_i \log r_i} Y_i \right).
\]

For the \( \Theta := (\theta_1, \cdots, \theta_n) \)-direction, if we consider the matter at the identity \( I_r \), the differential \( d_{\Theta} H_0 \) should be read as
\[
\frac{1}{2\pi} \sum_{i=1}^{k} N_i \theta_i;
\]
so,

\[ d\Theta H_0 = \exp\left(\frac{1}{2\pi} \sum_{i=1}^{k} \theta_i N_i \right) \exp\left(\sum_{i=1}^{k} \left(\frac{1}{4} \log |\log r_i| Y_i \right) \right) \frac{1}{2\pi} \sum_{i=1}^{k} N_i d\theta_i, \]

here we consider \( H_0 \) as \( \exp\left(\frac{1}{2\pi} \sum_{i=1}^{k} \theta_i N_i \right) \exp\left(\sum_{i=1}^{k} \left(\frac{1}{4} \log |\log r_i| Y_i \right) \right) \circ I_r \), denoted it by \( \sqrt{H_0} \circ I_r \). Translating everything into the complex coordinates \((z_1, \cdots, z_n)\), we have

\[ d_r H_0 = \frac{1}{4} \left( \exp\left(\frac{1}{2\pi} \sum_{i=1}^{k} \theta_i N_i \right) \right) \sum_{i=1}^{k} \left( \frac{dz_i}{z_i} + \frac{d\overline{z}_i}{\overline{z}_i} \right) \frac{Y_i}{|\log |z_i||}, \]

\[ d\Theta H_0 = \frac{1}{4\pi \sqrt{-1}} \left( \sqrt{H_0} \right) \sum_{i=1}^{k} N_i \left( \frac{dz_i}{z_i} + \frac{d\overline{z}_i}{\overline{z}_i} \right). \]

As showed before, under the product metric \( ds^2_\rho \) on \((\Delta^*)^k \times \Delta^{n-k}\) and the invariant metric on \( \mathcal{P}_r \), \( \|dH_0\| = \text{constant}. \) So, we have

\[ \|d_r H_0\|^2, \|d\Theta H_0\|^2 \leq C, \]

for some positive constant \( C \); in particular, \( \|N_i d_{z_i} z_i \|^2 \leq C. \) Consequently, since \( |\frac{d\overline{z}_i}{\overline{z}_i}|^2 = |\log |z_i||^2 \) under the positive metric \( ds^2_\rho \), \( N_i \), as an endomorphism of the bundle \( L_\rho \), has the following point-wise norm estimate\(^3\)

\[ \|N_i\|^2 \leq C |\log |z_i||^{-2}, \]

where we always consider \( \sum_{i=1}^{k} N_i \frac{dz_i}{z_i} \) as a 1-form homomorphism on \( L_\rho \).

**Remark:** We remark that such a norm estimate is not necessarily precise; for example, in the case of \( N_i = N_j \), one even has \( \|N_i\|^2 \leq C |\log |z_i||^{-2} \) \( |\log |z_j||^{-2}. \)

We now express the above estimates in terms of weight filtrations as follows. This is easy if we note the relation \( [N_i, Y_i] = 2N_i; \) corresponding to the diagonalisation of \( Y_i \), we can consider the weight filtration of \( N_i \): \( \{W_k\} \) satisfying \( N_i W_k \subset W_{k-2} \). If \( v \) is a flat section lying in \( W_k \), satisfying

\[ \|v\|^2_{H_0} \sim \prod_{j=1}^{k} |\log r_j|^a_j \]

and \( N_i v \neq 0 \), one then has

\[ \|N_i v\|^2_{H_0} \leq C |\log r_i|^{a_i-2} \prod_{j \neq i} |\log r_j|^{a_j}. \]

---

\(^3\)The norm of a tangent vector of \( \mathcal{P}_r \) is equivalent to its norm when it is considered as an endomorphism of \( L_\rho \), cf. [24].
By the above computation and argument, we now have

\[
\partial H_0 = \frac{1}{4} \left( \exp \left( \frac{1}{2\pi} \sum_{i=1}^{k} \theta \cdot N_i \right) \sum_{i=1}^{k} \frac{dz_i}{z_i} \frac{Y_i}{\log |z_i|} \right) + \frac{1}{4\pi \sqrt{-1}} \left( \sqrt{H_0} \right) \sum_{i=1}^{k} N_i \frac{dz_i}{z_i}
\]

and

**Proposition 2** Using \( \frac{dz_i}{z_i} \) as basis, up to some isometries, \( \partial H_0 \) is asymptotic to \( \sum_{i=1}^{k} N_i \frac{dz_i}{z_i} \) as \( |z_i| \) goes to zero. Also, from the previous discussion, we know that when considering \( \partial H_0 \) as a 1-form homomorphism on \( L_\rho \), it is point-wise bounded under the complete Kähler metric on \( X \), and hence (in applications) \( L^2 \)-bounded as an operator between certain \( L^2 \)-spaces.

4 The proofs of Theorem 1, 2

In order to deform the initial metric \( H_0 \) into a harmonic one with the same asymptotic behavior as \( H_0 \), from now on, we assume that the representation \( \rho \) is semisimple. Namely, for any boundary component \( \Sigma \) of \( P_r \), there exists an element \( \gamma \in \pi_1(X) \) satisfying \( \Sigma \cap \rho(\gamma)(\Sigma) = \emptyset \); in other words, the image of \( \rho \) does not fix any boundary component or is not contained in any proper parabolic subgroup \([6, 4, 24]\). For similar definitions cf. also \([9, 17]\).

Our strategy is first to deform \( H_0 \) to a harmonic metric, and then to prove that the harmonic metric has the same asymptotic behavior as \( H_0 \). The first step is standard with the assumption that the representation \( \rho \) is semisimple; we here give a sketch together with some necessary properties.

4.1 Harmonic metric obtained by deforming the initial metric \( H_0 \)

In the following, we consider \( X = X - D \) as a complete noncompact manifold with the Kähler metric constructed in Proposition 1; sometimes, we also need the Kähler metric to be locally of the product form \( ds^2_{P} \); we can do this in those cases that do not involve its derivatives.

Take a sequence of compact manifolds \( \{X_i\} \) (with smooth boundary) such that \( X_i \subset X_{i+1} \) and \( \bigcup_{i=1}^{\infty} X_i = X \). Then, according to Hamilton \([8]\) (the theory of Hamilton readily applies to the equivariant setting), we can find a harmonic metric \( H_i \) of \( L_\rho \) on \( X_i \) with \( H_i|_{\partial X_i} = H_0|_{\partial X_i} \) and \( E(H_i; X_i) \leq E(H_0; X_i) \leq E(H_0) \). Next, we need to prove that
there exists a subsequence of \( \{H_i\} \) which converges uniformly on any compact subset of \( X \). Fix a compact subset \( X_0 \) of \( X \) and a point \( p \in X_0 \). Then the uniform boundedness of the energies of \( \{H_i\} \) imply that the energy densities \( e(H_i) \) are also uniformly bounded in \( i \) on \( X_0 \). Using the semisimplicity of \( \rho \) and the uniform boundedness of the energy densities \( e(H_i) \), one can show that \( H_i(p) \in \mathcal{P}_r \) are uniformly bounded in \( i \) (here, considering \( H_i \) as equivariant map into \( \mathcal{P}_r \), cf. [24]); so, \( \{H_i(X_0) \subset \mathcal{P}_r\} \) are also uniformly bounded in \( i \). Taking a diagonal sequence, we can find a subsequence of \( \{H_i\} \), still denoted by \( \{H_i\} \), which converges uniformly on any compact subset of \( X \) to a harmonic metric \( H \) of \( L_\rho \) on \( X \). Furthermore, \( E(H) \leq E(H_0) < \infty \).

Due to the finiteness of energy of the harmonic metric \( H \), Siu’s Bochner technique for harmonic maps [19] (which also applies to the equivariant setting) implies

**Lemma 2** The harmonic metric \( H \), as an equivariant map into \( \mathcal{P}_r \), is pluriharmonic; namely, when restricted to any complex curve, especially a punctured disk transversal to the divisor \( D \), \( H \) is harmonic.

### 4.2 \( H \) having the same asymptotic behavior as \( H_0 \)

Now, we can show that \( H \) and \( H_0 \) have the same asymptotic behavior near the divisor; we only need to show that \( H \) has the same asymptotic behavior as every local construction \( H_0 \) in §3.1. For simplicity of discussion, we continue to restrict to the case that \( \dim \mathbb{C} X = 3 \), \( D = D_1 + D_2 \) and \( D_1 \cap D_2 \neq \emptyset \) and to use the notations in §3.1.2.

We first consider the situation near \( D_1 \cap D_2 \), i.e. in \( U_\beta^{12} - D_1 \cup D_2 \). Fix arbitrarily an open set \( U_\beta \) in the finite open covering \( \{U_\alpha\} \) of \( D_1 \cap D_2 \) and a point \( p \in U_\beta \subset U_\alpha \). By the choice of the open covering, we have a local coordinate \((z_1, z_2, z_3)\) at \( p \) of \( \overline{X} \) covering \((\pi^{12})^{-1}(U_\beta') \cap U_\beta^{12} \) with \( p = (0, 0, 0) \), \( D_1 = \{z_1 = 0\} \), and \( D_2 = \{z_2 = 0\} \). By the local construction of \( H_0 \) at \( p \) in §3.1, \( H_0 \) is harmonic and of finite energy when restricted to a punctured disk (in \((\pi^{12})^{-1}(U_\beta') \cap U_\beta^{12} - D_1 \cup D_2 \)) transversal to \( D_1 \) or \( D_2 \); on the other hand, \( H \) has the same properties by its pluriharmonicity and energy finiteness. Now, we claim

**Lemma 3** \( \text{dist}_{\mathcal{P}_r}(H, H_0)(z'_1, z'_2, 0) \) \(^4\) is uniformly bounded for \( (z'_1, z'_2, 0) \in (\pi^{12})^{-1}(U_\beta') \cap U_\beta^{12} - D_1 \cup D_2 \).

**Proof:** First, for a fixed \( z'_1 \), define

\[
S^2_{z'}(z'_1) = \{(z'_1, z, 0) \in (\pi^{12})^{-1}(U_\beta') \cap U_\beta^{12} - D_1 \cup D_2 \}.
\]

\(^4\) Considering \( H, H_0 \) as equivariant maps from the universal covering into \( \mathcal{P}_r \), the distance function between \( H, H_0 \) is still equivariant, so can be considered as a function on the base manifold.
w.l.o.g., we can assume it is a punctured disk which is transversal to $D_2$ at $(z'_1, 0, 0)$. As pointed out above, $H_0$ and $H$ are harmonic when restricted to $S^2_D(z'_1)$. We now prove
\[
\sup_{x \in S^2_D(z'_1)} \text{dist}(H, H_0)(x) \leq \sup_{x \in \partial U^{12}_i \cap S^2_D(z'_1)} \text{dist}(H, H_0)(x).
\]

In order to prove the above inequality, we consider the sequence of harmonic metrics $\tilde{H}_i$ on $S^2_D(z'_1) \cap X_i$ with
\[
\tilde{H}_i|_{\partial U^{12}_i \cap S^2_D(z'_1)} = H_i|_{\partial U^{12}_i \cap S^2_D(z'_1)}, \quad \tilde{H}_i|_{\partial X_i \cap S^2_D(z'_1)} = H_0|_{\partial X_i \cap S^2_D(z'_1)}.
\]

We remark that $H_i|_{\partial U^{12}_i \cap S^2_D(z'_1)}$ converges uniformly to $H|_{\partial U^{12}_i \cap S^2_D(z'_1)}$. It is easy to prove that the sequence $\{\tilde{H}_i\}$ (if necessary, go to a subsequence) converges uniformly on any compact subset of $S^2_D(z'_1)$ to a harmonic metric $\tilde{H}$ on $S^2_D(z'_1)$ and that, by the subharmonicity of $\text{dist}(\tilde{H}_i, H_0)$ on $S^2_D(z'_1) \cap X_i$ and the maximum principle,
\[
\sup_{x \in S^2_D(z'_1)} \text{dist}(\tilde{H}, H_0)(x) \leq \sup_{x \in \partial U^{12}_i \cap S^2_D(z'_1)} \text{dist}(\tilde{H}, H_0)(x);
\]
and hence $\tilde{H}$ also has finite energy. Thus, on $S^2_D(z'_1)$, we have two harmonic metrics $H, \tilde{H}$ with finite energy and $H|_{\partial U^{12}_i \cap S^2_D(z'_1)} = H_i|_{\partial U^{12}_i \cap S^2_D(z'_1)}$.

We now want to show $\tilde{H} \equiv H$ on $S^2_D(z'_1)$, and hence the required inequality. We observe the distance function $\text{dist}(\tilde{H}, H)$ on $S^2_D(z'_1)$, denoted by $w$. It is clear that $w$ is subharmonic and $w|_{\partial U^{12}_i \cap S^2_D(z'_1)} = 0$; on the other hand, by the finiteness of energy of $\tilde{H}$ and $H$, a simple computation shows that $w$ is of finite energy. Thus, the problem is reduced to show the following lemma (for its proof, cf. Appendix).

**Lemma 4** Let $\Delta^*$ be a puncture disk. Assume that $w$ is a non-negative subharmonic function vanishing on the exterior boundary and of finite energy, then $w \equiv 0$.

Assume that $\sup_{x \in \partial S^2_D(z'_1)} \text{dist}(H, H_0)(x)$ is attained at the point $(z'_1, z^0_2, 0)$. Define
\[
S^1_D(z^0_2) = \{(z, 0, 0) \in (\pi^{12})^{-1}(U'_\beta) \cap U^{12}_c \cup D_1 \cup D_2\};
\]
also assume that it is a punctured disk which is transversal to $D_1$ at $(0, 0, 0)$. The same argument as above shows
\[
\sup_{x \in S^1_D(z^0_2)} \text{dist}(H, H_0)(x) \leq \sup_{x \in \partial U^{12}_i \cap S^2_D(z'_1)} \text{dist}(H, H_0)(x).
\]

---

5Since $\tilde{H}_i, H_0$ are harmonic and $P_\gamma$ has non-positive curvature, a standard computation shows that $\text{dist}(\tilde{H}_i, H_0)$ is subharmonic; cf. e.g. [21].
Assuming that \( \sup_{x \in \partial U^{1,2} \cap S^1 (z_2)} \text{dist}(H, H_0)(x) \) is attained at the point \((z'_1, z'_2, 0)\), then we have
\[
\text{dist}(H, H_0)(z'_1, z'_2, 0) \leq \text{dist}(H, H_0)(z_1, z_2, 0)
\]
for \((z'_1, z'_2, 0) \in (\pi^{1,2})^{-1}(U_{\beta}^{1,2}) \cap U_{\epsilon}^{1,2} - D_1 \cup D_2\). Clearly, we can choose a fixed compact subset of \(X\), which is independent of \((z'_1, z'_2, 0)\) and always contains the corresponding \((z_1, z_2, 0)\). This proves the lemma.

The arbitrariness of \(p\) and the above lemma imply that \(H\) and \(H_0\) have the same asymptotic behavior at infinity on \(U^{1,2} \cap \epsilon \cup D_1 \cup D_2\). Using a similar argument as above, we can also show that \(H\) and \(H_0\) have the same asymptotic behavior at infinity on \(V_i \cap \epsilon \cup D_i\), \(i = 1, 2\). Thus, we have showed that \(H\) and \(H_0\) have the same asymptotic behavior at infinity on \(X\).

We remark that in the above proof, we only use two properties of the harmonic metric \(H\): finiteness of energy and pluriharmonicity. So we have

**Theorem 3** Any harmonic metric \(H\) on \(L_\rho\) (\(\rho\) is not necessarily semisimple) of finite energy must have the same asymptotic behavior as the local construction \((1)\) in the introduction near the divisor.

Now, we turn to the uniqueness problem. This is easy. Assume \(H_1, H_2\) are two harmonic metrics with the same asymptotic behavior as \(H_0\). Then, \(\text{dist}_{p_\epsilon}(H_1, H_2)\) is a bounded subharmonic function on \(X\). A well-known fact which says that a bounded positive subharmonic function on a complete Riemannian manifold with finite volume has to be constant implies
\[
\text{dist}_{p_\epsilon}(H_1, H_2) = \text{constant}.
\]
This shows that \(H_1\) and \(H_2\) are the same up to an isometry of \(P_\epsilon\).

**Appendix:** Since the punctured disk is conformal to a half cylinder, so Lemma 4 is equivalent to the following

**Lemma 5** Let \(C\) be a half cylinder with Euclidean metric. Assume that \(w\) is a non-negative subharmonic function vanishing on \(\partial C\) and of finite energy, then \(w \equiv 0\).

Take a global coordinate \((\theta, y)\) on \(C\) with \(0 \leq \theta \leq \pi\) and \(y \geq 0\); the Euclidean metric is \(d\theta^2 + dy^2\). Take a sequence of compact subsets \(C_n = \{ y \leq n \}, n = 1, 2, \ldots \), and consider a sequence of harmonic functions \(u_n\) with \(u_n|_{\partial C} = 0\) and \(u_n|_{y=n} = w_n|_{y=n}\). The maximum principle implies \(u \geq w\) on \(C_n\) and \(E(u_n; C_n) \leq E(w; C_n) \leq E(w)\). The standard elliptic estimate implies \(u_n\) converges to a harmonic function \(u\) on \(C\).
uniformly on any compact subset, which satisfies $u \geq w$, $u|_{\partial C} = 0$, and $E(u) \leq E(w)$. So, the proof of Lemma 5 can be reduced to the following fact which is known to analysts but whose proof we include here for completeness.

**Lemma.** Let $C$ be a half cylinder with Euclidean metric. Assume that $u$ is a non-negative harmonic function with $u|_{\partial C} = 0$ and finite energy, then $u \equiv 0$.

**Proof of Lemma.** Take a global coordinate $(\theta, y)$ on $C$ with $0 \leq \theta \leq \pi$ and $y \geq 0$; the Euclidean metric is $d\theta^2 + dy^2$. Take a smooth cut-off function $\psi_{r,\rho}$ with $\psi_{r,\rho}(\theta, y) = 1$, $y \leq r$, $\psi_{r,\rho}(\theta, y) = 0$, $y \geq r + \rho$, $\rho > 0$ and $|\nabla \psi_{r,\rho}| \leq \frac{1}{\rho}$.

Compute

$$0 = -\int_C \Delta u \psi_{r,\rho}^2$$

$$= -\int_C \nabla(\nabla u \psi_{r,\rho}^2) + \int_C |\nabla u|^2 \psi_{r,\rho}^2 + 2 \int_C \nabla u \psi_{r,\rho} \nabla \psi_{r,\rho} u$$

$$\geq \int_C |\nabla u|^2 (\psi_{r,\rho})^2 - (\frac{1}{2}) \int_C |\nabla u|^2 (\psi_{r,\rho})^2 + 2 \int_C |\nabla \psi_{r,\rho}|^2 u^2.$$ 

So, we have

$$\int_C |\nabla u|^2 (\psi_{r,\rho})^2 \leq 4 \int_C |\nabla \psi_{r,\rho}|^2 u^2.$$ 

It is clear that $\int_C |\nabla u|^2 (\psi_{r,\rho})^2 \to E(u)$, as $r$ goes to infinity. So, if we can choose appropriate $\psi_{r,\rho}$ so that $\int_C |\nabla \psi_{r,\rho}|^2 u^2$ can be arbitrarily small, then $E(u) = 0$ and hence $u = 0$.

Take a sufficiently small $\epsilon$. Since $E(u) < \infty$, there exists $r_0$ such that $\int_{y \geq r} |\nabla u|^2 \leq \epsilon$ for $r \geq r_0$. On the other hand,

$$\int_C |\nabla \psi_{r,\rho}|^2 u^2 \leq \frac{1}{\rho^2} \int_{r \leq y \leq r + \rho} u^2.$$ 

We now estimate $\int_{r \leq y \leq r + \rho} u^2$. Since $u$ is harmonic, $|\nabla u|^2$ is subharmonic. So by the average value inequality, we have

$$|\nabla u|^2(x) \leq c \int_{r + i - 1 \leq y \leq r + i + 2} |\nabla u|^2, \ r + i \leq y(x) \leq r + i + 1,$$

for $i = 0, 1, \cdots, n - 1$, where $c$ is a positive constant independent of $r, i$. 

First, estimate \(|u(\theta, r) - u(\theta, r + n)|\) as follows.

\[
|u(\theta, r) - u(\theta, r + n)|^2 = \left| \sum_{i=0}^{n-1} (u(\theta, r + i) - u(\theta, r + i + 1)) \right|^2
\]

\[
\leq \left( \sqrt{c} \sum_{i=0}^{n-1} \int_{r+i-1 \leq y \leq r+i+1} |\nabla u|^2 \right)^2
\]

\[
\leq 3cn \int_{r-1 \leq y \leq r+n+1} |\nabla u|^2
\]

\[
\leq 3cn \int_{y \geq r_0} |\nabla u|^2 \leq 3ce
\]

In general, we have

\[
|u(\theta, r) - u(\theta, r + s)|^2 \leq 3ces.
\]

On the other hand, we have

\[
|u(\theta, r + s)|^2 \leq 2|u(\theta, r + s) - u(\theta, r)|^2 + 2|u(\theta, r)|^2.
\]

Collecting the above, we have

\[
\frac{1}{\rho^2} \int_{r \leq y \leq r + \rho} u^2 \leq \frac{1}{\rho^2} \int_{r \leq y \leq r + \rho} (2|u(\theta, y) - u(\theta, r)|^2 + 2|u(\theta, r)|^2)
\]

\[
\leq \frac{1}{\rho^2} \int_{r \leq y \leq r + \rho} (6ce(y - r) + 2|u(\theta, r)|^2).
\]

Take \(r = r_0 + 1\) and \(\rho\) sufficiently large, we have

\[
\frac{1}{\rho^2} \int_{r \leq y \leq r + \rho} u^2 \leq c'e,\]

for a certain positive constant. Thus, when \(r, \rho\) are sufficiently large, \(\int_C |\nabla \psi_{r,\rho}|^2 u^2\) becomes arbitrarily small. The proof of the lemma is finished.

**References**


