Quadratic programs over the Stiefel manifold

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Abstract

We characterize the optimal solution of a quadratic program over the Stiefel manifold with an objective function in trace formulation. The result is applied to relaxations of HQAP and MTLS. Finally, we show that strong duality holds for the Lagrangian dual, provided some redundant constraints are added to the primal program.

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1. Introduction

In this paper we will study the following quadratic program with quadratic equality constraints in the form of column orthogonality:

\[ \text{QP} : \min_{X \in \mathcal{H}_{m,n}} \text{tr} AX^H BX, \]

where \( A \in C^{n \times n}, \ B \in C^{m \times m} \) are Hermitian matrices, and \( \mathcal{H}_{m,n} \) denotes the set of \( m \times n \) orthogonal matrices. Throughout the paper, we assume \( m \geq n \) and denote \( \mathcal{H}_{m,n} \) by \( \mathcal{H}_{n} \) if \( m = n \). This constraint set is known as the Stiefel manifold, and arises in such areas as the symmetric eigenvalue problem, nonlinear eigenvalue problem and signal processing. For theories and algorithms for general nonlinear optimization on the Stiefel manifold, see [7] and the references therein. When \( m = n \), QP covers a relaxation of the homogeneous quadratic assignment problem, and when \( A = I_n \), QP describes a relaxation of the multi-dimensional total least squares problem.

The quadratic assignment problem (QAP) is used to model the problem of allocating a set of \( n \) facilities to a set of \( n \) locations while minimizing the quadratic objective arising from the distance between the locations in combination with the flow between the facilities. The QAP in the trace formulation takes the form

\[ \text{QAP} : \min_{X \in \mathcal{P}} \text{tr} AX^T BX + CX, \]

where \( A, B \in R^{n \times n} \) are symmetric matrices, \( C \in R^{n \times n} \), and \( \mathcal{P} \) denotes the set of permutation matrices. The problem is of interest both for the applicability and its difficulty. It is well known that the travelling