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Spectra of Random Graphs

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Outline

- Introduction to the random graph $G(n,p)$ and random subgraph of hypercube.
- The largest eigenvalue of the two random graphs.
- Some conjectures about the way to estimate the largest eigenvalues of different random graph.

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Definition and Background

- Let $G = (V, E)$ be a simple graph (without loop and multiedges) with vertex set V and edge set $E(G)$.
- $A(G) = (a_{ij})$: the adjacency matrices of G with $a_{ij} = 1$ if $v_i \sim v_j$; $a_{ij} = 0$ otherwise.
- The eigenvalues of $A(G)$ are denoted by

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G);$$

called the *eigenvalues* of G. The family

$$\{\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)\}$$

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- The random graph G(n,p) is the discrete probability space composed of all labeled graphs on the vertices $\{1, \ldots, n\}$, where each edge $(i, j), 1 \leq i < j \leq n$, appears randomly and independently with probability $p = p(n)$.
  This model is named for Paul Erdős and Alfréd Rény, who first introduced it in 1959.

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- Let Q^n be a graph whose vertices are all the vectors $\{x = (x_1, \ldots, x_n) \mid x_i \in \{0, 1\}\}$ and two vectors x and y are adjacent if they differ in exactly one coordinate, i.e., $\sum_i \mid x_i - y_i \mid = 1$. We call Q^n the n-dimensional cube or simply the n-cube.
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Estimate of the Spectrum

Theorem 1

(M.Krivelevich and B.Sudakov 2002) Let $G = G(n, p)$ be a random graph and let Δ be the maximum degree of G. Then almost surely the largest eigenvalue of the adjacency matrix of G satisfies $\lambda_1(G) = (1 +_o (1))max\{\sqrt{\Delta}, np\}$, where the $_o(1)$ term tends to zero as max$\{\sqrt{\Delta}, np\}$ tends to infinity.

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Theorem 2

(A.Soshnikov and B.Sudakov 2002) Let $G(Q^n, p)$ *be a random subgraph of the n-cube and let* Δ *be the maximum degree of G. Then almost surely the largest eigenvalue of the adjacency matrix of G satisfies* $\lambda_1(G) = (1 +_O (1))max\{\sqrt{\Delta}, np\}$*, where the* $_O(1)$ *term tends to zero as max*$\{\sqrt{\Delta}, np\}$ *tends to infinity.*

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How to estimate Δ?

- For $G(n,p)$, let $\Delta_p = \max\{k : n\binom{n-1}{k}p^k(1-p)^{n-k} \geq 1\}$
- For $G(Q^n,p)$, let $\Delta_p = \max\{k : 2^n\binom{n-1}{k}p^k(1-p)^{n-k} \geq 1\}$
- Δ_p is the maximal k for which the expectation of the number of vertices of degree k in $G(n,p)$(or $G(Q^n,p)$) is still at least one.

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- In the proof, when p is very small, $\lambda_1(G) = (1 +_O (1))\sqrt{\Delta_p}$.
- When p is very large, $\lambda_1(G) = (1 +_O (1))np$.
- When p is in the middle part, the graph is divided into several parts whose vertices have different degrees.

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For $G(n,p)$:

- X_1:vertices with degree at least $\Delta_p^{3/4}$.
- X_2:vertices with degree larger than $np(1+1/loglogn)+\Delta_p^{1/3}$ but less than $\Delta_p^{3/4}$.
- Y_1:contains all vertices of $V(G)-X_1 \bigcup X_2$ with at least one neighbor in X_1.
- Y_2:$V(G)-X_1 \bigcup X_2 \bigcup Y_1$.
- The largest eigenvalue of each subgraph can be estimated relatively easily.
- The proof of Theorem2 is similar.

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Some Conjectures about the Different Models

Sudakov use this way to estimate the largest eigenvalues of random subgraphs of $G(n, p)$ and $G(Q^n, p)$. In other words, the graph models are got from randomly "deleting" edges from the complete graph and n-cube.

We think we can change the base graphs by other ones.

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Expander Graphs

- The Edge Boundary of a set S, denoted ∂S, is $\partial S = E(S, \bar{S})$. This is actually the set of outgoing edges from S.
- The Expansion Parameter of G, denoted $h(G)$, is defined as: $h(G) = \min_{\{S \mid |S| \leq \frac{n}{2}\}} \frac{|\partial S|}{|S|}$

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A family of Expander graphs $\{G_i\}$ where $i \in \mathbb{N}$ is a collection of graphs with the following properties:

- The graph G_i is a d-regular graph of size n_i (d is the same constant for the whole family).$\{n_i\}$ is a monotone growing series that doesn't grow too fast(e.g.$n_{i+1} \leq n_i^2$).
- For all i, $h(G_i) \geq \epsilon \geq 0$.

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- For all i, $h(G_i) \geq \epsilon \geq 0$.

If we randomly delete edges from an expander graph, we get a new random graph model. Note that the complete graph is a special expander graph.

The expander graph can be substituted for other graphs, too.

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Thank you very much for attention!