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Multiagent net and Algebraic connectivity

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This work is joined with my advisor Xiao-dong Zhang (张晓东)

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Outline

- Introduction .
- The multiagent net.
- The algebraic connectivity.
- The main theorem.

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Definition and Background

- Let $G = (V, E)$ be a simple graph (without loop and multiedges) with vertex set $V = [n]$ and edge set $E(G)$.
- $A(G) = (a_{ij})$: the adjacency matrices of G with $a_{ij} = 1$ if $v_i \sim v_j$; $a_{ij} = 0$ otherwise.
- $D(G) = diag(d_1, \cdots, d_n)$ degree diagonal matrix with d_i being the degree of vertex v_i.

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- The *Laplacian matrix* of G: $L(G) = D(G) - A(G)$.
- $L(G)$ is positive semi-definite and singular.
- The eigenvalues of $L(G)$ are denoted by

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G) = 0;$$

  called the *Laplacian eigenvalues* of G. In particular, $\lambda_{n-1}(G)$ is called the algebraic connectivity of G and denoted by $\alpha(G)$.
- $\mathbf{1}$ is an eigenvector corresponding $\lambda_n(G)$.

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Agreement protocol(**):

For a graph $G = (V, E)$, $V = [n]$, we define a agreement protocol as the following:

$$\dot{x}(t) = -L(G)x(t),$$

i.e.

$$\dot{x}_i(t) = \sum_{j \in N(i)} (x_j(t) - x_i(t)),$$

where $x_i(t)$ is a value at the time t and at the vertex i.

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Three Problems: 

- 1. Whether $x(t)$ is convergent?
- 2. If $x(t)$ is convergent, then what is the form of $x(t)$ if t tends infinity.
- 3. If $x(t)$ is convergent, then what is the convergent rate?

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- Using the spectral factorization of the Laplacian, one has

$$e^{-L(G)t} = e^{-\lambda_1(G)t} u_1 u_1^T + \ldots + e^{-\lambda_{n-1}(G)t} u_{n-1} u_{n-1}^T + e^{-\lambda_n(G)t} u_n u_n^T.$$

  where u_i is an eigenvector corresponding to λ_i and $u_i u_j^T = 0$, $j \neq i = 1, 2, \ldots, n$.

- The solution of (**),initialized from $x(0) = x_0$, is $x(t) = e^{-L(G)t} x_0$.
- Let $\mathcal{A} = \{x \in R^n | x_i = x_j$, for all $i, j\}$

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Theorem 1

Let G be a connected graph. Then the agreement protocol (**) converges to the agreement set $\mathcal{A}$ with a rate of convergence that is dictated by $\lambda_{n-1}(G)$.

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Spectral Turán Theorem

Theorem 2 (Jin-Zhang)

Let G be a non-complete graph of order n not containing K_{r+1}. Then

$$\alpha(G) \leq n - \lceil \frac{n}{r} \rceil = \alpha(T_{n,r}), \quad (1)$$

where $\lceil a \rceil$ is the least integer no less than a. Moreover, if $n = kr$ or $n = kr + r - 1$, then equality (1) holds if and only if G is Turán graph $T_{n,r}$. If $n = kr + t, 0 < t < r - 1$, then equality (1) holds if and only if there exist graphs $H_1, \ldots, H_t$ of order $k + 1$ with no edges and H of order $n - (k + 1)t$ not containing K_{r+1-t} such that

$$G = H_1 \vee H_2 \cdots \vee H_t \vee H$$

and $\alpha(H) \geq n - (k + 1)(t + 1)$.

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In the following, we give the graphs given clique number with minimum algebraic connectivity:

Theorem 3

Let G be a connected graph with the clique number $r \geq 2$. Then

$$\alpha(G) \geq \alpha(Ki_{n,r}), \tag{2}$$

where $Ki_{n,r}$ is a kite graph of order n which is obtained by adding a pendant path of length $n-r$ to a vertex of K_r. Moreover, equality (2) holds if and only if $G = Ki_{n,r}$.

Proof of the Theorem 3

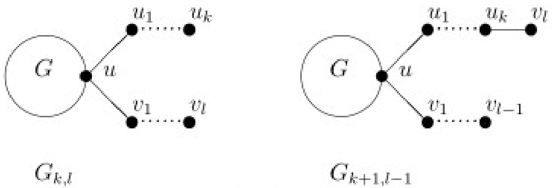

Fig. 1

Proof of the Theorem 3

Lemma 4 (Guo 2010)

Let G be a connected graph with at least two vertices and two paths $P : uu_1u_2 \ldots u_k$ *and* $Q : uv_1 \ldots v_l$ *of lengths* k, l $(k \geq l \geq 1)$. *If* $G_{k,l}$ *is the graph obtained from* G *by attached two paths* P, Q *at vertex* u *and* $G^{(1)}_{k+1,l-1} = G_{k,l} - v_{l-1}v_l + u_kv_l$ *(see Fig. 1) and* $k \geq l \geq 1$, *then*

$$\alpha(G_{k,l}) \geq \alpha(G_{k+1,l-1}). \quad (3)$$

Moreover, inequality (3) is strict if either $X(v_1) \neq 0$ *or* $X(u_1) \neq 0$.

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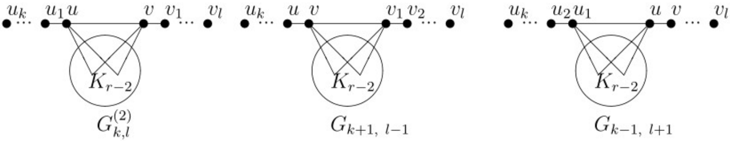

Fig.3

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Proof of the Theorem 3

Lemma 5 (Jin-Zhang)

Let $G^{(2)}_{k,l}$ *be the graph obtained from a complete graph* $K_r, r \geq 3$ *with vertex set* $V(K_r) = \{w_1, \ldots, w_{r-2}, u, v\}$ *by attached two paths* $P = uu_1 \ldots u_k$ *and* $Q = vv_1 \ldots v_l$ *at vertices* u *and* v*, respectively (see Fig. 3). If*

$G^{(2)}_{k+1,l-1} = G - \{uw_i, 1 \leq i \leq r-2\} + \{v_1 w_i, 1 \leq i \leq r-2\}$,

$G^{(2)}_{k-1,l+1} = G - \{vw_i, 1 \leq i \leq r-2\} + \{u_1 w_i, 1 \leq i \leq r-2\}$.

Then

$$\alpha(G^{(2)}_{k,l}) > \min\{\ \alpha(G^{(2)}_{k+1,l-1}),\ \alpha(G^{(2)}_{k-1,l+1})\ \}. \qquad (4)$$

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Proof of the Theorem 3

Corollary 6 (Jin-Zhang)

Let $G^{(2)}_{k,l}$ be the graph obtained from a complete graph $K_r, r \geq 3$ with vertex set $V(K_r) = \{w_1, \ldots, w_{r-2}, u, v\}$ by attached two paths $P = uu_1 \ldots u_k$ and $Q = vv_1 \ldots v_l$ at vertices u and v, respectively. If $k \geq l \geq 1$, then

$$\alpha(G^{(2)}_{k,l}) > \alpha(G^{(2)}_{k+1,l-1}). \quad (5)$$

Proof the above Corollary:

If $k = l$, the assertion follows from Lemma 5 and $G^{(2)}_{k+1,l-1} = G^{(2)}_{k-1,l+1}$. If $k > l$, suppose that $\alpha(G^{(2)}_{k,l}) > \alpha(G^{(2)}_{k+1,l-1})$ does not hold, i.e., $\alpha(G^{(2)}_{k,l}) \leq \alpha(G^{(2)}_{k+1,l-1})$. Then by Lemma 5, we have

$$\alpha(G^{(2)}_{k,l}) > \min\{\alpha(G^{(2)}_{k-1,l+1}), \alpha(G^{(2)}_{k+1,l-1})\} = \alpha(G^{(2)}_{k-1,l+1}). \quad (6)$$

By repeated uses of Lemma 5 and (6), we have

$$\alpha(G^{(2)}_{k-1,l+1}) > \min\{\alpha(G^{(2)}_{k-2,l+2}), \alpha(G^{(2)}_{k,l})\} = \alpha(G^{(2)}_{k-2,l+2}). \quad (7)$$

By repeated uses of Lemma 5, we obtain

$$\alpha(G^{(2)}_{k,l}) > \alpha(G^{(2)}_{k-1,l+1}) > \alpha(G^{(2)}_{k-2,l+2}) > \cdots > \alpha(G^{(2)}_{k-(k-l),l+(k-l)}) = \alpha(G$$

which is a contradiction, since $G^{(2)}_{k,l} = G^{(2)}_{l,k}$. So the assertion holds.

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Proof of the Theorem 3

Corollary 7 (Jin-Zhang)

Let $G^{(2)}_{k,l}$ *be a graph of order* n *obtained from the complete graph* $K_r, r \geq 3$ *with vertex set* $V(K_r) = \{w_1, \ldots, w_{r-2}, u, v\}$ *by attached two paths* $P = uu_1 \ldots u_k$ *and* $Q = vv_1 \ldots v_l$ *at vertices* u *and* v*, respectively, where* $n = r + k + l$*. If* $k > 0, l > 0$*, then*

$$\alpha(G^{(2)}_{k,l}) > \alpha(Ki_{n,r}). \quad (8)$$

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Corollary 8

Let G be a graph of order n with clique number r. Then

$$\frac{n}{n-\alpha(G)} \leq r \leq n+1-\frac{4}{n\alpha(G)}.$$

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Thanks For your Attentions!