Pointwise upper estimates for transition probability of continuous time random walks on graphs

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Continuous Time Random Walk

Let $\Gamma = (\mathbb{V}, \mathbb{E})$ be a connected, locally finite graph without double edges.

Let $\mu = (\mu_{xy})$ be an edge weight function on $\mathbb{E}$, such that $\mu_{xy} = \mu_{yx} > 0$ for each $(x, y) \in \mathbb{E}$, while $\mu_{xy} = 0$ for each $(x, y) \notin \mathbb{E}$.

Let $\nu = (\nu_x)$ be a positive vertex weights on $\mathbb{V}$.

Denote by $X = \{X_t : t \geq 0\}$ a continuous time random walk on Γ with generator

$$\mathcal{L}f(x) = \frac{1}{\nu_x} \sum_{y \in \mathbb{V}} (f(y) - f(x))\mu_{xy}.$$

If $\nu_x = \sum \mu_{xy}$ and $\mu_{xy} \in \{0, 1\}$ for all x, y, then the process X is called continuous time simple random walk on $\mathbb{V}$.
It is a process that waits an exponential time mean 1 at each vertex and then jumps along one of its neighbor uniformly.

If $\nu_x \equiv 1$, then the expected waiting time of each jump may vary greatly. Moreover, such a process may explode in finite time.
For example, $\Gamma = \mathbf{Z}^+$, $\mu_{x-1,x} = 2^x$.

Pairwise Estimates

Fix vertices x_1, x_2 and functions f_1, f_2 such that

$$\mathbf{P}_{x_i}(X_t = x_i) \leq \frac{1}{f_i(t)}.$$

Our interest is, under what circumstance $\mathbf{P}_{x_1}(X_t = x_2)$ will have Gaussian upper bounds.

For example, Let X be a continuous time simple random walk on $\mathbf{Z}^d$. We have

$$\mathbf{P}_x(X_t = x) \leq \frac{C}{t^{d/2}};$$

and

$$\mathbf{P}_x(X_t = y) \leq \frac{c}{t^{d/2}} \exp\left(-\frac{C\|x-y\|^2}{t}\right).$$

for any $t \geq \|x-y\|$.

History

The problem of getting a Gaussian upper bound from two point estimates was introduced in the manifold case by Grigor'yan (1997). In the subsequent researches, Coulhon, Grigor'yan & Zucca (2005) studied the problem for discrete time random walks on graphs, while Folz (2011) studied in the continuous time random walks. The current paper considers the same problem, however, it improves the result of Folz(2011) by $\cdots$.

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Adapted Metric on Γ

Let $d_\nu(\cdot,\cdot)$ be a metric of Γ such that

$$\begin{cases} \frac{1}{\nu_x}\sum_y d_\nu(x,y)^2\mu_{xy} \le 1 & \text{for all } x \in \mathbb{V}, \\ d_\nu(x,y) \le 1 & \text{whenever } x,y \in \mathbb{V} \text{ and } x \sim y. \end{cases} \tag{1}$$

Metrics satisfying (1) are called adapted metrics. Such metrics were initiated by Davies (1993), and are closely related to the intrinsic metric associated with a given Dirichlet form.

Intuition: If μ_{xy} is larger, then more quickly for a particle transfers from x to y and so $d_\nu(x,y)$ should be smaller.

$(A, \gamma)-$regular

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$. Let $A \geq 1$ and $\gamma > 1$. We say that f is $(A, \gamma)-$regular if the function f is increasing and satisfies that

$$\frac{f(\gamma s)}{f(s)} \leq A \frac{f(\gamma t)}{f(t)} \quad \text{for all } t \geq s \geq 0, \tag{2}$$

which was introduced by Grigor'yan (1997).

For example, $t^n, e^{t^{1/2}}, e^t$;

$$f(t) = \begin{cases} ct & \text{if } \; t \leq T_1, \\ Ct^2 & \text{if } \; t > T_1; \end{cases}$$

In particular, if Volume Doubling holds, that is, $V(x, 2r) \leq CV(x, r)$ then

$$f(t) = V(x, \sqrt{t}).$$

is regular.

Theorem (Grigor'yan) Let x_1, x_2 be distinct points on a smooth Riemannian manifold M, and suppose that there exist $(A, \gamma)-$regular functions g_1, g_2 such that, for all $t > 0$ and $i \in \{1, 2\}$,

$$q_t(x_i, x_i) \leq \frac{1}{g_i(t)}$$

Then for any $D > 2$ and all $t > 0$, the Gaussian upper bound

$$q_t(x_1, x_2) \leq \frac{4A}{g_1(\delta t) g_2(\delta t)} \exp\left(-\frac{d(x_1, x_2)^2}{2Dt}\right)$$

holds, where $\delta = \delta(D, \gamma)$.

Theorem (Coulhon, Grigor'yan and Zucca) Let (Γ, μ) be a weighted graph satisfying condition $\inf_{x\in\mathbb{V}} \mathbf{P}_x(X_1 = x) \geq \alpha > 0$. Let x_1, x_2 be two fixed vertices in Γ, and assume that there are two $(A, \gamma)-$regular functions g_1, g_2 such that, for all $k \in \mathbb{N}$,

$$h_{2k}(x_i, x_i) \leq \frac{1}{g_i(k)}.$$

Then, for all $k \in \mathbb{N}$,

$$h_k(x_1, x_2) \leq \frac{C_0}{g_1(\eta k)g_2(\eta k)} \exp\left(-\frac{d(x_1, x_2)^2}{2D_0 k}\right),$$

where $\eta = \eta(\gamma) > 0, D_0 = D_0(\alpha, \gamma) > 0$ and $C_0 = C_0(A, \alpha, \gamma)$.

Theorems

Let $p_t(x,y) = \frac{\mathbf{P}_x(X_t=y)}{\nu_y}$.

Theorem A(Folz 2011)

Suppose there exists $C_\nu > 0$ such that $\nu_x \geq C_\nu$ for all x. Let g_1, g_2 be $(A,\gamma)-$regular functions satisfying

$$g_i(t) \leq Ae^{t^{1/2}}. \tag{3}$$

Suppose that

$$p_t(x_i, x_i) \leq \frac{1}{g_i(t)}. \tag{4}$$

Then there exists $C_1(A,\gamma,C_\nu)$, $C_2(\gamma)$ and $\alpha(\gamma) > 0$, such that for all $t \geq d_\nu(x_1,x_2)$,

$$p_t(x_1,x_2) \leq \frac{C_1}{\sqrt{g_1(\alpha t) g_2(\alpha t)}} \exp\left(-\frac{C_2 d_\nu(x_1,x_2)^2}{t}\right). \tag{5}$$

However, our work improves the result of Folz (2011) by no longer requiring a lower bound on ν_x. The improvement comes from imposing conditions on the transition probabilities $\mathbf{P}_x(X_t = x)$ instead of the heat kernels $p_t(x, x)$. Note that the transition probabilities are invariant under the transformation from (μ, ν) to $(c\mu, c\nu)$, where $(c\mu)_{xy} = c\mu_{xy}$ and $(c\nu)_x = c\nu_x$.

Theorem 1

Suppose

$$\mathbf{P}_{x_i}(X_t = x_i) \le \frac{1}{f_i(t)}. \tag{6}$$

Let $\delta \ge 1$*. If each* f_i *is* $(A,\gamma)-$*regular and satisfies*

$$f_i(t) \le Ae^{\delta t} \text{ for all } t \in \mathbb{R}_+, \tag{7}$$

then there exist universal positive constants C_1 *and* θ*, such that for any* $t \ge d_\nu(x_1, x_2)$ *we have*

$$\mathbf{P}_{x_1}(X_t = x_2) \le \frac{C_1 A^\beta (\nu_{x_2}/\nu_{x_1})^{1/2}}{\sqrt{f_1(\alpha t) f_2(\alpha t)}} \exp\left(-\theta \frac{d_\nu(x_1, x_2)^2}{t}\right), \tag{8}$$

where $\alpha = \min\{(2\gamma)^{-1}, (64\delta)^{-1}\}$ *and* $\beta = \lceil \frac{\log \gamma}{\log 2} \rceil$.

Remark 1. The condition (7) is quite natural. Note that $\mathbf{P}_x(X_t = x) \geq \exp\left(-\frac{\mu_x}{\nu_x}t\right)$, where $\mu_x = \sum_y \mu_{xy}$. It implies that (7) holds if $A = 1$ and $\delta = \max\{\frac{\mu_{x_1}}{\nu_{x_1}}, \frac{\mu_{x_2}}{\nu_{x_2}}\}$. In particular, for CSRW one can take $\delta = 1$.

Remark 2. One can also trace the values of C_1 and θ. Indeed, we select $\theta = 10^{-7}$ in our proof.

Remark 3. It is potentially very useful for random walks in random environments where one may lack global regularity.

Application

Barlow(2004), Random walks on supercritical percolation clusters, Ann. Probab.

Lemma. There exist random variables $T_x(\omega) < \infty$ and non-random constants c_1, c_2 such that almost surely, for all $x_2 \in G(\omega)$ and $t > 0$,

$$q_t^\omega(x,x) \leq \begin{cases} c_1 t^{-1/2} & \text{if } 0 < t \leq T_x(\omega) \\ c_2 t^{-d/2} & \text{if } t > T_x(\omega). \end{cases}$$

The theorem above shows that if $t \geq C_1(T_x \vee T_y) \vee d(x,y)$ then we have the Gaussian upper bound

$$q_t(x,y) \leq C_2 t^{-d/2} \exp\left(-C_3 \frac{d(x,y)^2}{t}\right).$$

Positive Subsolution

Let $\mathbb{I}$ be an interval of $\mathbb{R}_+$. We say that $u : \mathbb{I} \times \mathbb{V} \mapsto \mathbb{R}_+$ is a positive subsolution on $\mathbb{I} \times \mathbb{V}$ if

$$\frac{\partial}{\partial t} u \leq Lu \quad \text{on} \;\; \mathbb{I} \times \mathbb{V}.$$

Furthermore, we define a set of functions:

$$\begin{aligned} \mathcal{H}(\mathbb{I}) = & \{u : u \text{ is a positive subsolution on } \mathbb{I} \times \mathbb{V} \text{ and} \\ & |\{z \in \mathbb{V} : u(t,z) \neq 0, t \in \mathbb{I}\}| < \infty\}. \end{aligned}$$

Let $o \in B \subseteq \mathbb{V}$ with $|B| < \infty$. Set

$$u_B(t,z) = \frac{\nu_o^{1/2}}{\nu_z} \mathbf{P}_o(X_t = z, \;\inf\{s \geq 0 : X_s \notin B\} > t). \tag{9}$$

Then $u_B \in \mathcal{H}(\mathbb{R}_+)$.

Question: Let u be a positive subsolution. Does there exist a sequence of positive subsolutions $u_n, n = 1, 2, \cdots$, such that, each u_n has finite support and (u_n) converges to u in some sense?

For any functions f, g on $\mathbb{V}$, define

$$\langle f, g \rangle = \sum_{x \in \mathbb{V}} f(x) g(x) \nu_x.$$

Integral Maximum Principle

Let h be a positive function on $\mathbb{I} \times \mathbb{V}$ and $u \in \mathcal{H}(\mathbb{I})$. If for each $t \in \mathbb{I}$ one has

$$\frac{1}{\nu_y} \sum_x \frac{|h(t,x) - h(t,y)|^2}{4h(t,x)h(t,y)} \mu_{xy} \leq -\frac{\partial}{\partial t} \log h(t,y) \quad \text{for all} \quad y \in \mathbb{V}, \tag{10}$$

then $J(t) = \langle u^2(t,\cdot), h(t,\cdot) \rangle$ is decreasing on $\mathbb{I}$.

Owing to the Adapted metric d_ν, it leads immediately to Corollary as follows. Define a set of functions:

$$\mathcal{F}(\mathbb{I}) = \Big\{ h : h \text{ is a positive function on } \mathbb{I} \times \mathbb{V} \text{ and for each } t \in \mathbb{I}, x \sim y, \\ \frac{|h(t,x) - h(t,y)|^2}{4h(t,x)h(t,y)} \leq -d_\nu(x,y)^2 \frac{\partial}{\partial t} \log h(t,y) \Big\}.$$

Corollary

Let $u \in \mathcal{H}(\mathbb{I})$ and $h \in \mathcal{F}(\mathbb{I})$. Then $J(t) = \langle u^2(t,\cdot), h(t,\cdot) \rangle$ is decreasing on $\mathbb{I}$.

Let $\rho(\cdot) = d_\nu(o, \cdot) \wedge R$ for some $o \in \mathbb{V}$ and $R \geq 0$.

Example 2

Fix $a \in [0, \frac{1}{4}]$. *Let* $h_1(t, x) = e^{a\rho(x) - \frac{a^2}{2}t}$. *Then* $h_1 \in \mathcal{F}(\mathbb{R}_+)$.

Example 3

Fix $\tau > 0$. *For each* $t \geq 0$ *and* $z \in \mathbb{V}$, *set*

$$h(t, z) = \exp\left\{ \left(\rho(z) - 4^{-1}e\ (t + \tau)\right)\ \log\left(1 \vee \frac{\rho(z)}{4^{-1}e\ (t+\tau)}\right) - \frac{t}{\tau} \right\}.$$

Then $h(t, z) \in \mathcal{F}(\mathbb{R}_+)$.

Some result

Set $\mathcal{H}_o = \{u \in \mathcal{H}(\mathbb{R}_+) : u(0,z) = \nu_o^{-1/2} 1_{\{o\}}(z)$ for each $z \in \mathbb{V}\}$.

Proposition 4

Let $u \in \mathcal{H}_o$. For any $t, R > 0$, we have

$$\langle u(t,\cdot)^2, 1 - 1_{B_R} \rangle \le \exp\left(-\frac{R^2}{8t}\right) \text{ if } t \ge R.$$

Proof. Let $a = \frac{4R}{t}$. Then $a \in [0, \frac{1}{4}]$. Obviously,

$$\langle u(t,\cdot)^2, 1 - 1_{B_R} \rangle \le \langle u(t,\cdot)^2, e^{a\rho(x) - \frac{a^2}{2}t} \rangle e^{-aR + \frac{a^2}{2}t}.$$

By the Integer Maximum Principle,

$$\langle u(t,\cdot)^2, e^{a\rho(x) - \frac{a^2}{2}t} \rangle \le \langle u(0,\cdot)^2, e^{a\rho(x)} \rangle = 1.$$

Corollary 5

For any $z \in \mathbb{V}$,

$$\mathbf{P}_o(X_t = z) \le (\nu_z/\nu_o)^{1/2} \exp\left\{-\frac{r^2}{16t}\right\} \textit{if } t \ge r > 0,$$

where $r = d_\nu(o, z)$.

The intuition of our main theorem can be seen from Corollary 5, if $f_1 = f_2 \equiv 1$ are selected as the trivial upper bounds.

Corollary

For any $z \in \mathbb{V}$,

$$\mathbf{P}_o(X_t = z) \le (\nu_z/\nu_o)^{1/2} \exp\left(-\frac{r}{2}\log\left(\frac{1.01r}{t}\right) + 60\right) \text{ if } r \ge t > 0,$$

where $r = d_\nu(o, z)$.

Compared with Theorems B and C, Corollaries 5 and 6 work more efficiently when $t \in [0.9r, 1.1r]$ and $r = d_\nu(o, z)$ is large.

Theorem B(Folz 2011)

If $x, y \in \mathbb{V}$, then for all $t > 0$,

$$p_t(x,y) \le (\nu_x \nu_y)^{-1/2} \exp\left(-\frac{r^2}{2t}\left(1 - \frac{r}{t}\right) - \Lambda t\right),$$

where $r = d_\nu(x,y)$ and $\Lambda \ge 0$ is the bottom of the L^2 spectrum of the operator $\mathcal{L}_\nu$.

Theorem C(Folz 2011)

If $x, y \in \mathbb{V}$, then for all $t > 0$,

$$p_t(x,y) \le (\nu_x \nu_y)^{-1/2} \exp\left(-\frac{r}{2}\log\left(\frac{r}{2et}\right) - \Lambda t\right).$$

$(A,\gamma)-$regular on an interval

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$. Let $A \geq 1$ and $\gamma > 1$. We say that f is $(A,\gamma)-$regular on $[a,b)$ if the function f is increasing on $\mathbb{R}_+$ and satisfies that

$$\frac{f(\gamma s)}{f(s)} \leq A\frac{f(\gamma t)}{f(t)} \quad \text{for all } a \leq s < t < \gamma^{-1}b. \tag{11}$$

Theorem 7

If each f_i *is* $(A, \gamma)-$*regular on* $[T_1, T_2)$ *and satisfies*

$$f_i(t) \leq Ae^{\delta t} \text{ for all } t \in [T_1, T_2) \tag{12}$$

then there exist universal positive constants C_1 *and* θ*, such that for any* $t \in [\widetilde{T}_1, T_2)$ *we have*

$$\mathbf{P}_{x_1}(X_t = x_2) \leq \frac{C_1 A^{\beta} (\nu_{x_2}/\nu_{x_1})^{1/2}}{\sqrt{f_1(\alpha t) f_2(\alpha t)}} \exp\left(-\theta \frac{d_\nu(x_1, x_2)^2}{t}\right), \tag{13}$$

where $\widetilde{T}_1 = (8\alpha^{-2} T_1^2) \vee d_\nu(x_1, x_2)$.

Thank you!