Metastable Densities for the Contact Process on Power Law Random Graphs

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Introduction
Newman-Strogatz-Watts random graphs

- The graph: \( G^n := (V^n, E^n) \).
- Vertex set: \( V^n := \{1, 2, \cdots, n\} \).
- Degree of vertex \( i \) is denoted by \( d_i \) (\( i = 1, 2, \cdots, n \)), which follows
  (1) \( d_1, d_2, \cdots, d_n \) i.i.d.
  (2) \( p_k := P(d_1 = k) \sim Ck^{-\alpha} \) (\( \alpha > 1, C > 0 \)) if \( k \) is large. i.e.

\[
\lim_{k \to \infty} k^\alpha \cdot p_k = C \in (0, \infty).
\]
How can we construct the graph once given a **suitable** realization of the degree sequence $(d_1(\omega), d_2(\omega), \cdots, d_n(\omega))$?
Newman-Strogatz-Watts random graphs

  each vertex $i$ was issued with $d_i$ half edges and these half edges were matched up in a \textit{uniformly} chosen manner.

- If $\alpha > 3$, then $P(\text{no loops or multiple edges}) \to 1$ as $n \to \infty$. If $2 < \alpha \leq 3$, not so.

- We treat both cases in our work.
Basic definitions of contact processes

- First introduced by T. E. Harris (1974).
- A model to describe the spread of diseases.
- Two classical books:
Basic definitions of contact processes

- The process \((\xi_t : t \geq 0)\): a continuous-time Markov process.
- State space: \(\{A : A \subseteq V^n\}\).
- At each \(t\), each vertex is either healthy or infected. \(\xi_t\) is the collection of infected vertices at time \(t\).
- Transition rates:
  \[
  \begin{align*}
  &\xi_t \rightarrow \xi_t \setminus \{x\} \text{ for } x \in \xi_t \text{ at rate 1}, \\
  &\xi_t \rightarrow \xi_t \cup \{x\} \text{ for } x \notin \xi_t \text{ at rate } \lambda \cdot |\{y \in \xi_t : x \sim y\}|.
  \end{align*}
  \]
- \((\xi_t^A : t \geq 0)\): the process with initial state \(A\).
- Absorbing state: \(\emptyset\).
Self-duality

For any $A, B \in V^n$ and $t > 0$, we have

$$P(\xi_t^A \cap B \neq \emptyset) = P(\xi_t^B \cap A \neq \emptyset).$$

Especially, for any $x \in V^n$ and $t > 0$, we have

$$P(\xi_t^x \neq \emptyset) = P(x \in \xi_t^{V^n}).$$
Problem and Previous Results
Consider the contact process on the Newman-Strogatz-Watts random graph.

**Assumptions:**

1. $p_0 = p_1 = p_2 = 0$;
2. Conditioned on $E_n := \{d_1 + \cdots + d_n \text{ is even}\}$.

$$\left( \lim_{n \to \infty} P(E_n) = \frac{1}{2} \right)$$
Assumptions and notation

- Given $\delta \in (0, 1)$, $\lambda > 0$, and $x \in V^n$ randomly chosen. Define
  \[
  \rho_n(\lambda, \delta) := P\left(\xi^{x}_{\exp(n^{1-\delta})} \neq \emptyset\right).
  \]

- By self-duality,
  \[
  \rho_n(\lambda, \delta) = P\left(x \in \xi^{V^n}_{\exp(n^{1-\delta})}\right) = E\left(\frac{|\xi^{V^n}_{\exp(n^{1-\delta})}|}{n}\right).
  \]

- Define
  \[
  \underline{\rho}(\lambda, \delta) := \lim_{n \to \infty} \inf \rho_n(\lambda, \delta),
  \]
  \[
  \bar{\rho}(\lambda, \delta) := \lim_{n \to \infty} \sup \rho_n(\lambda, \delta).
  \]
Problem: What is the asymptotic behavior of $\underline{\rho}(\lambda, \delta)$ and $\overline{\rho}(\lambda, \delta)$ if $\lambda > 0$ is small?
Previous works

  If $\alpha > 3$, then $\forall \delta \in (0, 1)$,
  \[
  b\lambda^c \leq \underline{\rho}(\lambda, \delta) \leq \overline{\rho}(\lambda, \delta) \leq B\lambda^c
  \]
  when $\lambda$ is small enough.

- S. Chatterjee and R. Durrett (2009):
  If $\alpha > 3$, then $\forall \delta \in (0, 1)$, $\varepsilon > 0$,
  \[
  c\lambda^{1+2(\alpha-2)+\varepsilon} \leq \underline{\rho}(\lambda, \delta) \leq \overline{\rho}(\lambda, \delta) \leq C\lambda^{1+(\alpha-2)+\varepsilon}
  \]
  when $\lambda$ is small enough.
What are we interested in?

- Improve the bounds (find the \textbf{actual} behavior of $\underline{\rho}(\lambda, \delta)$ and $\overline{\rho}(\lambda, \delta)$)
- Extend the result to $\alpha > 2$.

Reference:
S. Chatterjee and R. Durrett: Contact processes on random graphs with power law degree distributions have critical value 0, \textit{Ann. Probab.} \textbf{37} 2332-2356 (2009).
Our main results

- If $\alpha > 3$, there exist $m_1, M_1 > 0$ so that, for any $\delta \in (0, 1)$ and small enough $\lambda > 0$,
  \[
  m_1 \frac{\lambda^{1+2(\alpha-2)}}{\log^{2(\alpha-2)} \left( \frac{1}{\lambda} \right)} \leq \rho(\lambda, \delta) \leq \overline{\rho}(\lambda, \delta) \leq M_1 \frac{\lambda^{1+2(\alpha-2)}}{\log^{2(\alpha-2)} \left( \frac{1}{\lambda} \right)}.
  \]

- If $2^{\frac{1}{2}} < \alpha \leq 3$, there exist $m_2, M_2 > 0$ so that, for any $\delta \in (0, 1)$ and small enough $\lambda > 0$,
  \[
  m_2 \frac{\lambda^{1+2(\alpha-2)}}{\log^{\alpha-2} \left( \frac{1}{\lambda} \right)} \leq \rho(\lambda, \delta) \leq \overline{\rho}(\lambda, \delta) \leq M_2 \frac{\lambda^{1+2(\alpha-2)}}{\log^{\alpha-2} \left( \frac{1}{\lambda} \right)}.
  \]

- If $2 < \alpha \leq 2^{\frac{1}{2}}$, there exist $m_3, M_3 > 0$ so that, for any $\delta \in (0, 1)$ and small enough $\lambda > 0$,
  \[
  m_3 \lambda^{1+\frac{\alpha-2}{3-\alpha}} \leq \rho(\lambda, \delta) \leq \overline{\rho}(\lambda, \delta) \leq M_3 \lambda^{1+\frac{\alpha-2}{3-\alpha}}.
  \]
Idea of Proof
Some basic definitions

- Denote $\mu := \sum_n np_n$, then $\mu \in (0, \infty)$ since $\alpha > 2$.

- Denote the law $\tilde{q}$ by $\tilde{q}_n = \frac{np_n}{\mu}$.

- Denote the law $q$ by $q_n = \tilde{q}_{n+1}$.
  ($q$ is called the **size-biased law**.)

- Given a graph $G$ containing vertex $x$, denote by $B_G(x, K)$ the set of vertices in $G$ at distance less than or equal to $K$ from $x$, including $x$ itself.
Connection with Galton-Watson trees

**Proposition:** For each $n \in \mathbb{N}$, let $G^n$ be a Newman-Strogatz-Watts random graph with degree law $p$ with associated exponent $\alpha > 2$. Assume that $x$ is uniformly chosen in $\{1, \ldots, n\}$, independently of the graph. Then for any $K \in \mathbb{N}$ fixed, as $n \to \infty$, the law of $B_{G^n}(x, K)$ converges to the law of $B_T(o, K)$, where $T$ is a Galton-Watson tree such that

(i) the degree of the root $o$ is chosen $\sim p$.

(ii) the degrees of subsequent vertices are i.i.d. $\sim \tilde{q}$ (in other words, the offspring distribution for subsequent generations is equal to $q$).

References:


Notation of measures

- Denote by $\widetilde{P}_{(p,q)}$ the law under which a random rooted tree $T$ is obtained by choosing the degrees of the vertices independently: that of the root $o$ according to probability $p$ and those of “subsequent” vertices according to $q$.

- Use $\widetilde{P}_q$ for the law under which every vertex has degree i.i.d. according to $q$.

- Notation $\widetilde{P}_{(p,q),\lambda}$ denotes the joint law of the contact process with parameter $\lambda$ on a tree independently generated according to law $\widetilde{P}_{(p,q)}$.

- Notation $\widetilde{P}_{q,\lambda}$ denotes the joint law of the contact process with parameter $\lambda$ on a tree independently generated according to law $\widetilde{P}_q$. 
Two useful estimations

**Proposition 1:** For any $\varepsilon, \delta > 0$, there exists $\lambda_1 > 0$ such that, for any $\lambda < \lambda_1$ and $R > 1$,

\[
\bar{\rho}(\lambda, \delta) \geq \tilde{P}(p,q), \lambda \left( \exists y \in B_T(o, R), \ t \in \mathbb{R}_+ : \ \deg(y) > \frac{1}{\lambda^{2+\varepsilon}}, \ y \in \xi_t^o, \right. \\
\left. \text{and the infection path from} \ o \ \text{to} \ y \ \text{lies entirely in} \ B_T(o, R) \right).
\]

**Proposition 2:** For any $\delta > 0$,

\[
\bar{\rho}(\lambda, \delta) \leq \tilde{P}(p,q), \lambda \left( \xi_t^o \neq \emptyset \ \forall \ t \right).
\]
The case $\alpha > 3$

- **Lower bound:** When $\alpha > 3$, sites of “supercritical” degree, that is, degree larger than $1/\lambda$ to some power strictly larger than 2, are typically very far from each other and far from the root of the tree. However, if the infection starts at a site whose degree is a large multiple of $\frac{1}{\lambda^2} \log^2 \left( \frac{1}{\lambda} \right)$, then the infection is maintained for a long time and can therefore reach distant sites. We argue that at this distance, supercritical sites can be found.

- **Upper bound:** Argue the opposite direction. That is: sites of degree smaller than a small multiple of $\frac{1}{\lambda^2} \log^2 \left( \frac{1}{\lambda} \right)$ do not maintain the infection for a time that is sufficient to reach “supercritical” sites.
The case $2^{1/2} < \alpha \leq 3$

- The expectation associated to the size-biased distribution $q$ is infinite.

- **Lower bound:** Similar to the lower bound of the case $\alpha > 3$.

- **Upper bound:** In contrast to the case $\alpha > 3$, there is a non-negligible probability that “supercritical” sites are close to the root. Then, if the infection is sustained close to the root for some time of order $1/\lambda^\varepsilon$, where $\varepsilon$ is not very small, these supercritical sites will be reached. The proof of the upper bound depends on controlling the probability of this event.
The case $2 < \alpha \leq 2\frac{1}{2}$

- Here the bulk of the density no longer comes from sites which are neighbors of sites with degree of the order (neglecting log terms) $\frac{1}{\lambda^2}$.

- **Idea:** Compare with branching processes.
References


S. Chatterjee and R. Durrett: Contact processes on random graphs with power law degree distributions have critical value 0, Ann. Probab. 37 2332-2356 (2009).


Thank you!

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