002-(2016-2017-2)MS107, Probability Theory, 2017/05/11

Name:

Student ID:

1. (**10 points**) For $n = 1, 2, 3, 4, 5$, let p_n be the probability of winning the following game: Throw a single die $6n$ times; You win whenever you get the number 6 at least n times. Compare the five numbers p_1, p_2, p_3, p_4, p_5.

2. (**10 points**) For every positive integer n, find an example of $n+1$ random variables such that every n of them are independent but not all of them.

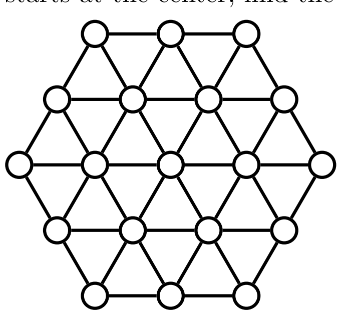
3. (**10 points**) Let $x_1, \dots, x_8$ be uniformly distributed 8 points on a unit line. Let I_i be the segment connecting x_{2i} and x_{2i-1} for $i = 1, \dots, 4$. Calculate the probability that there exists $k \in \{1, 2, 3, 4\}$ such that $I_k \cap I_i \neq \emptyset$ for all $i = 1, \dots, 4$.

4. (**10 points**) Let $X_n, n \geq 1$, be a sequence of independent random variables of constant finite expectation μ and constant finite variance. Apply Chebyshev's inequality to prove the Law of Large Numbers: $\lim_{k\to\infty} P(|\frac{\sum_{n=1}^{k} X_n}{k} - \mu| < \epsilon) = 1$ for any $\epsilon > 0$.

5. (**10 points**) Let G be a graph without isolated vertices. For each vertex v of G, let $D(v)$ denote the number of neighbors of v in G. Pick uniformly a random vertex X_0 of G and then let X_1 be a uniformly random neighbor of X_0 in G. Show that the expectation of $D(X_1) - D(X_0)$ is nonnegative.

6. (**15 points**) Let $A_n, n \geq 1$, be a sequence of events in a probability space satisfying $A_n \subseteq A_{n+1}$ for all $n \geq 1$. Let $A = \cup_{n=1}^{\infty} A_n$. Suppose that there exists $\epsilon > 0$ such that $P(A \mid A_n^c) \geq \epsilon$. Show that $P(A) = 1$.

7. (**15 points**) Consider the random walk on the following graph. Every unit time one moves to an adjacent vertex with equal probability. Suppose one starts at the center, find the mean recurrence time for the center.



8. (**20 points**) Consider a random walk on $\mathbb{Z}$ in which one has probability $1/2$ to move one step to the right/left. Let u_{2m} be the probability of a return to the origin at time $2m$. Show that the probability that a random walk

of length $2m$ has a last return to the origin at time $2k$, where $0 \leq k \leq m$, equals $u_{2k}u_{2m-2k}$.