Algebraic Combinatorics
Homework #2
Mar 2, 2016

- A simplicial complex Δ is *pure* if its maximal faces all have the same dimension.
- An ordering of the maximal faces of a simplicial complex, say $F_1, F_2, \ldots, F_n$, is a *shelling order* provided, for every $i \in [n]$, the set

$$2^{F_i} \setminus (2^{F_1} \cup \cdots \cup 2^{F_{i-1}})$$

has a unique minimal element. A simplicial complex Δ is *shellable* if its maximal faces have a shelling order.

- A simplicial complex Δ is *partitionable* if Δ can be written as a disjoint union

$$\Delta = [R_1, F_1] \cup \cdots \cup [R_n, F_n], \qquad (1)$$

where $F_1, \ldots, F_n$ are all maximal faces of Δ.

1. Let Δ be a d-dimension simplicial complex with f-factor $(f_{-1}, f_0, \ldots, f_{d-1})$ and h-factor $(h_0, h_1, \ldots, h_d)$. Define $f_{i,j}$ inductively as follows:

- $f_{1,j} = f_j$ for $j \in \{-1, 0, \ldots d-1\}$;
- $f_{i,-1} = 1$ for $i \in \{1, 2, \ldots, d+1\}$;
- $f_{i,j} = f_{i-1,j} - f_{i,j-1}$ for $i \geq 2, j \geq 0$ and $i + j \leq d+1$.

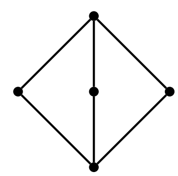
Prove that $h_j = f_{d+2-j,j-1}$.

2. Find a simplicial complex which is partitionable but not shellable.

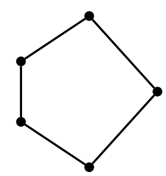
3. Let Δ be a pure partitionable simplicial complex. For the sets $R_1, \ldots, R_n$ displayed in a partition (1), show that the number $|\{j : |R_j| = k, j \in [n]\}|$ is totally determined by k and Δ. Will this still hold when Δ is not pure?

4. Let L be a lattice. A *sublattice* of L is a subposet of L closed under taking meets and joins. Show that L does not contain the *diamond* M_5 and the *pentagon* N_5 as a sublattice if and only if it satisfies one of the following:

$$\begin{aligned}
x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z), && \forall x, y, z \in L; \\
x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z), && \forall x, y, z \in L; \\
(x \vee y) \wedge (y \vee z) \wedge (z \vee x) &= (x \wedge y) \vee (y \wedge z) \vee (z \wedge x), && \forall x, y, z \in L; \\
(z \wedge x, z \vee x) &\neq (z \wedge y, z \vee y), && \forall x, y, z \in \binom{L}{3}.
\end{aligned}$$



M_5



N_5