Algebraic Combinatorics
Homework #1
Feb 23, 2016

- For any positive integer n, let $[n]$ be the set of the first n positive integers, namely $[n] = \{1, \ldots, n\}$.
- For any binary relation $\leq$ on $[n]$, namely $\leq \subseteq [n] \times [n]$, we write $i \leq j$ whenever $(i, j) \in \leq$ and we put the characteristic matrix of $\leq$ to be the $n \times n$ Boolean matrix $\mathcal{M}(\leq)$ whose entries $M_{i,j}$ are given by

$$M_{i,j} = \begin{cases} 1, & \text{if } i \leq j, \\ 0, & \text{otherwise.} \end{cases}$$

- We call a Boolean matrix M antisymmetric if $(M_{i,j}, M_{j,i}) \neq (1, 1)$ when $i \neq j$.
- The Hasse diagram of a poset $(P, \leq)$ is the digraph on vertex set P such that (x, y) is an arc whenever $x \neq y$, $y \leq x$ and there is no $z \in P \setminus \{x, y\}$ such that $y \leq z \leq x$. A poset P is ranked whenever it possesses a function r such that $r(x) - r(y) = 1$ whenever (x, y) is an arc in the Hasse diagram of P.
- Let P be the set of non-increasing functions defined on the set of positive integers which have finite supports. For $\mu, \nu \in P$, we define $\mu \leq \nu$ whenever $\mu(i) \leq \nu(i)$ for all positive integers i. The poset $(P, \leq)$ is called the Young's lattice.

1. Take an integer $n \geq 3$. Let $\leq$ be a binary relation on $[n]$ and $M = \mathcal{M}(\leq)$. Assume that M is antisymmetric, has no zero lines and satisfies $M^2 = M$. Show that $([n], \leq)$ may not be a poset.

2. Let H be a digraph with vertex set $V_H = [n]$ and A be its adjacency matrix. If

- $\sum_{k=1}^{\infty} A^k$ is a nilpotent matrix, and
- $(A)_{i,j} = 1 \Rightarrow (\sum_{k=2}^{\infty} A^k)_{i,j} = 0$,

then H is the Hasse diagram of a poset.

3. Let $(P, \leq)$ be a poset such that for all $x, y \in P$, every saturated chain from x to y, if any, has the same length, which is denoted by $\mathrm{d}(x, y)$. Prove or disprove the following: If for all $a, b, x, y \in P$, we have

$$\mathrm{d}(x, a) - \mathrm{d}(x, b) = \mathrm{d}(y, a) - \mathrm{d}(y, b),$$

then $(P, \leq)$ is a ranked poset.

4. Determine the automorphism group of the Hasse diagram of the Young's lattice.