MA323
Spring 2015
Final Exam
06/18/15
Time: 4 pm – 5:40 pm

Name (Print): ____________________

Student ID (Print): ____________________

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1. (15 points) Let G be a group and let $\delta_g, g \in G$, be the function on G such that $\delta_g(h) = \delta_{g,h}$. Show that the linear transformation on $\mathbb{C}[G]$ that sends δ_g to $\delta_{g^{-1}}$ for all $g \in G$ induces an isomorphism from the left regular representation of G to the right regular representation of G.

2. (15 points) Show that each character of a finite group G which is zero for all $g \in G \setminus \{1\}$ is an integral multiple of the character of the right regular representation.

3. (15 points) Fix a basis $\{e_1, \ldots, e_n\}$ of an n-dimensional complex vector space V. For any $g \in S_n$, the permutation representation ρ of S_n sends g to the linear map $\rho(g)$ on V that maps e_j to $e_{g(j)}$ for all j. Let W be the linear subspace of V consisting of those elements $\sum_{i=1}^{n} x_i e_j$ where $\sum_{i=1}^{n} x_i = 0$. Let χ be the character of the representation ρ restricted on W. Calculate $\sum_{g \in S_n} \chi(g)^2$.

4. (15 points) Let H be a subgroup of a finite group G. Let χ be an irreducible character of G, and let $Res_H^G\chi$ be the restriction of χ to H. Assume that $Res_H^G\chi = \sum_i c_i\chi_i$, where χ_i runs through all irreducible complex representations of H. Show that $\sum_i c_i^2 \leq |G : H|$.

5. (10 points) Let T be an n-dimensional irreducible complex representation of the finite group G. Prove that $\frac{n}{|G|}\sum_{g\in G}\overline{\chi_T(g)}T(g)$ is the identity map.

6. (10 points) State the Frobenius Reciprocity Theorem for class functions and give a proof of it.

7. (10 points) Calculate the characters of all real irreducible representations of the group A_4.

8. (10 points) Let A be a $d \times d$ complex matrix. For any positive integer N and any $\sigma \in S_N$ consisting of disjoint cycles of length $N_1, \ldots, N_r$, let $Tr_\sigma(A) = Tr(A^{N_1}) \cdots Tr(A^{N_r})$.

   (a) (5 points) Show that $\det(A) = \frac{1}{d!} \sum_{\sigma \in S_d} sgn(\sigma) Tr_\sigma(A)$.

   (b) (5 points) Let ρ be a 2-dimensional complex representation of a finite group G and let $X_g, g \in G$ be a set of variables. Show that $\det(\sum_{g \in G} X_g \rho(g)) = \frac{1}{2} \sum_{g \in G} (\chi_\rho(g)^2 - \chi_\rho(g^2)) X_g^2 + \sum_{\{g,h\} \in \binom{G}{2}} (\chi_\rho(g)\chi_\rho(h) - \chi_\rho(gh)) X_g X_h$.