1. Let G be a connected graph and let T be a tree. Suppose that for each $v \in V(T)$ we have a subset $B(v)$ of $V(G)$ such that $\cup_{v \in V(T)} B(v) = V(G)$, $\cup_{v \in V(T)} \binom{B(v)}{2} \supseteq E(G)$ and that $B(v) \cap B(u) \subseteq B(w)$ if w lies in the unique path connecting v and u in T. Suppose that $\max_{v \in V(T)} \max_{x,y \in B(v)} d_G(x,y) \leq 10$. Estimate $\max_{x,y,p,q \in V(G)} \delta_G(x,y,p,q)$, where $\delta_G(x,y,p,q)$ is the absolute value of the difference between the largest and the second largest of the three sums: $d_G(x,y) + d_G(p,q), d_G(x,p) + d_G(y,q), d_G(x,q) + d_G(y,p)$.

2. Let $I_1, I_2, \ldots, I_n$ be a set of intervals on the real line such that each point is covered by at most two of them. Assign positive integers to these intervals inductively as follows. Set $f(I_1) = 1$. For $t > 1$, set $f(I_t)$ to be the minimum positive integer that is not contained in the set $\{f(I_p) : 1 \leq p < t, I_p \cap I_t \neq \emptyset\}$. Estimate $\max\{f(I_t) : t = 1, 2, \ldots, n\}$.

3. Let $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{Z}$ the set of integers. Let L be a subgroup of $\mathbb{Z}^n$ such that $L \cap \mathbb{N}^k = \{0\}$. For any $u \in \mathbb{Z}^k$, set $F(u) = (u + L) \cap \mathbb{N}^k$. A Markov basis for L is a set $B \subseteq L$ such that $B = -B$ and that the graph F_u^B with vertex set F_u and edge set $\{xy : x - y \in B\}$ is connected for all $u \in \mathbb{Z}^k$ such that $F_u \neq \emptyset$. Prove that L has a finite Markov basis.

4. Let $B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Let $A_0, A_1, A_2, \ldots$ be a series of matrices such that $\{i : A_i = B_1\}$ and $\{i : A_i = B_2\}$ are both infinite sets and that $\{i : A_i = B_1\} \cup \{i : A_i = B_2\} = \mathbb{N}$. Show that $\lim_{n \to \infty} A_0 A_1 \cdots A_n$ exists and determine its value.

5. Let $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_i = \pm 1\} = A \cup B$. Show that there exists $y \in \mathbb{R}^4$ and $\lambda \in \mathbb{R}$ such that $xy^\top < \lambda$ for $x \in A$ and $xy^\top > \lambda$ for $x \in B$ if and only if $(A + A) \cap (B + B) = \emptyset$.

6. Let $[n] = \{1, 2, \ldots, n\}$. Let A be an $n \times n$ matrix such that it holds $\{A_{i,1}, A_{i,2}, \ldots, A_{i,n}\} = [n]$ for any $i \in [n]$. For any $j \in [n]$, put f_j to be the element of $[n]^{[n]}$ such that $A_{i, f_j(i)} = j$ holds for each $i \in [n]$. Let $\mathscr{A}$ be a subset of $[n]^{[n]}$ such that $\{f_1, \ldots, f_n\} \subseteq \mathscr{A}$ and that if $g, h \in \mathscr{A}$ then their composition $g \circ h \in \mathscr{A}$. Show that $rank(\mathscr{A}) \mid n$, where $rank(\mathscr{A})$ is defined to be $\min\{|Image(f)| : f \in \mathscr{A}\}$.

The points you get from each assignment will be inversely proportional to the performance of your classmates on that one and, needless to say, proportional to your own performance there.