上 海 交 通 大 学 试 卷 (A 卷)

(2008 至 2009 学年 第 1 学期)

班级号 ______________ 学号 ______________ 姓名 ______________

课程名称 概率论及其应用（ ACM 班） 成绩 ______________

1. (15 分) Please prove Murphy's Law, namely if something bad can happen it eventually will. Here is its more precise statement:

Let $A_n, n \geq 1$ be any sequence of events satisfying the nesting condition $A_n \subseteq A_{n+1}$ for any $n \geq 1$ – you can think of A_n as the event that the bad thing happens on or before day n. Let $A = \cup_{n=1}^{\infty} A_n$ and suppose that for some $\epsilon > 0$ and any $n \geq 1$ we have $P(A|A_n^c) \geq \epsilon$. Then $P(A) = 1$. (Hint: $P(A) = P(A|A_n^c)(1 - P(A_n)) + P(A|A_n)P(A_n)$.)

题号	一	二	三	四	五	六	七	八	九	十	总分
得分											
批阅人 (流水阅卷教师签名处)											

2. (15 分) Let $[n]=\{1,2,\ldots,n\}$ and let $X_i, i\in[n]$, be a collection of random variables. For any $S\subseteq[n]$, put X_S to be the collection of random variables $(X_i:\ i\in S)$. Let $S_1,\ldots,S_m$ be a family of nonempty subsets of $[n]$. For any $i\in[m]$, take $s_i=\max_{j\in S_i} r_j$ where r_j stands for $\sharp\{S_k:\ j\in S_k, k\in[m]\}$. Prove the following:

$$\sum_{i\in[m]}\frac{H(X_{S_i}|X_{\{j:j\notin S_i,j<\max S_i\}})}{s_i}\le H(X_{[n]}).$$

3. (15 分) Let $A_i, i \in [n]$, be a family of sets. For any $S \subseteq [n]$, put $A_S = \cap_{i \in S} A_i$ and $A^S = \cup_{i \in S} A_i$. Make use of the inclusion-exclusion principle (namely $1_{A^{[n]}} = \sum_{S \subseteq [n], S \neq \emptyset} (-1)^{\sharp S - 1} 1_{A_S}$) and De Morgan's rule to deduce the following: $1_{A_{[n]}} = \sum_{S \subseteq [n], S \neq \emptyset} (-1)^{\sharp S - 1} 1_{A^S}$.

4. (15 分) Show that a graph on 50 vertices has no more than 1225 minimal edge cutsets.

5. (20 分) Given a fair die, each of its six faces marked with a different number among 1, 2, ..,6, we roll it repeatedly and independently and record the result as a sequence $x_1 x_2 \cdots$ where x_i is the number which falls uppermost. Compute the expectation of the random variable $X = \min\{i : \ x_{i+1} = 3x_i\}$.

6. (10 分) Let $X_1, \ldots, X_n$ be a set of collectively independent random variables satisfying $P(X_i = 1 - p_i) = p_i$ and $P(X_i = -p_i) = 1 - p_i$. Prove that $P(|\sum_{i\in[n]} X_i| \geq a) \leq 2exp(-2a^2/n)$.

7. (5 分) Show that the relative entropy (Kullback Leibler distance) between any two probability mass functions must be nonnegative.

8. (5 分) Give the statement of Lovász Local Lemma.