Graph Theory Midterm II

Name: ____________ Student ID: ____________ Score: ____________

Please say what you mean and mean what you say!

1. (10 marks) Let $\phi : \mathbb{Z}_2^{\mathbb{Z}} \to \mathbb{Z}_2^{\mathbb{Z}}$ be the sliding block code defined by $\phi(x)_i = x_i + x_{i+1}$ for $i \in \mathbb{Z}$. Find the number of points x satisfying $\phi^3(x) = x$.

2. (20 marks) Let G be the digraph on 4 vertices with all possible arcs other than self-loops. Consider the vertex shift $(\widehat{X_G}, \sigma)$. Show that there is no sliding block code ϕ with memory 0 and anticipation 1 from $\widehat{X_G}$ to itself such that $\phi^2 = \sigma$. [1]

3. (10 marks) Let G be the digraph with $V(G) = \{a, b, c, d, e\}$ and $A(G) = \{\overrightarrow{ab}, \overrightarrow{bc}, \overrightarrow{ca}, \overrightarrow{ad}, \overrightarrow{de}, \overrightarrow{ea}\}$. Please make a sequence of four elementary out-splittings to yield a digraph with maximum out-degree 3.

4. (20 marks) Let X be a shift space. The memory of X is the smallest integer N, if any, such that for any symbol b and any word $w \in B(X)$ of length at least N, we have $wb \in B(X)$ if and only if $w'b \in B(X)$ where w' is the suffix of w of length N. For a labelled digraph $\mathcal{G}$, its memory is the smallest integer N, if any, such that all words of $X_{\mathcal{G}}$ of length N are synchronizing. (a). Prove that $X_{\mathcal{G}}$ is an N-step shift of finite type when $\mathcal{G}$ is right-resolving and has memory N. (b). Show that the memory of any sofic shift is no greater than the memory of any of its presentations and there is a presentation whose memory is the same with that of X provided X has finite memory. (c). Let X be an irreducible shift of finite type which has a right-resolving presentation $\mathcal{G}$ with r vertices. Show that X is $(\frac{r^2-r}{2})$-step.

5. (15 marks) For a sofic shift X, call a word $w \in B(X)$ intrinsically synchronizing if whenever wz and $z'w$ are in $B(X)$, so is $z'wz$. Suppose X

[1] This result even holds when 4 is replaced by general $n > 1$.

is irreducible. Prove that any intrinsically synchronizing word of X must be a synchronizing word of any minimal presentation of X.

6. (15 marks) Suppose that X and X' are sofic shifts with right-resolving presentations $\mathcal{G}$ and $\mathcal{G}'$, respectively. If $X \subseteq X'$ and $\mathcal{G}$ is irreducible, show that for every vertex I in $\mathcal{G}$ there must be a vertex I' in $\mathcal{G}'$ such that $\mathrm{F}_{\mathcal{G}}(I) \subseteq \mathrm{F}_{\mathcal{G}'}(I')$.

7. (10 marks) Find the minimal right-resolving presentation for the sofic shift presented by the following symbolic adjacency matrix

$$\begin{bmatrix} \emptyset & b & \emptyset & a & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & a & b & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & a & b \\ b & \emptyset & \emptyset & a & \emptyset & \emptyset \\ b & \emptyset & \emptyset & \emptyset & \emptyset & a \\ b & \emptyset & \emptyset & \emptyset & a & \emptyset \end{bmatrix}.$$

8. (10 marks) Show that $X_{[m]}$ factors onto $X_{[n]}$ if and only if $m \geq n$.