8: 00 – 9:50 am, October 15, 2007, Xia Yuan 105

Graph Theory Midterm I

Name: ____________ Student ID: ____________ Score: ____________

If you have doubts about the wording of a problem, please ask for clarification. In no case should you expect that I will try to understand a too ugly or too succinct solution and assign you any credit for it. You should say what you mean and mean what you say! The last three questions are adapted from a working paper of Andreas Dress et al.. If you have interest in knowing more details of that work or even some further research directions, contact him at andreas@picb.ac.cn.

0. (10 marks) Use basic linear algebra to prove that a graph is bipartite if and only if it contains no odd cycle.

1. (10 marks) Prove that the faces of a planar map can be 2-colored if every node of it has even degree.

2. (15 marks) Construct explicitly a surjective mapping from the set of unlabelled trees on n vertices to the set of n-tuples of positive integers $(d_1, d_2, \ldots, d_n)$ satisfying $d_1 \leq d_2 \leq \ldots \leq d_n$ and $\sum_{i=1}^{n} d_i = 2n-2$. (Your task is not only to give me a construction!)

3. (15 marks) Let G be a bipartite graph with vertex classes X and Y having arbitrary cardinalities. Let $A \subseteq X$ and $B \subseteq Y$. Suppose there are complete matchings from A into Y and from B into X. Prove that G contains a set of independent edges covering all the vertices of $A \cup B$. (This is the Schröder-Berstein Theorem reformulated in the language of graph theory.)

4. (10 marks) Prove that 6 points in the plane, no 3 on a line, form at least 3 convex quadrilaterals.

5. (10 marks) Let V be a finite set and $\mathbb{R}$ the set of real numbers. We call $W \in \mathbb{R}^{\binom{V}{2}}$ an ultrametric provided $W(uv) \leq \max(W(uw), W(wv))$

holds for any $uv, uw, wv \in \binom{V}{2}$. For any $W, W' \in \mathbb{R}^{\binom{V}{2}}$ we write $W \geq W'$ whenever $W - W'$ takes nonnegative value everywhere. Prove that for any $W \in \mathbb{R}^{\binom{V}{2}}$ there is a unique ultrametric W', called the subdominant ultrametric for W, such that $W \geq W' \geq W^*$ holds for every ultrametric W^* satisfying $W \geq W^*$. (*Hint: Consider* W' *defined by setting* $W'(uv) = \min_{v_1,\ldots,v_n} \max\{W(uv_1), W(v_1v_2), \ldots, W(v_{n-1}v_n), W(v_nv)\}$ *where* $v_1, \ldots, v_n$ *run through all possible, maybe empty, sequences of different members of* $V \setminus \{u, v\}$.)

6. (15 marks) Show that the following algorithm produces the subdominant ultrametric W' for $W \in \mathbb{R}^{\binom{V}{2}}$:

- Input: $W \in \mathbb{R}^{\binom{V}{2}}$
- Initialize: $S = \emptyset$
- Repeat untill $S = \binom{V}{2}$: Choose a $uv \in \binom{V}{2} \setminus S$ such that $W(uv) = \min_{xy \in \binom{V}{2} \setminus S} W(xy)$ and put $S := S \cup \{uv\}$; Define

  $$W'(uv) = \min_{v_1,\ldots,v_n} \max\{W(uv_1), W(v_1v_2), \ldots, W(v_{n-1}v_n), W(v_nv)\}$$

  where $v_1, \ldots, v_n$ run through all possible, maybe empty, sequences of different members of $V \setminus \{u, v\}$ satisfying $uv_1, v_1v_2, \ldots, v_{n-1}v_n, v_nv \in S$.
- Output: $W' \in \mathbb{R}^{\binom{V}{2}}$

7. (15 marks) Let G denote the complete graph on the vertex set V whose edges are weighted by $W \in \mathbb{R}^{\binom{V}{2}}$. For any edge uv of any minimum spanning tree of G, show that $W'(uv) = W(uv)$ where W' is the subdominant ultrametric for W.