# Graph Theory

Yaokun Wu
Department of Mathematics
Shanghai Jiao Tong University
Shanghai, 200240, China
ykwu@sjtu.edu.cn

Course slides for ACM class, Fall 2005

http://www.math.sjtu.edu.cn/teacher/wuyk/acm.pdf

## Class times and locations

Odd week:

Monday 16:00 – 17:40, Shangyuan 106

Wednesday 8:00 – 9:40, D-502

Even week:

Wednesday 8:00 – 9:40, E-122

Our official textbook is

Douglas B. West, Introduction to Graph Theory, China Machine Press, 2004.

Here are some additional recommended readings:

W.T. Tutte, Graph Theory As I Have Known It, Clarendon Press, Oxford, 1998.

J.H. van Lint and R.M. Wilson, A Course in Combinatorics, China Machine Press, 2004.

B. Bollobas, Modern Graph Theory, World Publishing Corporation, 2003.

S. Jukna, Extremal Combinatorics: With Applications in Computer Science, Springer-Verlag, 2001.

This is an introductory course. It demands certain mathematical maturity, a nodding acquaintance with linearly thinking and probabilistically thinking. But what matters more is interest in the material and a commitment to work hard, as Prof. West advised.

Many undergraduates begin graph theory with little practice at presenting explanations, and this hinders their appreciation of graph theory and other mathematics. The intellectual discipline of justifying an argument is valuable independent of mathematics; I hope that students will become comfortable with this. In writing solutions to exercises, students should be careful in their use of language ("say what you mean"), and they should be intellectually honest ("mean what you say"), which includes acknowledging when they have left gaps. – D.B. West

If you want to hand in any mathematical arguments to earn your grades, please write clearly and mathematically. It is not enough to give a right answer. The derivation and quality of writing counts!

William Thomas Tutte (May 14, 1917 – May 2, 2002) is known as a master code-breaker and was one of the driving forces in combinatorics. The term 'Graph theory' first appears in English in the following paper of Tutte:

A ring in graph theory. Proc. Cambridge Philos. Soc. 43, (1947). 26–40.

The idea appeared in this paper is of fundamental significance and the very important concept of Tutte polynomial began to be the key to study many parts of mathematics since then. We copy a review of it in the next page.

Let the complexity of a graph $L$ be defined as the number of trees which can be formed by taking all the nodes and some (or all) of the branches; e.g., the vertices and edges of a tetrahedron form a graph of complexity 16. Let $L_A{}'$ be derived from $L$ by suppressing a given branch $A$, and $L_A$" by identifying the two ends of $A$ while suppressing $A$ and any other branches that may have joined those ends. The complexity of $L$ is equal to the sum of the complexities of $L_A{}'$ and $L_A$"; e.g., $16 = 8 + 8$. The author seeks to characterize those numerical properties of a graph which are additive in this sense. He then considers "cubical" graphs whose nodes are all of degree 3, and describes a simple transformation by means of which any such graph may be reduced to a standard form consisting of the same (even) number of nodes joined in sequence by single and double branches alternately, with a loop at each end of the whole chain.

Reviewed by H. S. M. Coxeter

The PhD thesis of Tutte takes two strands, one of algebra and one of combinatorics, and spins them into one thread – matroid theory.

My thesis attempted to reduce Graph Theory to Linear Algebra. It showed that many graph-theoretical results could be generalized as algebraic theorems about structures I called 'chain groups'. Especially, I was discussing a theory of matrices in which elementary operations could be applied to rows but not to columns. – W. T. Tutte, An Algebraic Theory of Graphs, PhD thesis, Cambridge University.

The following paper by Tutte himself describes how he became acquainted with the Tutte polynomial, beginning from his study of a recreational problem, squaring the square.

W.T. Tutte, Graph-polynomials, Advances in Applied Mathematics, 32 (2004) 5–9.

Search for simple unifying mathematical principles, as exemplified by his work on graph polynomials, underlies much of Tutte's work. – U.S.R. Murty, Dedication: Professor W.T. Tutte, Journal of Combinatorial Theory B 92 (2004) 191–192.

Paul Seymour of Princeton University writes:

"Professor Tutte has been for many years the dominant figure in graph theory, and his contributions to the subject outweigh those of any other individual (in every sense except perhaps quantity). There are numerous instances when Tutte has found a beautiful result in a hitherto unexplored branch of graph theory, and in several cases this has been a breakthrough', leading to the development of a major new subject."

Lászlo Lovász of Microsoft writes:

"Few theorems in mathematics are honored by the general public by naming them after the mathematician who proved them. In Tutte's case, however, there are several such results: for somebody working in matching theory, Tutte's theorem is his characterization of graphs having a perfect matching – for a matroid theorist, it means his characterization of regular matroids – for somebody studying Hamiltonian cycles it means his result that 4-connected planar graphs have a Hamilton cycle. And there is also the Tutte polynomial of a graph (and a matroid), which is again a household word for many combinatorialists."

Alan Turing is well-known for deciphering Enigma codes. But that success was only with the naval and air force versions; the army version of Enigma, a set of machine-ciphers named Fish, proved to be more resistant to analysis.

One of Tutte's great contribution is to uncover, from samples of the messages alone, the structure of the machines which generated these codes. Tony Sale, who first described this work in a 1997 article in New Scientist, characterized it as the "greatest intellectual feat of the whole war."

In those Bletchley Park days, rather than break a specific code, Tutte indeed put himself to creating a general algorithm to find from the enciphered messages the initial settings of the machine wheels. In 1943, the electronic computer COLOSSUS was designed and built by the British Post Office just to run the algorithms that Tutte developed, the "Statistical Method".

"...He was one of many who regarded signing the Official Secrets Act as a lifelong obligation, and when stories of the great deeds done at Bletchley began to leak out, he did not immediately leap on the bandwagon. It was probably a relief to him when, in the 1990s, it became clear that at least some of the secrets were no longer official. At his 80th birthday celebrations in 1997 he felt able to tell me some of the details, and in 1998 he gave a talk entitled "Fish and I" (now available on the internet). He tells how, others having failed, he was asked to work on the cipher system, known in Britain as "Tunny", used by the German Army High Command. He had an idea and, although not optimistic, he "thought it best to seem busy". So he copied out the ciphertext onto squared paper, using chunks of various lengths, noticed certain patterns, and was able to infer the structure of the system. ..."

– Norman Biggs

# Schedule of Lectures

(Green part is tentative)

- (5/9, 7/9, 14/9) Basic proof techniques: A playground for double counting, induction, extremality, bijection...

- (19/9, 21/9, 28/9, 12/10) Electrical network and potential theory on graph

- (8) Matroid, duality and Tutte polynomial

- (10) Marriage problems
- (6) Flows
- (6) Perfect graph and intersection representation
- (6) Probabilistic method

## I. Basic proof techniques: A playground for double counting, induction, extremality, bijection...

Counting pairs is the oldest trick in combinatorics*... Every time we count pairs, we learn something from it. – Gil Kalai

*The commutative law for the multiplication of positive integers just follows from Double Counting!

When you played with the puzzles that ask if you can trace a figure without lifting your pencil, you were working with an Euler path. Leonhard Euler is a very early player of such a puzzle and the puzzle he met is the famous Königsberg 7-Bridge Problem. In 1736 Euler published a paper on the solution of the Königsberg Bridge problem entitled ' The solution of a problem relating to the geometry of position', which is now considered as the beginnings of topology. In this paper, Euler stated the following theorem but gave no proof, perhaps because the suitable definitions * needed for such a proof did not exist then. The first published proof was produced by Hierholzer in 1873.

**Theorem 1** *A graph is Eulerian if and only if it is connected and even.*

*A definition is the enclosing of a wilderness of idea within a wall of words.
– Samuel Butler (1835-1902)

*Proof.* An inclusion-maximal closed trail is an Eulerian cycle*. ∎

**Exercise 2** *A graph is even if and only if it has no edge cuts of odd size; a graph is bipartite if and only if it has no odd circuit.*

There is a beautiful monograph of Herbert Fleischner addressing the important class of Eulerian graphs, which consists of three volumes and only the first two volumes are available now:

H. Fleischner, Eulerian Graphs and Related Topics, Part 1, Vol. 1, Elsevier Science Publishers B. V., Amsterdam-New York, 1990.

H. Fleischner, Eulerian Graphs and Related Topics, Part 1, Vol. 2, Elsevier Science Publishers B. V., Amsterdam-New York, 1991.

*A circuit is a closed trail passing through any vertex at most once. A cycle is an even graph, namely a disjoint union of circuits; if connected, it is an Eulerian cycle. Our use of these concepts is different from the use of them by West.

You may wonder how can one write three big books on such a trivially-looking concept of Eulerian graphs. Yes, there have been lots of deep results related to this concept. A good scientist is able to make some nontrivial beautiful statement about some familiar (trivial?) object. To do so, he/she sometimes needs to introduce (recognize?) additional structures and concepts. With the discovery of those hidden facts, we have more understanding of an object and will have the possibility to make good use of our understanding in the so-called ‘practical world’.

In a sense, making an extremal choice goes directly to the important case.

**Theorem 3 (Mantel 1907)** *The maximum number of edges in an $n$-vertex triangle-free simple graph $G$ is $\lfloor \frac{n^2}{4} \rfloor$.*

*Proof.* Choose a largest independent set $A$ of $G$ and put $B = V(G) \setminus A$. Since no edge has both endpoints in $A$, every edge of $G$ meets $B$. Moreover, considering that $G$ is triangle-free and $A$ is a maximum size independent set, we infer that $\mathbf{deg}_G(v) \leq |A|$ for each vertex $v$. It then follows $|E| \leq \sum_{v \in B} \mathbf{deg}_G(v) \leq \sum_{v \in B} |A| = |A||B| \leq \lfloor \frac{n^2}{4} \rfloor$, where equality can be achieved if and only if $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. ∎

**Exercise 4** *Prove Turán's Theorem*[*]*: Let $n \geq p \geq 3$. The unique simple graph $G$ on $n$ vertices without $p$-cliques and the maximum number of edges is the complete multipartite graph $K_{n_1,\cdots,n_{p-1}}$*[†]*, where $\sum n_i = n$ and $|n_i - n_j| \leq 1$.*

For any graph $G$ and any $S \subseteq E(G)$, $M^S(G)$ and $m^S(G)$ denote the sets of all inclusion-maximal and inclusion-minimal, respectively, cycles of $G$ that cover all edges in $S$.

**Theorem 5** *If $m^S(G)$ is odd for each singleton set $S \subseteq E(G)$, then $G$ is even.*

[*] For some of its geometric applications, read [West, 5.2.10, 5.2.11].

[†] This class of graphs are also knows as Turán graphs, due to Turán's Theorem.

*Proof.* Regard each subset of $E(G)$ as its indicator vector lying in $\mathbb{F}_2^{E(G)}$. The assertion that $|m^S(G)|$ is odd for each singleton set says $\sum C = E(G)$, where $C$ runs through all circuits of $G$. Since each circuit is even and the sum of even graphs is even, the claim follows. * ∎

**Exercise 6** *If $m^S(G)$ is odd for each two element set $S \subseteq E(G)$, can we conclude that $G$ is even?*

The next result is a converse of Theorem 5. Note that the first result being equivalent to Theorem 7 was obtained by Toida (Exercise 8). But the proof given by Toida cannot be generalized to get more general results on binary matroid. You can check that our arguments presented below indeed proves something stronger than the statement of the theorem.

*A graph $G$ is even if and only if $E(G)$ belongs to the cycle space.

**Theorem 7 (T.A. McKee 2005)** *For any even graph $G$ and $S \subseteq E(G)$, $|m^S(G)|$ is odd.*

*Proof.* We prove the result by assuming the contrary and finding a contradiction. Suppose $S$ is an inclusion-maximal subset of $E(G)$ that has $|m^S(G)|$ even.

Clearly, $S$ does not belong to the cycle space, as otherwise $|m^S(G)| = |\{S\}| = 1$. But we know that the cycle space is the orthogonal complement of the cut space. This implies that there is a cutset $D$ with $|D \cap S|$ being odd. The fact that $G$ is even amounts to saying that $E(G)$ is a member of the cycle space and so $|D|$ has to be even. Combining the previous two observations, we get $D \setminus S$ contains an odd number of edges, say $e_1, \cdots, e_\ell$. It is easy to find that $|S^* \cap (D \setminus S)|$ is odd for every $S^* \in m^S(G)$. Write $S_j$ for $S \cup \{j\}$. By the maximality of $S$, we know that $|m^{S_j}(G)|$ is odd for each $j$.

Define $\zeta_j = \{(C_1, C_2) : C_1 \cap C_2 = \emptyset, S \subseteq C_1, j \in C_2, C_2$ is a cycle$\}$ and $\sigma_j = \{(C_1, C_2) \in \zeta_j : C_1 \cup C_2 \in \mathrm{m}^{S_j}(G)\}$. Note that for any $(C_1, C_2) \neq (C_1', C_2') \in \sigma_j$, we have $C_1 \cup C_2 \neq C_1' \cup C_2'$, as otherwise $C_1 \triangle C_2'$ will be a proper subset of $C_1 \cup C_2$ containing $S_j$, which is impossible. We now deduce $1 = \sum_{j=1}^{\ell} \sum_{C \in m^{S_j}(G)} 1 = \sum_{C \in m^S(G)} \sum_{e_j \in C} 1 + \sum_{j=1}^{\ell} \sum_{(C_1,C_2) \in \sigma_j} \sum_{e_j \in C_2} 1 = \sum_{C \in m^S(G)} 1 = 0$*, yielding a contradiction, as required. ∎

**Exercise 8** [†] *The number of paths between any two distinct vertices in an Eulerian graph is even.*

**Exercise 9** *Generalize Theorem 7 to be a theorem on linear space.*

*Is the third equality here obvious? More details can be found in the proof of Theorem 30.

†S. Toida, Properties of a Euler graph, *Journal of the Franklin Institute* **295** (1973), 343–345.

**Exercise 10** *Prove that for any graph* $G$ *and* $S \subseteq E(G)$, $\left|M^S(G)\right| - \left|m^S(G)\right|$ *is an even number.*

**Theorem 11** *Every loopless graph* $G$ *has a bipartite subgraph with at least* $\frac{e(G)}{2}$ *edges.*

**Theorem 12 (Dirac 1952)** *If* $G$ *is a simple graph with at least three vertices and* $\delta(G) \geq \frac{n(G)}{2}$, *then* $G$ *is Hamiltonian.*

**Exercise 13** *Prove that each simple graph* $G$ *contains at least* $\frac{e(G)(4e(G)-n(G)^2)}{3n(G)}$ *triangles.*

**Exercise 14** *Suppose* $G$ *is a simple graph with* $\lfloor\frac{m^2}{4}\rfloor - n$ *edges where* $n, m$ *are positive integers. Prove that if* $G$ *contains a triangle then it contains at least* $\lfloor\frac{m}{2}\rfloor - n - 1$ *triangles.*

We come to the First Theorem of Graph Theory, also called the Hand-shaking Lemma.

**Theorem 15** *There are an even number of odd nodes in every graph.*

*Proof.* Assign weight 1 to each edge–end point pair and then collect the weights in two ways, edge by edge or vertex by vertex. Or observe that each hand-shaking (adding of an edge) preserves the parity of the number of odd nodes. ∎

**Theorem 16 (Smith's Theorem)** *In every cubic graph $G$ the number of Hamiltonian circuits passing through any specified edge is even.*

*Proof.* A Tait coloring $T$ of $G$ colors $E(G)$ in three colors $\alpha, \beta$ and $\gamma$ such that no two of the same color meet at a vertex *. We do not consider a permutation of the three colors as giving a new Tait coloring.

*This is just a 1-factor factorization. In directed case, it corresponds to the so-called Road Coloring in the study of synchronizing automata.

Taking the edges of two colors, say $\alpha$ and $\beta$, we obtain a Tait cycle $T_{\alpha,\beta}$. A 2-regular subgraph $K$ of $G$ whose components all have even sizes is said to be good. A subgraph $K$ is good if and only if it can interpreted as the Tait cycle $T_{\alpha\beta}$ of some Tait coloring $T$. Indeed, there are exactly $2^{k-1}$ distinct Tait colorings having $K$ as a Tait cycle, where $k(K)$ stands for the number of components of $K$.

Note that $T_{\alpha\beta} + T_{\beta\gamma} + T_{\gamma\alpha} = 0$. Sum this equation over all Tait colourings and exchange the order of summation yields $\sum_K 2^{k(K)-1} K = 0$ where $K$ runs through all good subgraphs. Since we are working over $\mathbb{F}_2$, only those good graphs $K$ with $k(K) = 1$, namely Hamiltonian circuits, make a nonzero contribution. This proves that the sum of all Hamiltonian circuits is zero, as was to be shown. ∎

**Exercise 17** *Let* $e$ *be an edge of* $G$ *with* $u, v$ *as its endpoints. If every vertex of* $G$ *other than* $u, v$ *has odd degree, then there is an even number of Hamiltonian circuits passing through* $e$.

**Exercise 18** *Denote by* $t_n$ *the maximum number* $k$ *such that each 2-coloring of* $E(K_n)$ *contains* $k$ *monochromatic triangles. Prove the following.*

(i) $t_{2n} = 2\binom{n}{3}$.

(ii)

$$t_{2n+1} = \begin{cases} \frac{(n-2)n(2n+1)}{6}, & \text{if } n \equiv 0 \pmod 2, \\ \frac{(n-2)n(2n+1)}{6} + \frac{1}{2}, & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

Everything of importance has been said before by somebody who did not discover it. – Alfred North Whitehead

The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thoughts of concrete fact. – Alfred North Whitehead, English mathematician & philosopher, 1861-1947.

End of Lesson One 5/9/05

There are a class of theorems in graph theory asserting that for any input $G$ satisfying certain conditions, $G$ has an even (or odd) number of $H$'s satisfying some specified conditions. A typical way of doing double counting implicitly is to construct an 'exchange graph' $X$, maybe quite larger compared to $G$, such that the odd nodes of $X$ are the objects $H$ we want to show there is an even number of (or such that all but an odd number of the odd nodes of $X$ are the objects which we want to show there is an odd number of).

**Theorem 19** *Any Eulerian graph $G$ with $n(G)$ even has evenly many spanning trees.*

*Proof.* Consider the graph $X$ whose nodes are spanning trees of $G$ and two vertices are connected by an edge if the corresponding trees differ by a swap, namely if one is obtained from the other by adding an edge and then deleting another edge in the edge cut corresponding to the remaining two components of the tree. Note that Exercise 2 says that the cut is of even size. ∎

**Exercise 20** *Any bipartite graph* $G$ *with* $n(G) + e(G)$ *even has an even number of spanning trees.*

**Exercise 21** *Any bipartite Eulerian graph has an even number of spanning trees.*

We now move on to Berman's generalization * of the famous Smith's Theorem (Theorem 16).

**Theorem 22** *For any loopless graph $G$ with at least three vertices and a specified pair of nonnegative integers $(h(v), k(v))$ for each vertex $v$ of $G$ such that $h(v) + k(v) = \mathbf{deg}_G(v)$ and such that either $h(v) = 1$ or $k(v)$ is odd (perhaps both), there is an even number of good spanning trees of $G$, that is, spanning trees $H$ with $\mathbf{deg}_H(v) = h(v)$ for each $v \in V(G)$.*

*Kenneth Berman, Parity results on connected $f$-factors, Disc. Math. 59 (1986) 1–8.

*Proof.* If no good spanning tree exists, that even number is just 0. Assume that such a tree does exist. Since there are at least three vertices, there is $w$ with $h(w) > 1$ and hence $k(w)$ is odd. Fix a $w$ with odd $k(w)$ and we construct the exchange graph $X$ as follows.

Define a spanning tree $H$ of $G$ to be bad provided $\mathbf{deg}_H(w) = h(w) + 1$ and there is one vertex $v$ with $\mathbf{deg}_H(v) = h(v) - 1$ and $\mathbf{deg}_H(u) = h(u)$ for all $u \neq v, w$. The nodes of $X$ consist of all bad spanning trees and good spanning trees of $G$ and there is an edge between two trees if and only if each can be obtained from the other by exchanging one edge in the tree for one edge of the other tree. We will show that bad trees and good trees are respectively even nodes and odd nodes in $X$ and then the theorem will follow from the fact of 'even number of odd nodes'.

Observe how the degree sequence of a spanning tree is affected after an edge swapping. We find that for each edge of $X$, the pair of edges of $G$ appeared in the corresponding swapping must have a common end point. As a consequence, we are able to characterize the parity of nodes in $X$.

On the one hand, for any bad tree $H$ with $\mathbf{deg}_H(v) = h(v){-}1 \geq 1$, the neighbors $H'$ of $H$ in $X$ is in bijection with the set of edges $e \in N_G(v) \backslash N_H(v)$, namely $H'$ is obtained by adding $e$ and deleting the other edge $e'$ in the unique circuit in $H{+}e$ that is incident with $u$, where $u$ is the endpoint of $e$ other than $v$.* But $h(v) - 1 \geq 1$ says that $k(v)$ is odd and so $\mathbf{deg}_X(H) = |N_G(v) \backslash N_H(v)| = k(v){+}1$ is even.

On the other hand, for any good tree $H$, we have $\mathbf{deg}_X(H) = |N_G(w) \setminus N_H(w)| = k(w)$ which is odd. Indeed, each neighbor $H'$ of $H$ is obtained by adding an edge $e \in N_G(w) \setminus N_H(w)$, say the other end of $e$ is $u$, and deleting the unique edge $e' \neq e$ in the circuit of $H + e$ and incident to $u$. ∎

*Suppose the endpoint of $e'$ other than $u$ is $\mu$. If $\mu = w$, then $H'$ becomes a good tree; otherwise, it is a bad tree with $\mathbf{deg}_{H'}(\mu) = h(\mu) - 1$.

Our treatment of Theorem 22 follows

K. Cameron, J. Edmonds, Some graphic use of an even number of odd nodes, Ann. Inst. Fourier, Grenoble 49 (1999) 815–827.

In this paper, the authors assert that 'using the exchange graphs, the theorem seem(s) suitable for the first hour of an introduction to graph theory'. They also discuss the interesting concept of "existentially polytime theorem" as a generalization of "good characterization", a key idea in computing science which Jack Edmonds presented in the 1960's.

**Exercise 23** *Illustrate that Theorem 16 follows from Exercise 17 while Exercise 17 follows from Theorem 22.*

**Exercise 24** *Can you find a generalization of Theorem 22 for linear space?*

Consider the linear space of $1 \times n$ vectors over $\mathbb{F}_2$, which can be viewed as $\mathbb{F}_2^{[n]}$, the set of functions from $[n] = \{1, \cdots, n\}$ to $\mathbb{F}_2$. Let $W$ be a subspace of $\mathbb{F}_2^{[n]}$, $A \subseteq [N]$ and $x \in \mathbb{F}_2^{[n]}$. Define $p_A(x)$ to be an element of $\mathbb{F}_2^A$ with $\mathbf{supp}(x) \cap A = \mathbf{supp}(p_A(x))$ and put $W_A = \{x \in W : \ x(i) = x(j), \forall i, j \in A\}$. Say a vector in $W$ is maximal if its support is maximal under inclusion among all those vectors of $W$. Say a nonzero vector in $W$ is minimal if its support is minimal under inclusion among all those nonzero vectors of $W$. Denote by M($W$) (m($W$)) the set of maximal (minimal) vectors of $W$.

**Theorem 25** * *$|M(W)|$ is odd.*

*P. Hoffmann, Counting maximal cycles in binary matroids, *Discrete Mathematics* **162** (1996), 291–292.

*Proof.* The claim is trivial when $n = 1$. So we assume $n > 1$ and proceed with the assumption that the claim holds when $n$ is smaller. If $\mathbf{dim} W = 0$, then $\mathsf{M}(W) = W = \{0\}$ and hence we are home. Now suppose $\mathbf{dim} W > 0$. Then $W$ is the disjoint union of $\mathsf{M}(W)$, $\{0\}$ and $\mathcal{M}(W)$, where $\mathcal{M}(W) = W \setminus (\mathsf{M}(W) \cup \{0\})$. Since $|W| = 2^{\mathbf{dim} W}$ is even, it suffices to show that $|\mathcal{M}(W)|$ is even.

Construct a graph $G$ with $V(G) = \mathcal{M}(W)$ and there is an edge between $x$ and $y$ if and only if $\mathbf{supp}(x) \cap \mathbf{supp}(y) = \emptyset$ and $x + y \in \mathsf{M}(W)$. Take arbitrarily an $x \in \mathcal{M}(W)$. Our task is to demonstrate that $x$ is an odd node of $G$. Let $B$ be the set of maximal vectors of $p_{[n] \setminus \mathbf{supp}(x)}(W_{\mathbf{supp}(x)})$ in $\mathbb{F}_2^{[n] \setminus \mathbf{supp}(x)}$. Note that $0 < |[n] \setminus \mathbf{supp}(x)| < n$. Thus, by inductive assumption, $|B|$ is odd. However, the neighbors of $x$ are exactly those vectors $y$ whose supports coincide with the support of one element of $B$ and hence $\mathbf{deg}_G(x) = |B|$, finishing the proof. ∎

**Exercise 26** *Deduce Theorem 7 from Theorem 25 and vice versa.*

**Theorem 27 (Paul Erdős 1965)** *Every set $B = \{b_1, \cdots, b_n\}$ of nonzero integers contains a sum-free subset of size $> \frac{n}{3}$.*

*Proof.* Since there are infinitely many primes which are congruent 2 modulo 3, we can take such a prime $p = 3k+2$ which does not divide any element of $B$. Put $C = \{k+1, k+2, \cdots, 2k+1\}$. $C$ is a sum-free subset of $\mathbb{Z}_p$ with $\frac{|C|}{p-1} = \frac{k+1}{3k+1} > \frac{1}{3}$. Consider the $n \times (p-1)$ matrix $A$ over $\mathbb{Z}_p$ with $A_{ij} = jb_i$. Checking the elements of $A$ row by row, we see that more than $\frac{n(p-1)}{3}$ entries of $A$ fall in $C$. This says that there is a column of $A$, say the $j$th one, more than $\frac{n}{3}$ of whose entries lie in $C^*$. The set $\{i : A_{ij} \in C\}$ is the required sum-free subset. ∎

*We are using the so-called Averaging Principle, also known as Pigeon-Hole Principle.

A theory-builder will tend to say that Theorem A is deep because it uses Theorem B which uses Theorem C etc., all of which were, individually, significantly results. A problem-solver may well not have a long chain of logical dependences of this kind. However, if we consider a more general kind of dependence, based on general principle again, then the picture changes. It will often be the case that, while there is no formal dependence between two results, there would have been no hope of proving one of them if one was unaware of the general principle introduced in the proof of the other. Chains of this kind of dependence can be quite long, so combinatorialists too can have the satisfaction of solving problems that would have been well out of reach a generation ago. In this way, one feels that the subject as a whole is progressing. – William T. Gowers, The two cultures of mathematics, in: *Mathematics: frontiers and perspectives*, 65–78, Amer. Math. Soc., Providence, RI, 2000.

If one understands one's painting in advance, one might as well not paint anything. – Salvador Dali

An artist is not one who is inspired but one who can inspire others. – Salvador Dali, Spanish Surrealist Painter, 1904-1989.

End of Lesson Two 7/9/05

Denote by $tK_2$ the graph with two vertices and with $t$ parallel edges between them. Clearly, the existence of $3K_2$ says that the answer to Exercise 6 should be NO. But the next result tells us that a more meaningful answer to Exercise 6 should be 'almost YES', as in a sense $3K_2$ represents the only type of obstructions for a positive answer.

**Theorem 28** *If $|m^S(G)|$ is odd for each two element set $S \subseteq E(G)$, then $G$ is not even if and only if $G$ is $tK_2$ for some odd number $t$.*

*Proof.* Clearly, for each odd $t$, $tK_2$ is not even and satisfies the given condition. In the remaining we check that any other graph $G$ satisfying the condition must be even.

Take $v \in V(G)$ and let $e_1, \cdots, e_k$ be all edges incident with $v$ (Note that an edge will appear twice in this list if it is a loop at $v$). Our task is to prove that $k$ is even.

First suppose that there is $f \in E(G) \setminus \{e_1, \cdots, e_k\}$. Consider the bipartite graph $X$ with a vertices bipartition $V(X) = V_1 \cup V_2$, where $V_1 = \{e_1, \cdots, e_k\}$ and $V_2 = \{C : \exists i, C \in m^{\{e_i, f\}}(G)\}$, and there is an edge between $e_i \in V_1$ and $C \in V_2$ if and only if $C \in m^{\{e_i, f\}}(G)$. Note the the given assumption means that $V_1$ consists of odd nodes of $X$. Let us prove that $V_2$ consists of even nodes, from which we can conclude that $V_1$ is exactly the set of all odd nodes of $X$ and thus $k$ is even, as wanted.

To achieve it, we need to note that any $C \in m^{\{e_i,f\}}(G)$ is either a circuit or a disjoint union of two circuits, say $C_1$ and $C_2$, such that $e_i \in C_1, f \in C_2$, and these two circuits can at most pass through one common vertex in $G$ (otherwise there is a cycle properly contained in $C$ which includes both $e_i$ and $f$.)*. For the first case, since $C \in m^{\{e_i,f\}}(G)$ if and only if $C$ passes through $e_i$ in $X$ and since $C$ must pass through an even number of elements $e_i$ of $V_1$, we know that $C$ is an even node in $X$. For the other case, we observe that $C \in m^{\{e_j,f\}}(G)$ if and only if $e_j \in C_1$. But $C_1$ also passes through an even number of elements of $V_1$ and hence we still deduce that $C$ is an even node of $X$.

*Lemma 31 presents a more general argument valid for arbitrary binary matroid.

Now consider the case that $E(G) = \{e_1, \cdots, e_k\}$. If there are more than one vertices in $G$ which are different from $v$ and adjacent to $v$. Then our preceding arguments show that all these neighbors are even nodes in $G$ and henceforth $v$ itself has to be even. If one of $e_i$ is a loop, we can consider subdivide $e_i$ into a path of length three and thus create a new graph in which $v$ has more than one neighbors and we can easily see that $v$ is an even node of the original graph $G$ by applying the former argument to the new graph. Finally, there remain the case that $k = 0$ and the case that all $e_i$ go from $v$ to $w \neq v$. Since the possibility of $G = tK_2$, $t$ being odd, is excluded, we find that $G$ is even in this case. The proof is ended. ∎

**Exercise 29** *Work out a generalization of Theorem 28 in the context of linear space (binary matroid).*

Let $W$ be a subspace of $\mathbb{F}_2^n$. We follow Woodall * to prove the following generalization of Theorem 7 (Exercise 8, Theorem 25). Recall the definition of maximal vector and minimal vector of the subspace $W$ in the whole space $V = \mathbb{F}_2^n$ and some other convention made before Theorem 25.

**Theorem 30** $\sum_{x \in \mathsf{m}(W)} x = \sum_{x \in \mathsf{M}(W)} x.$

*Proof.* When $n = 1$, it happens either $\mathsf{M}(W) = \mathsf{m}(W) = \{e_1\}$ or $\mathsf{M}(W) = \{0\}$ and $\mathsf{m}(W) = \emptyset$, and in each of these two cases the theorem is obvious. We proceed by induction and assume now $n > 1$ and the theorem holds when the whole space is of smaller dimension.

*D.R. Woodall, A proof of McKee's Eulerian-bipartite characterization, *Discrete Mathematics* **84** (1990), 217–220.

There are two possibilities, either there is $i \in [n]$ such that $e_i \in W$, and hence $e_i \in \mathsf{m}(W)$, or there is no unit vector in $W$.

In the former case, all minimal vectors $x \neq e_i$ will have $x(i) = 0$ and all maximal vectors $x$ will have $x(i) = 1$. Consequently, we get the result by applying Theorem 25 and the induction hypothesis.

We now address the latter case. We fix an $i \in [n]$ and intend to prove

$$\sum_{x \in \mathsf{M}(W)} x(i) - \sum_{x \in \mathsf{m}(W)} x(i) = 0. \tag{1}$$

Observe that $W_{ij}$ is a subspace of $V_{ij}$. Let $\mathsf{M}_{ij}(W) = \{x \in \mathsf{M}(W) : \ x(i) = x(j)\}$, $\mathsf{m}_{ij}(W) = \{x \in \mathsf{m}(W) : \ x(i) = x(j)\}$, $S^{ij}(W) = \{\mathbf{supp}(x) : \ x \in W, x(i) = x(j) = 1\}$, $\mathsf{m}^{ij}(W) = \{x \in W_{ij} : \ \mathbf{supp}(x)$ is inclusion-minimal among elements of $S^{ij}(W)\}$, and $\sigma_{ij}(W) = \{(y, z) : \ \mathbf{supp}(y) \cap \mathbf{supp}(z) = \emptyset, \ i \in \mathbf{supp}(y), j \in \mathbf{supp}(z), \ y, z \in W, \ y + z \in \mathsf{m}^{ij}(W)\}$.

It is not hard to see that a vector $x$ satisfying $x(i) = 1$ is a maximal vector of $W_{ij}$ with respect to $V_{ij}$ if and only if $x \in \mathsf{M}_{ij}(W)$; correspondingly, an vector $x$ with $x(i) = 1$ is a minimal vector of $W_{ij}$ with respect to $V_{ij}$ if and only if $x \in \mathsf{m}_{ij}(W)$ or $x = y + z$ for $(y, z) \in \sigma_{ij}(W)$. We now prepare a lemma for later use.

**Lemma 31** *Assume that $(y, z) \in \sigma_{ij}(W)$. There is no $y' \in W$ such that $\mathbf{supp}(y') \subseteq \mathbf{supp}(y+z)$, $i \in \mathbf{supp}(y')$, and $y' \notin \{y, y+z\}$.*

*Proof.* If $j \in \mathbf{supp}(y')$, then we conclude that $\mathbf{supp}(y') \subsetneq \mathbf{supp}(y+z)$ are both elements of $S^{ij}(W)$, contradicting to $y+z \in \mathsf{m}^{ij}(W)$; while if $j \notin \mathbf{supp}(y')$, we find that $\mathbf{supp}(y'+z) \subsetneq \mathbf{supp}(y+z)$ are both elements of $S^{ij}(W)$, a contradiction again. ∎

*Proof.* (of Theorem 30 continued) A corollary of Lemma 31 is that $y_1+z_1 \neq y_2+z_2$ whenever $(y_1,z_1) \neq (y_2,z_2) \in \sigma_{ij}(W)$. Therefore, we can infer from our induction hypothesis that

$$\sum_{x \in \mathsf{M}_{ij}(W)} x(i) = \sum_{x \in \mathsf{m}_{ij}(W)} x(i) + \sum_{(y,z) \in \sigma_{ij}(W)} 1. \tag{2}$$

Since $W = (W^{\perp})^{\perp *}$, we know from the current assumption $e_i \notin W$ that there is an element $w \in W^{\perp}$ such that $w(i) = 1$. Another key corollary of Lemma 31 is that for any any $(y, z) \in \sigma_{ij}(W)$ and any $k \in \mathbf{supp}(z)$, we have $(y, z) \in \sigma_{ik}(W)$. Taking into account $w \perp z$, it then follows that for any fixed $(y, z)^{\dagger}$

$$\sum_{\substack{j \in \mathbf{supp}(w) \setminus \{i\} \\ (y,z) \in \sigma_{ij}(W)}} 1 = \sum_{k \in \mathbf{supp}(w) \cap \mathbf{supp}(z)} 1 = 0. \tag{3}$$

Note that for any $x \in W$, $x(i) = 1$ if and only if $|(\mathbf{supp}(x) \cap \mathbf{supp}(w)) \setminus \{i\}|$ is odd. This means that the LHS of Eq. (1) is

$^*$This is true for any space equipped with a non-degenerate symmetric form.
$^{\dagger}$Remember that $i$ is already fixed here.

$$\sum_{x \in \mathsf{M}(W)} \sum_{j \in (\mathbf{supp}(x) \cap \mathbf{supp}(w)) \setminus \{i\}} x(i)$$

$$- \sum_{x \in \mathsf{m}(W)} \sum_{j \in (\mathbf{supp}(x) \cap \mathbf{supp}(w)) \setminus \{i\}} x(i).$$

This in turn leads to the following equalities that

LHS of Eq. (1) $= \sum_{j \in \mathbf{supp}(w) \setminus \{i\}} \sum_{x \in \mathsf{M}_{ij}(W)} x(i)$

$$- \sum_{j \in \mathbf{supp}(w) \setminus \{i\}} \sum_{x \in \mathsf{m}_{ij}(W)} x(i) \quad \text{double counting}$$

$$= \sum_{j \in \mathbf{supp}(w) \setminus \{i\}} \left( \sum_{x \in \mathsf{M}_{ij}(W)} x(i) - \sum_{x \in \mathsf{m}_{ij}(W)} x(i) \right)$$

$$= \sum_{j \in \mathbf{supp}(w) \setminus \{i\}} \sum_{(y,z) \in \sigma_{ij}(W)} 1 \quad \text{by Eq. (2)}$$

$$= \sum_{(y,z)} \sum_{\substack{j \in \mathbf{supp}(w) \setminus \{i\} \\ (y,z) \in \sigma_{ij}(W)}} 1 \quad \text{double counting}$$

$$= 0, \qquad \text{by Eq. (3)}$$

as was to be shown. ∎

**Corollary 32** *A graph is even if and only if each edge lies in an odd number of circuits.*

**Corollary 33** *A graph is bipartite if and only if each edge lies in an odd number of cocircuits*[*].

**Exercise 34** *Use Corollary 32 to give a proof of Exercise 8.*

**Exercise 35** *Use Theorem 30 to give a proof of Exercise 10.*

**Exercise 36** *Compare the proofs of Theorems 7, 25, 28 and 30.*

**Exercise 37** [†] *For any graph $G$, $E(G)$ is a disjoint union of circuits and cocircuits. (Hint: Consider the cycle space $N$ and the cocycle space $N^{\perp}$. $E(G) \in (N \cap N^{\perp})^{\perp} = N^{\perp} + N$.)*

[*]Cocircuit is a dual concept of circuit. It is called bond in [West].

[†]W-K. Chen, On vector spaces associated with a graph, *SIAM J. Appl. Math.* **20** (1971), 526–529.

Let $W$ be a linear subspace of $\mathbb{F}_2^n$. Consider the set of supports of elements from $W$. They form a poset under the inclusion relation, denoted $\mathbb{P}(W)$.

Theorem 30 (as well as its corollaries, Theorems 7 and 25) is some assertion on the poset $\mathbb{P}(W)$, namely, for every element $x$ in the ground set $[n]$, the number of maximal elements of $\mathbb{P}(W)$ containing $x$ has the same parity with the number of minimal elements of $\mathbb{P}(W)$ containing $x$.

**Question 38** *Can we say something more about the structure of $\mathbb{P}(W)$? What about its Möbius function? What about its order complex?*

Theorem 7 can be found in T.A. McKee, $S$-minimal unions of disjoint cycles and more odd eulerian characterizations, Congressus Numerantium, to appear.

McKee maintains the following interesting webpage: Graph Duality/Dualities Papers, `http://www.math.wright.edu/People/Terry_McKee/Button_32_Dual.html`

Seems that the duality between maximal vectors and minimal vectors is not fully understood.

There are various dualities and there are various connections between these dualities:

Even graph and bipartite graph (Especially, consider the planar case); Matroid and its dual matroid; Subspace and its orthogonal complement (Code and dual code); Clutter and its blocker; Homology and cohomology; Linear program and its dual linear program, Poincare duality

———————

If things are nice there is probably a good reason why they are nice; and if you do not know at least one reason for this good fortune, then you still have work to do. — Richard Askey

An $[n,k]$ binary linear code $\mathcal{C}$ is just a $k$-dimensional linear space of $\mathbb{F}_2^n$. The study of various combinatorial regularities of linear codes is a fascinating field.

The weight of a vector is the size of its support. The Hamming distance between two vectors is the weight of their difference. The covering radius * of a code $\mathcal{C}$, denoted $\rho(\mathcal{C})$, is the smallest integer $\rho$ such that every vector has distance $\rho$ or less from at least one codeword in $\mathcal{C}$. Let $T$ be a tree. For each $v \in V(T)$, denote by $x[v] \in \mathbb{F}_2^{V(T)}$ the vector whose support coincides with the open neighborhood of $v$ in $T$. Let $\mathcal{C}_T$ be the linear code generated by $x[v], v \in V(T)$.

*For a negative result on the relation between covering radius and matroid, see: T. Britz, C.G. Rutherford, Covering radii are not matroid invariants, *Discrete Mathematics* **296** (2005), 117–120.

**Exercise 39** *Let $\mathcal{C}$ be a $k$-dimensional subspace of $\mathbb{F}_2^n$. Let $H$ be an $(n-k)\times n$ matrix whose rows constitute a basis of $\mathcal{C}^{\perp}$*. Prove that the covering radius of $\mathcal{C}$ is the smallest number $s$ such that every $(n-k)\times 1$ vector is a linear combination of $s$ columns of $H$. In particular, $\rho(\mathcal{C}) \leq n-k$.*

Here is a question formulated in the language of coding theory but whose real motivation comes from the classification of real simple Lie algebra.

**Exercise 40** *Let $\ell$ be the number of leaves of the tree $T$. Prove that $\rho(\mathcal{C}(T)) \leq \lfloor \frac{\ell}{2} \rfloor$. In particular, show that $\rho(\mathcal{C}_T) = 0$ when $T$ is a path of odd length and $\rho(\mathcal{C}_T) = 1$ when $T$ is a path of even length. Can you get any other estimate of the covering radius of $\mathcal{C}_T$ for a general tree $T$?*

*In coding theory, $H$ is called a parity check matrix for the linear code $\mathcal{C}$.

**Exercise 41** *An acyclic graph $G$ is a tree if and only if $e(G) = n(G) - 1$.*

**Exercise 42** *A binary phylogenetic $n$-tree is an ordered pair $(T, \phi)$, where $T$ is a tree with only degree one and degree three nodes and $\phi$ is a bijection from $\{1, \cdots, n\}$ to the leaves of $T$. Prove that the total number of binary phylogenetic $n$-trees is $\frac{(2n-4)!}{(n-2)!2^{n-2}}$. (Hint: Use induction on $n$.)*

**Exercise 43** *Let $\mathcal{T}$ be the set of spanning forests of a graph $G$. Let $H$ be the graph with vertex set $\mathcal{T}$ in which $T_1, T_2 \in \mathcal{T}$ are joined by an edge if and only if $|E(T_1) \triangle E(T_2)| = 2$. Show that (i) $H$ has diameter at most $n(G) - c$, where $c$ is the number of components of $G$; (ii) Every edge of $H$ is contained in a Hamilton circuit of $H$.*

**Exercise 44** *(i) Suppose* $D$ *is a digraph with chromatic number* $\chi$*. Then its line digraph* $\mathbf{L}(D)$ *has chromatic number at least* $\log_2(\chi)$ *(Hint: From any proper vertex coloring of* $\mathbf{L}(D)$ *we can color each vertex* $v$ *of* $D$ *using the set of colors received by the out-going arcs at* $v$*.);*

*(ii) Let* $G_n$ *be the graph with* $V(G_n) = \{(i,j) : 0 \le i < j \le n\}$ *and* $E(G_n) = \{((i,j),(j,k)) : 0 \le i < j < k \le n\}$*. Prove that* $\chi(G_n) = \lceil \log_2(n) \rceil$*. (Hint: color* $(i,j) \in V(G_n)$ *with the minimum integer* $t$ *such that* $2^t \nmid (j-i)$*.)*

**Exercise 45** *There are $n$ girls $g_1, \cdots, g_n$ and $m$ boys $b_1, \cdots, b_m$ such that each $g_i$ knows $h_i < m$ boys and each $b_j$ knows $a_j < n$ girls. We adopt the convention that $b$ knows $g$ if and only if $g$ knows $b$. Suppose that whenever $b_j$ does not know $g_i$, we have $a_j \geq h_i$. Show that $m \leq n$.*

Do the following exercises in [West]: 1.2.42, 1.4.40, 2.1.29, 2.1.54, 2.1.58, 2.1.72, 2.3.30, 2.3.31.

The existence of analogies between the central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to their central features. – Eliakim Moore*, 1862-1932.

End of Lesson Three 14/9/05

*Eliakim Moore was an extraordinary genius, vivid, imaginative, sympathetic, foremost leader in freeing American mathematicians from dependence on foreign universities, and in building up a vigorous American School, drawing unto itself workers from all parts of the world. He is not a relative of Robert Moore, who is famous for "Moore Method" of teaching mathematics.

## II. Electrical network and potential theory on graph

We mainly follow the first part of N. Biggs, Algebraic potential theory on graphs, *Bull. London Math. Soc.* **29** (1997), 641–682, and Chap. 2 of Russell Lyons, Yuval Peres, Probability on Trees and Networks, available at `http://php.indiana.edu/~rdlyons/prbtree/prbtree.html`

This paper encompasses a motley of ideas from several areas of mathematics, including, in no particular order, random walks, the Picard group, exchange rate networks, chip-firing games, cohomology, and the conductance of an electrical network. The linking threads are the discrete Laplacian on a graph and the solution of the associated Dirichlet problem. Thirty years ago, this subject was dismissed by many as a trivial specialization of cohomology theory, but it has now been shown to have hidden depths. Plumbing these depths leads to new theoretical advances, many of which throw light on the diverse applications of the theory. – Norman Biggs, Algebraic potential theory of graphs, *Bull. London Math. Soc.* **29** (1997), 641–682.

$\cdots$ Algebra is concerned with manipulation in time, and geometry is concerned with space. These are two orthogonal aspects of the world, and they represent two different points of view in mathematics$\cdots$ If you are blind, you do not see space, if you are deaf, you do not hear, and hearing takes place in time. On the whole, we prefer to have both faculties. $\cdots$Geometrically, you think of cycles that you can add and subtract and you get what is called the homology group of a space. Homology is a fundamental algebraic tool that was invented in the first half of the century as a way of getting some information about topological space; some algebra extracted out of geometry.$\cdots$ – Sir Michael Atiyah, Mathematics in the 20th century, *Bull. London Math. Soc.* **34** (2002), 1–15.

The main thing that interests me in mathematics always is the interconnection between different parts of mathematics, the fact that one problem may have half a dozen different ways of being looked at in different subjects, a bit of algebra, a bit of geometry, a bit of topology. – Sir Michael Atiyah

In a narrow sense, representation theory studies a system in terms of its symmetries. Duality is just bilateral symmetry. In a wide sense, representation means a morphism, namely a mapping preserving some structure. Morphism (representation) is a generalized symmetry.

What we learn from our whole discussion and what has indeed become a guiding principle in modern mathematics is this lesson: Whenever you have to do with a structure-endowed entity $\Sigma$, try to determine its group of automorphisms, the group of those element-wise transformations which leave all structural relations undisturbed. You can expect to gain a deep insight into the constitution of $\Sigma$ in this way. – Hermann Klaus Hugo Weyl (9/11/1885-9/12/1955), *Symmetry*, Princeton University Press, 1952.

It has been said that when you do not quite understand the properties of new mathematical objects, you should try to put a group structure on them. This seems like a whim, but in fact it has more than once succeeded. – Jean Dieudonnė (1906 – 1992), *Mathematics – the Music of Reason*, Springer, 1992, p. 154.

To achieve good understanding of an object, we need to examine it from various viewpoints, that is to say, we look for various representations of it and hope that the study of some suitable representation will help to expose something we need which might hide too deep to be noticed when we look at the object from other directions.

The knowledge on an object is a play kept by the God. To appreciate the play, we have to invite some actors to play it on stage. Finding a good actor is often the key to understand the story.

Various useful representations for graphs have been found, including intersection representation, topological representation, polyhedral representation, geometric representation*, polynomial representation, analytic representation, symbolic dynamics representation, matrix representation, probabilistic representation (random walk on graphs), and many many others. Do not forget that it is often helpful to use graphs to represent graphs. Graph homomorphism † is now a very important and interesting research topic.

*L. Lovasz, Geometric representations of graphs, Lecture notes available at: `http://research.microsoft.com/users/lovasz/geommain.pdf`

†J. Nesetril, P. Winkler (editors), Graphs, Morphisms, and Statistical Physics, American Mathematical Society, 2004.
J. Nesetril, P. Hell, Graphs and Homomorphisms, Oxford University Press, 2004.
Graph Homomorphism papers of L*á*szl*ó* Lov*á*sz available at: `http://research.microsoft.com/users/lovasz/papers.htm`

In this chapter, electrical currents will be the main actor on the stage of graphs. Here are some interesting books in which you can appreciate much more wonderful performances than what we are able to show to you here.

Peter G. Doyle, J. Laurie Snell, Random Walks and Electric Networks, Mathematical Association of America, 1984.

András Recski, Matroid Theory and its Applications in Electric Network Theory and in Statics, *Algorithms and Combinatorics* **6** Springer-Verlag, 1989.

H. Narayanan, Submodular Functions and Electrical Networks, *Annals of Discrete Mathematics* **54** Elsevier, 1997.

Ladislav Novak, Alan Gibbons, Hybrid Graph Theory and Network Analysis, *Cambridge Tracts in Theoretical Computer Science* **49** Cambridge University Press, 1999.

Kazuo Murota, Matrices and Matroids for Systems Analysis, *Algorithms and Combinatorics* **20** Springer, 1999.

All graphs $G$ discussed in this chapter are assumed to be finite and loopless.

Each edge $e = \{a, b\}$ corresponds to two oriented edges, $ab$ and $ba$. We denote the set of oriented edges of $G$ by $\overline{E}(G)$. Note that a current $w$ flows through edge $e$ from its endpoint $a$ to the other endpoint $b$ can be equivalently described as a current $-w$ flows from $b$ to $a$. To reflect this fact, we construct the abelian group $C_1(G, \mathbb{R}) = \{\sum_i r_i e_i : \ e_i \in \overline{E}(G), r_i \in \mathbb{R}\}$ generated by all those oriented edges of $G$ with the only defining relation $ab = -ba$ for all unoriented edge $e = \{a, b\}$. $C_1(G, \mathbb{R})$ is called the first chain group of $G$, or the chain group of $G$ in dimension 1, with coefficients $\mathbb{R}$.

To describe the flow of electrical currents, we fix an orientation $A$ of $G$, namely for each edge $\{a, b\}$ we choose exactly one of $ab$ and $ba$ to be a member of $A$ and think of that the edge $\{a, b\}$ is oriented as the chosen oriented edge. This is just to choose a basis for $C_1(G)$ which makes it easy to do algebraic manipulations below. Recall that we do the same thing of fixing a basis in many situations and the final wanted geometric conclusion should be coordinate-free, that is, independent of the choice of the basis.

Since $E(G)$ is finite, each elements of $C_1(G, \mathbb{R})$ can be naturally identified with an element of $C^1(G, \mathbb{R}) = \mathbb{R}^A$, where $f \in C_1(G, \mathbb{R})$ corresponds to the mapping which sends $a \in A$ to the coefficient of $a$ in $f$ (w.r.t. the given basis $A$). This says nothing but the isomorphism between a finite dimensional linear space and its dual space (space of 1-forms). Note that $C^1(G, \mathbb{R})$ is also called the first cochain group of $G$ with coefficients $\mathbb{R}$.

For an oriented edge $e$, use $e^+$ to denote its terminal vertex and $e^-$ its initial vertex. $\nabla_D^+$ and $\nabla_D^-$ are two matrices whose columns are indexed by $V(G) = V(D)$ and rows by $A(D)$ such that

$$\nabla^+(e,x) = \begin{cases} 1 & \text{if } x = e^+, \\ 0 & \text{otherwise;} \end{cases} \quad \nabla^-(e,x) = \begin{cases} 1 & \text{if } x = e^-, \\ 0 & \text{otherwise.} \end{cases}$$

Put $\nabla_D = \nabla_D^+ - \nabla_D^-$. Define the incidence matrix of $D$ to be $\mathbb{I}_D = \nabla_D^\top$.

The structure of the oriented graph $D = (V, A)$ is completely described by its incidence matrix $\mathbb{I}_D$. Note that the information of loops can not be reflected in $\mathbb{I}_D$ and this is the reason that we restrict here to discuss graphs without loops.

The map $f \to \nabla^\top f = \partial f$ is known as the boundary mapping of the graph $G$ and the map $g \to \nabla g = \delta g$ is known as the coboundary mapping of $G$. The coboundary mapping can be viewed as a kind of difference or "discrete differential" operator on $G$. The geometric meaning of boundary mapping and coboundary mapping is very clear and is obviously independent of the choice of a basis of $C^1(G) = C_1(G)$. So, be sure that though the matrix representations $\nabla$ and $\nabla^\top$ depend on the chosen oriented graph $D$, what we are interested are those coordinate-free properties of their geometric counterparts. Looking at boundary mapping and coboundary mapping proves to be a good way to get information about the graph $G$.

It is William Rowan Hamilton (1805-1865) who introduced the Nabla symbol in 1853 in his lectures on Quaternions, to refer to the gradient of a function.

The other persona I have a lot of admiration for in a personal way was Walter Hamilton, Walter Rowan Hamilton, who was a mathematical physicist. Hamiltonian mechanics, Hamiltonians, is specific to physics, but he also invented quaternions, which is a great part of mathematics, which I'm very fond of as well. He was an original mathematician in many ways, a slightly difficult character as a person. But I like the unusual. – Michael Atiyah

It is time to review some basic physical facts about electrical networks, which will be reformulated using the Nabla operator $\nabla$ and its adjoint one $\nabla^\top = \mathbb{I}$.

We begin with two famous laws of Gustav Robert Kirchhoff*

Kirchhoff's Voltage Law (KVL) asserts that the voltage changes around a closed path in a circuit add up to zero.

Kirchhoff's Current Law (KCL) states that the sum of the currents entering any node (i.e., any junction of wires) equals the sum of the currents leaving that node.

*He was born in Königsberg, Germany, 12th March 1824, and died in Berlin, Germany, 17th October 1887. While being a student in Albertus University of Königsberg, he announced his voltage law between 1845-1846. He graduated in 1847. Two years later, following the experiments of Kohlrausch, he introduced his current law.

The relationship between currents and voltage changes is given by Ohm's Law (OL), saying that the voltage change is the sum of externally applied voltages and the produce of the currents and the resistance.

Let $n, p, r, w \in \mathbb{R}^A$ be the externally applied voltages vector, the voltage change vector, the resistance vector, and the electrical current vector, respectively. We now present formal mathematical statements of the above physical laws.

KCL: $\nabla^\top w = 0$.

OL: $n + \mathbf{Diag}(r) w = p$.

If you think of the voltage differences as a vector field in the network, KVL says that the line integral in this vector field is Path Independent, which, as you know in your calculus course, is equivalent to saying that there is a well-defined potential field $\mathbb{V}$ in the network and $p$ is just the gradient of $\mathbb{V}$*. That is, the voltage change $p(ab)$ along an edge from $a$ to $b$ is just the potential difference $\mathbb{V}(b) - \mathbb{V}(a)$. So, we arrive at

KVL: $p \in \mathbf{Im}(\nabla)$.

By now, we see that the only physical restriction on the currents in a network is

$$\begin{cases} \nabla^\top w = 0; \\ n + \mathbf{Diag}(r)w \in \mathbf{Im}(\nabla). \end{cases} \tag{4}$$

*Recall what happen with Newton-Leibnitz formula, Stokes' formula, Cauchy integral theorem !

If you believe that Eq. (4) really characterizes a physical system with some natural properties, you will believe that for any given parameters $n$ and $r$, the solution to Eq. (4) exists and is unique.

Weil [*] gave probably the first complete proof of the existence and uniqueness of currents in a resistive network subject to KVL, KCL, and OL. Much earlier, Kirchhoff [†] gave an explicit expression for these currents in terms of spanning trees of the related graph – even before he announced his current law!

[*] H. Weyl, Repartición de corriente en uno red conductora, *Rev. Mat. Hisp-Amer.* **5** (1923), 153–164.

[†] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Stromer gefuhrt wird, *Poggendorf's Ann. Phys. Chem.* **72** (1847), 497–508.

Adding externally applied voltages has the same effect with adding some external sources and sinks where currents flow into or out of the network.

Take $w' = \mathbf{Diag}(r)^{-1}n$ and $x = \nabla^\top w' \in \mathbf{Im}\nabla^\top$. Replace $w$ by $\widetilde{w}-w'$ in Eq. (4), the task of determining the currents distribution becomes solving the following equation for a fixed element $x \in \mathbf{Im}\nabla^\top$:

$$\begin{cases} \nabla^\top \widetilde{w} = x \in \mathbf{Im}(\nabla^\top); \\ \mathbf{Diag}(r)\widetilde{w} \in \mathbf{Im}(\nabla). \end{cases} \tag{5}$$

Let us carry some elementary discussion on the spaces associated with the 'gradient' $\nabla$ and its adjoint $\nabla^\top = \mathbb{I}$.

**Theorem 46** $\mathbf{Im}\nabla^\top$ *is the orthogonal complement* of* $\mathbf{Ker}\nabla$ *in* $\mathbb{R}^V$; $\mathbf{Im}\nabla$ *is the orthogonal complement of* $\mathbf{Ker}\nabla^\top$ *in* $\mathbb{R}^E$. *In particular,* $\mathbf{dimIm}\nabla^\top + \mathbf{dimKer}\nabla = n(G)$ *and* $\mathbf{dimIm}\nabla + \mathbf{dimKer}\nabla^\top = e(G)$.

*Proof.* The assertions are all standard facts in linear algebra. To have a warm-up, let us prove $\mathbf{Ker}\nabla = (\mathbf{Im}\nabla^\top)^\perp$:

$x \in \mathbf{Ker}\nabla \iff \langle \nabla x, z\rangle = 0, \forall z \iff \langle x, \nabla^\top z\rangle = 0, \forall z \iff x \in (\mathbf{Im}\nabla^\top)^\perp$. ∎

*The orthogonal complement of a subspace $Y$ of $V$ is $Y^\perp = \{x \in V : \langle x, y\rangle = 0, \forall y \in Y\}$. It is not necessarily true that $Y \cap Y^\perp = \{0\}$ when the space is not over $\mathbb{R}$.

**Exercise 47** *Construct natural isomorphisms from* $\mathbf{Ker}(\nabla^\top)$ *to* $\mathbb{R}^E/\mathbf{Im}\nabla$ *and from* $\mathbf{Ker}(\nabla)$ *to* $\mathbb{R}^V/\mathbf{Im}(\nabla^\top)$ *.

$\mathbf{Ker}\nabla^\top = Z(G)$ and $\mathbf{Im}\nabla = B(G)$ are called the cycle space and the cut space, respectively, of $G$. Note that they are again intrinsically geometric objects, namely a different choice of the orientation of $G$ only changes their coordinate representation. This leads to the interpretation:

KVL: $p \in B(G)$; KCL: $w \in Z(G)$

Remember that the Laplacian $\Delta = \nabla^2$ is the divergence of the gradient field and has the form $\sum_i \frac{\partial^2}{\partial x_i^2}$ in orthogonal coordinates. The physical significance of the divergence of a vector field is the rate at which "density" exits a given region of space. What is its combinatorial counterpart like?

*This means that the homology and the cohomology of a graph coincide.

In view of $\Delta = \nabla^2$, we define the vertex Laplacian to be $L_0 = \nabla^\top \nabla$ and the edge Laplacian to be $L_1 = \nabla\nabla^\top$. We often simply write $L$ for $L_0$.

$\mathbb{D}^+$: the out-degree matrix of $D$; $\mathbb{D}^-$: the in-degree matrix of $D$; $D = D^+ + D^-$ : the degree matrix of $G$; $\mathbb{A}_1$: the forward adjacency matrix of $D = (V, A)$, $\mathbb{A}_1(x, y) = |e \in A : \ e^- = x, e^+ = y|$; $\mathbb{A}_2 = \mathbb{A}_1^\top$: the forward adjacency matrix of $\overleftarrow{D} = (V, \overline{E} \setminus A)$; $\mathbb{A} = \mathbb{A}_1 + \mathbb{A}_2$: the adjacency matrix of $G$

Observe that $\mathbb{D}^+ = (\nabla^-)^\top\nabla^-$, $\mathbb{D}^- = (\nabla^+)^\top\nabla^+$, $\mathbb{A}_1 = (\nabla^-)^\top\nabla^+$, and $\mathbb{A}_2 = (\nabla^+)^\top\nabla^-$. So, we get another expression for $L$:

$$\mathbb{D} - \mathbb{A} = (D^+ + D^-) - (\mathbb{A}_1 + \mathbb{A}_2) = (\nabla^+ - \nabla^-)^\top(\nabla^+ - \nabla^-) = \nabla^\top\nabla.$$

On the other hand, in view of $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$, a possible discretization of the continuous Laplacian operator should be something like a two order difference. As in continuous case, we first choose some directions at each point $x \in V$. Suppose $x$ is incident with $2k$ edges in $G$, oriented to be $e_{1-}, e_{1+}. \cdots, e_{k-}, e_{k+}$, where $e_{i,+}^{-} = e_{i,-}^{+} = x$. Then locally we can regard that there are $k$ directions, the $i$th direction $C_i$ being determined by passing from $e_{i-}$ to $e_{i+}$. Now, for $f \in \mathbb{R}^V$, we have $(\partial_{C_i}^2 f)(x) = (f(e_{i+}^{+}) - f(e_{i+}^{-})) - (f(e_{i-}^{+}) - f(e_{i-}^{-})) = (f(e_{i+}^{+}) - f(x)) + (f(e_{i-}^{-} - f(x))$. Thus, formally, it is reasonable to define $(\Delta f)(x) = \sum_i (\partial_{C_i}^2 f)(x) = (\mathbb{A} - \mathbb{D})(f)(x) = -(Lf)(x)$.

Hereafter, we use $\Delta$, the same symbol with that of continuous Laplacian, for $-L$, and reserve the name (discrete) Laplacian for $L$.

Green's Formula:

$$\int_{\Omega}(\nabla f \cdot \nabla g + f\Delta g)dS = \int_{\partial\Omega} f\frac{\partial g}{\partial n}ds$$

If you think of a graph without boundary, you will agree that the following is just the Discrete Green's Formula:

$-\sum_{x\in V} f(x)(\Delta g)(x) = \sum_{x\in V} f(x)(Lg)(x) = f^\top Lg = f^\top \nabla^\top \nabla g = \sum_{e\in E}(\nabla f)(e)(\nabla g)(e)$, $\forall f, g \in \mathbb{R}^V$;

or, more concisely, $-\langle f, \triangle g\rangle = -\langle \triangle f, g\rangle = \langle \nabla f, \nabla g\rangle$, $\forall f, g \in \mathbb{R}^V$.

Especially, taking $f = g$ yields

$$f^\top Lf = \sum_{e\in A}(f(e^+) - f(e^-))^2. \tag{6}$$

Modern computers operate in a discrete fashion both in time and space, and much of classical mathematics must be "discretized" before it can be implemented on computers as, for example, in the case of numerical analysis. – L. Lovász, D.B. Shmoys, É. Tardos, Combinatorics in Computer Science, in: Handbook of Combinatorics (Eds. R.L. Graham, M. Grötschel, L. Lovász) Vol II, pp. 2003–2038, The MIT Press, 1995.

At this point, we get a new understanding of $B(G)$ and $\mathbf{Ker}\nabla$. Recall the definition of a harmonic function (generally, harmonic $p$-form). Correspondingly, any element in $\mathbf{Ker}\nabla\nabla^\top = \mathbf{Ker}\nabla^\top = B(G)$ is referred to as a harmonic 1-form on $G$; while any element in $\mathbf{Ker}\boldsymbol{\Delta} = \mathbf{Ker}(L) = \mathbf{Ker}\nabla^\top\nabla = \mathbf{Ker}\nabla$ is said to be a harmonic 0-form on $G$. Recall that an (ordinary) harmonic function defined on $R^2$ which is bounded above or bounded below must be a constant (Also recall Liouville's theorem in complex analysis!).

**Theorem 48** *Let $G$ be a graph with $c$ connected components $V_1, \cdots, V_c$. Define $\sigma : C_0(G) \to C_{-1}(G) = \mathbb{R}^c$ by putting $\sigma(g)(i) = \sum_{x \in V_i} g(x)$. Then $\mathbf{Ker}\nabla$, the set of harmonic 0-forms, consists of those $f \in \mathbb{R}^V(G)$ taking constant value in each $V_i$, while its orthogonal compliment $\mathbf{Im}(\nabla^\top)$ coincides with $\mathbf{Ker}(\sigma)$.*

*Proof.* The first reading follows directly from the definition of $\nabla$. The second reading can be checked by looking at $\nabla^\top$ and using double counting. We can also deduce it from the first reading and the fact that $\mathbf{Im}(\nabla^\top) = (\mathbf{Ker}\nabla)^\perp$. ∎

**Exercise 49** *Prove the following for a graph with $c$ components:* $\mathbf{dim}B(G) = \mathbf{rank}(\nabla) = \mathbf{rank}(L) = n(G) - c$; $\mathbf{dim}C(G) = e(G) - n(G) + c$; $\mathbf{dimKer}\nabla = c$; $\mathbf{dimIm}\nabla^\top = n(G) - c$.

**Exercise 50** *Let $G$ be a graph with $e(G) = n(G) - 1$. Prove the equivalence of the following: (i) $\nabla_G$ has full row rank; (ii) Any $(n(G) - 1)$-minor of $\nabla_G$ is invertible; (iii) $G$ is a tree. (Hint: Exercise 41)*

Suppose $G$ is a connected graph. For $s, t \in V(G)$, define $\eta^{st} \in \mathbb{R}^V$ by

$$\eta^{st}(v) = \begin{cases} 1 & \text{if } v = s, \\ -1 & \text{if } v = t, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

We know from Theorem 48 that $\eta^{st} \in \mathbf{Im}(\nabla^\top)$. Indeed, $\eta^{st}$ corresponds to 1 current flowing into the network at $s$ and 1 current flowing out of the network at $t$.

To solve Eq. (5), it suffices to solve it for $x = \eta^{st}$ as they clearly span the space $\mathbf{Im}(\nabla^\top)$ and Eq. (5) is a linear system.

Picking $x = \eta^{st}$ in Eq. (5), by the obvious physical background, we get a solution $\widetilde{w}$ to Eq. (5). The effective resistance between $s$ and $t$ in the electrical network $(G, r)$ is the potential difference between them when the current is given as $\widetilde{w}$. When $r$ always takes value 1, the effective resistances of the network $(G, r)$ is defined to be the effective resistances of the graph $G$.

Rayleigh's Monotonicity Law*: The effective resistance does not increase after a cutting operation and does not decrease after a shortening operation.

*The method of applying shorting and cutting to get lower and upper bounds for the resistance of a resistive network was introduced by Lord Rayleigh in his paper "On the Theory of Resonance". It was first applied to the study of random walk by C. St J. A. Nash-Williams in: Random walk and electric currents in networks, *Proc. Camb. Phil. Soc.* **55** (1959), 181–194.

Perhaps you should now convince yourself that the above seemingly self-evident assertions, Rayleigh's monotonicity law, and the existence and uniqueness of solution to Eq. (4) (Eq. (5)), are not so easy to prove mathematically.

We will provide several proofs of the latter fact in the following and present one proof of the former at the end of this chapter.

We remark that the effective resistance of a network contains rich information of the graph and its determination is important for many purposes. The sum of effective resistances between all pairs of vertices in a connected graph is called its Kirchhoff index; a close-form formula for this parameter can be found here

José Luis Palacios, Closed-form formulas for Kirchhoff index, *International Journal of Quantum Chemistry* **81** (2001), 135–140.

Let $T$ be a forest. For $u, v \in V(T)$, define $T(u \to v)$ to be the unique element $x$ in $C_1(T)$ with $\partial x = v - u \in C_0(T)$ when $u, v$ lie in the same component of $T^*$ and put $T(u \to v)$ to be 0 when $u$ and $v$ lie in different components of $T$. We use the shorthand $T^e_{uv}$ for $\langle e, T(u \to v)\rangle$. Note that $T^{-e}_{uv} = -T^e_{uv} = T^e_{vu}$. For any oriented edge $e$, denote by $N(s, e, t)$ the number of spanning forests $T$ of $G$ satisfying $T^e_{st} = 1$. Denote by $k(G)$ the number of spanning forests of a graph $G$.

## A double counting proof

**Theorem 51** *Let $s, t$ be two vertices of $G$ lying in the same component. Then Eq. (5) for $x = \eta^{st}$ has a unique solution $\widetilde{w}$, which is given by $\widetilde{w}(e) = \frac{N(s,e,t) - N(s,-e,t)}{k(G)}$.*

*This is well-defined as $\mathbf{Ker}\partial$, the cycle space of the tree $T$, has dimension zero.

*Proof.* Compare counting by trees and counting by 2-trees. ∎

**Exercise 52 (Konheim and Weiss, 1966)** *There are $n$ parking spots $1, 2, \cdots, n$ on a one-way street. Cars $1, 2, \cdots, n$ arrive in this order. Each car $i$ has a favorite parking spot $f(i)$. When a car arrives, it first goes to its favorite spot. If the spot is free, the car will take it, if not, it goes to the next spot. Again, if that spot is free, the car will take it and will move further otherwise. If a car had to leave even the last spot and did not find the space, then its parking attempt is unsuccessful. If, at the end of this procedure, all cars have a parking spot, we say that $f$ is a parking function on $[n]$. Prove that the number of parking functions on $[n]$ is $(n+1)^{n-1}$. (Hint: Consider $n+1$ parking spots distributed in a circle. Determine which parking processes in the circle correspond to successful parking attempts in the one-way street. Use double counting.)*

The French mathematician, Henri Poicar$\acute{e}$, in trying to isolate the distinction between first-rate and second-rate mathematics, said " There are problems that one poses, and there are problems that pose themselves." – Ian Richards, Number Theory, in: Mathematics Today – Twelve Informal Essays, Edited by L.A. Steen, Springer, 1978.

Science is facts; just as houses are made of stones, so is science made of facts; but a pile of stones is not a house and a collection of facts is not necessarily science. – Henri Poincar$\acute{e}$, French mathematician & physicist, 1854-1912.

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For simplicity, we assume that the resistance of each edge is always 1*. Note that the solution $\tilde{w}$ to Eq. (5) for $x = \eta^{st}$ is almost a bicycle. Indeed, if we add an oriented edge $f$ with $f^+ = s$ and $f^- = t$, let $D' = (V, A \cup \{f\})$ and $G'$ the underlying (unoriented) graph of $D'$, let $\overline{w} \in C_1(G')$ be the extension of $\tilde{w}$ such that $\overline{e} = \begin{cases} \tilde{w}(e) & \text{if } e \in A, \\ 1 & \text{if } e = f, \end{cases}$ we see that Eq. (5) amounts to saying $\tilde{w}$ is a cocycle of $G$ and its extension $\overline{w}$ is a cycle of $G'$.

*In general case, we just consider generalized bicycle!

How can we get such a 'bicycle'? It is easy to assure you its existence! Since $C_1(G') = Z(G') \perp B(G')$, we know the existence of the decomposition $f = \overline{w} + \widehat{w}$, where $\overline{w} \in Z(G')$ and $\widehat{w} \in B(G')$. Note that

$$\overline{w}(e) = -\widehat{w}(e), \forall e \in A(G). \tag{8}$$

This says that not only $\overline{w} \in Z(G')$ but also $\widetilde{w} = \overline{w}|_{A(G)} \in B(G)$. Thus, the existence of the decomposition $C_1(G') = Z(G') \perp B(G')$ implies the existence of the solution to Eq. (5).

We are not only satisfied with knowing the existence of an almost bicycle. Let us construct one! More precisely, let us find a way to construct the decomposition of the edge space into the orthogonal sum of cycle space and cocycle space. The construction we present will provide more insight to the mysterious construction used in the double counting proof of Theorem 51.

Let $G$ be a graph and $T$ one of its spanning forests*. There is a linear mapping $f_T$ from $C_1(G)$ to $C_1(T)$ given by $e \to T(e^- \to e^+)$. Since $\partial T(e^- \to e^+) = e^+ - e^- = \partial e$, we know that $\partial f_T = \partial$. Therefore, we have $x \in \mathbf{Ker} f_T \Leftrightarrow f_T(x) = 0 \Leftrightarrow \partial f_T(x) = 0 \Leftrightarrow \partial x = 0 \Leftrightarrow x \in Z(G)$. This gives

**Lemma 53** $Z(G) = \mathbf{Ker} f_T$ *and* $\mathbf{Im} f_T = C_1(T)$. *Especially,* $f_T|_{C_1(T)} = Identity.$

Observe that $f_T$ induces a natural isomorphism between $C_1(T)$ and $C_1(G)/Z(G) = B(G)$. In other words, $f_T$ projects $C_1(G)$ onto $C_1(T)$ along the cycle space $Z(G)$.

*A spanning forest is a maximal subgraph whose cycle space dimension is zero.

**Lemma 54** *$f_T^\top x = x$ for any $x \in B(G)$.*

*Proof.* To prove $f_T^\top x = x$, we need to prove that for any $y \in C_1(G)$, it holds $0 = \langle f_T^\top x - x, y\rangle = \langle x, f_T y - y\rangle$. Since $C_1(G)$ is the direct sum of $C_1(T)$ and $Z(G)$, it suffices to consider two cases, $y \in C_1(T)$ or $y \in Z(G)$. If $y \in C_1(T)$, $f_T y - y = 0$ and so the claim follows. If $y \in Z(G)$, Lemma 53 guarantees that $f_T y - y = -y \in Z(G)$, which must be orthogonal to $x \in B(G)$. ∎

$f_T$ is a vectorial representation of some independence structure on $E(G)$. We say that a subset $S \subseteq E(G)$ is independent (in the given representation $f_T$) if $f_T(s), s \in S$, are independent in the linear space $\mathbb{R}^{E(T)}$.

The following theorem says that the independence structure induced from $f_T$ is independent of the specified spanning forest $T$.

**Theorem 55** *For any spanning forest $T$, $S \subseteq E(G)$ is independent in the representation $f_T$ if and only if $G(S)$ is acyclic, equivalently, if and only if $\mathbb{R}^S \cap Z(G) = \{0\}$.*

*Proof.* $\sum r_e f_T(e) = 0 \Leftrightarrow \sum r_e e \in \mathbf{Ker} f_T = Z(G)$. ∎

We note that $f_T$ is not a geometric object associated with the graph $G$. It is an invariant for the pair $G$ and $T$ and is invariant under orientation changes. We consider the average of all these representations $f_T$ for the same independence structure, namely

$$P = \frac{\sum_T f_T}{k(G)}, \tag{9}$$

where $T$ runs over all spanning forests of $G$. The operator $P$ turns out to be an invariant of the graph $G$ and has some really beautiful properties.

**Lemma 56** *For any $x \in Z(G)$ we have $Px = 0$; For any $x \in B(G)$ we have $x^\top P = x^\top$.*

*Proof.* Follows from Lemmas 53 and 54. ∎

**Lemma 57** $P = P^{\top}$.

*Proof.* Fix a basis $A$ for $C_1(G)$. For any $e, f \in A$, we intend to prove $\Sigma = \sum_T (f_T(e,f) - f_T(f,e)) = 0$. We need only consider the case that $e$ and $f$ lie in the same component of $G$. There are three kinds of spanning forests $T$: those containing both $e$ and $f$, those containing none of $e$ and $f$, and those containing exactly one of $e$ and $f$. Clearly, those former two kinds of spanning forests make no contributions to $\Sigma$. The third kind of spanning forests can be paired off in such a way that the two spanning forests in a pair can be obtained from each other by swapping the edges $e$ and $f$. It is not hard to verify that the contributions of the two forests in a pair cancel each other and so we are done. ∎

**Theorem 58** *$P$ is the orthogonal projection from $C^1(G)$ to $B(G)$, namely $\mathbf{Ker}P = Z(G), \mathbf{Im}P = B(G)$.*

*Proof.* Directly from Lemmas 56 and 57. ∎

According to our earlier discussion of almost bicycle, Theorem 58 will lead to another proof of Theorem 51.

**Remark 59** *Putting $S = I - P$, we know that $S$ is an orthogonal projection from $C^1(G)$ to $Z(G)$. In a sense, the 'orthogonal complement' of the spanning forest $T$ in $E(G)$ is $E(G) \setminus E(T)$. Is there any linear mapping $g_{E\setminus T}$ from $C_1(G)$ to $C_1(G \setminus T)$ playing a role dual to $f_T$? It is a good exercise to develop the dual theory of getting $S$ by averaging over $g_{E\setminus T}$ yourselves. Going through this process, you may appreciate the power of abstraction and have more motivation for the concept of a matroid and its dual matroid.*

**Exercise 60** *Let $T$ be a tree with $V(T) = \{v_1, \cdots, v_n\}$ and $X$ be the $(n-1) \times (n-1)$ matrix obtained from $\nabla_T$ by deleting the column indexed by $v_n$. Then $X$ is nonsingular and the $(v_i, e_j)$ entry of $X^{-1}$ is $T^{e_j}_{v_n v_i}$. Especially, deduce that $\mathbf{det}(X) \in \{1, -1\}$.*

**Exercise 61** *Let $M \in \mathbb{R}^{m \times p}, N \in \mathbb{R}^{m \times (m-p)}$ be two matrices of full column rank. Suppose that $M^\top N = 0$, namely the column space of $M$ is the orthogonal complement of that of $N$ in $\mathbb{R}^m$. Prove that $I - N(N^\top N)^{-1}N^\top = M(M^\top M)^{-1}M^\top$ is the orthogonal projection to the column space of $M$. Try to find a basis for the cocycle space and a basis for the cycle space of a graph $G$, respectively, and deduce matrix solutions for the projection operator $P$. Compare with the tree solution for $P$ as displayed in Eq. (9) and see what can be asserted.*

**Exercise 62** * *Let $G$ be a connected graph on $n$ vertices and $L$ its Laplacian. Prove that there is $U \in \mathbb{R}^{n\times(n-1)}$ such that $L = UU^\top$. Prove that the Steiner circumscribed ellipsoid* † *of the simplex spanned by the $n$ columns of $U^\top$ in $\mathbb{R}^{n-1}$ is defined by the equation $x^\top(U^\top U)^{-1}x = \frac{n-1}{n}$ for an unknown $x \in \mathbb{R}^{n-1}$.* ‡ *Discuss the relationship between the eigenvectors of $L$ and this ellipsoid. What can be said when $G$ is not connected?*

*Miroslav Fiedler, Geometry of the Laplacian, *Linear Algebra and its Applications,* **403** *(2005), 409–413.*

†The Steiner circumscribed ellipsoid of a simplex in $\mathbb{R}^n$ is the unique quadric in $\mathbb{R}^n$ which contains all the vertices of the simplex and the tangent hyperplane at each of its vertices is parallel to the hyperplane containing the remaining $n$ vertices.

‡Hint: Consider the orthogonal projection in $\mathbb{R}^n$ to the column space of $U$ and compare the two matrix expressions of the projection operator described in Exercise 61. Note that the orthogonal complement of the column space of $U$ is generated by the vector of all ones.

L. Ja. Beresina, Applications of the theory of surfaces to the theory of graphs, Lecture Notes in Mathematics, 792, Springer, (1980), 20–23,

L. Ja. Beresina, The graph as cone, *Journal of Geometry* **14** (1980), 154–158.

L. Ja. Beresina, The normal curvature of a graph, *Journal of Geometry* **18** (1982), 54–56.

**Exercise 63** *For any two spanning forests $T$ and $T'$ of a graph $G$, prove that $f_{T'}f_T = f_{T'}$ * and $f_T P = f_T$.*

**Exercise 64** *Let $G$ be a graph and $T, T'$ two of its spanning forests. For a matrix $M$ whose columns are indexed by $E(G)$, let $M^T$ be the matrix consisting of the columns of $M$ indexed by $E(T)$ and $M^N$ be the matrix obtained from $M$ be deleting columns indexed by $E(T)$. Prove that $f_{T'}^N = f_{T'}^T f_T^N$ and $P^N = P^T f_T^N$. (Hint: Exercise 63)*

*Use the facts that $f_T$ and $f_{T'}$ are projections along the same subspace $Z(G)$ and that $f_T|_{C^1(T)} = Id$ to show that $f_{T'}f_T$ and $f_{T'}$ coincide when applying to $Z(G)$ and $C^1(T)$.

It is not a large overstatement to claim that mathematics has traditionally arisen from attempts to understand quite concrete events in the physical world. The accelerated sophistication of the mathematical community has perhaps obscured this fact, especially during the present century, with the abstract becoming the hallmark of much of respectable mathematics. – J.K. Percus, June 30, 1971, Preface of J.K. Percus, Combinatorial Methods, *Applied Mathematical Sciences* **4**, Springer, 1971.

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For any $c \in C^1(G)$, define $\phi \in C^0(G)$ to be a potential induced by $c$ provided

$$P_G(c) = \nabla_G \phi. \tag{10}$$

The existence of a potential is obvious as $\mathbf{Im}P = B(G) = \mathbf{Im}\nabla$.

But why we call such a function a potential? Other then first getting $P_G(c)$ and then integrating it over $G$, how to find a potential of $c$ directly?

Return to our discussion of solving Eq. (5) for $x = \eta^{st}$. A bit different from what we did at the beginning of Lesson Five, we consider the oriented graph $D^*$ obtained from $D$ by adding an oriented arc $g$ from $s$ to $t$ rather than an arc from $t$ to $s$. Now the externally applied voltage corresponding to $\eta^{st}$ is nothing but 1 electric current flowing along the arc $g$, which instead may be regarded as an element of $\mathbb{R}^{A(D^*)}$. As in last lesson, we turn to the decomposition $g = Pg + Sg^*$. Since $Pg$ is a cocycle, it satisfies KVL and so does its restriction in $A(D)$. Moreover, since $Sg$ is a cycle, at each vertex the balance situation of $Pg$ is the same with $g$, namely $\partial Pg = \partial g$. This means that the restriction of $Pg$ to $A(G)$ is just the electric current (as well as the voltage difference) vector determined by $\eta^{st}$.

*Here, both $P$ and $S$ are corresponding operators on $G^*$.

We remark that by virtue of the fact $g = -f$ and Eq. (8), the above deduction is essentially the same as our earlier discussion on Eq. (5).

After getting the voltage difference vector (=the electric current vector), KVL allows us determine the the voltage of each vertex by fixing the voltage at one vertex and integrating the potential differences†.

A voltage vector $v \in \mathbb{R}^{V(G)} = \mathbb{R}^{V(G^*)}$ is just a vector with

$$P_{G^*} g = p = \nabla_{G^*} v. \tag{11}$$

Comparing with Eq. (10), this explains the name of the potential of a 1-form over a graph. We next turn to the problem of calculating the potential directly.

†We assume that the graph is connected.

If we have a factorization $P = \nabla X^\top$, then clearly $v = X^\top g$ is what we want. Recall that $P$ is symmetric (Lemma 57). So, we should expect a factorization $P = X\nabla^\top$. Since $\mathbf{Im}P = B(G) = \mathbf{Im}\nabla$ and $\mathbf{Im}\nabla^\top$ surely have the same dimension, we are looking for the construction of an isomorphism $X$ from $\mathbf{Im}\nabla^\top$ to $\mathbf{Im}\nabla$, whose existence is again trivial, as with the case of solving Eq. (5).

In the following, we present a canonical construction of $P = X\nabla^\top$. In view of the fact that $P = \frac{\sum_T f_T}{k(G)}$, we try to get a factorization $f_T = X_T\nabla^\top$ first. The existence of $X_T$ is still trivial, as Lemma 53 guarantees that $\mathbf{Im}f_T$ has the same dimension with $\mathbf{Im}\nabla^\top$ and so are isomorphic.

Consider a graph $G$ with a fixed orientation $A \subseteq \overline{E}(G)$. For any spanning forest $T$ of $G$, we introduce a linear mapping $X_T$ from $C^0(G)$ to $C^1(T)$ by requiring $X_T(v) = \sum_{e \in A} T^e_{e^- v} e$. Observe that $X_T$ is a construction depending on the given orientation. Indeed, $X_T(v)$ is just the sum of those arcs of $A$ which point towards $v$.

However, the next lemma shows that for any two vertices $u$ and $v$ appearing in the same component of $T$, $X_T(v-u)$ is independent of the orientation and thus has a geometric meaning.

**Lemma 65** *For any two vertices $u$ and $v$ in the same component of $T$, we have $X_T(v-u) = X_T(v) - X_T(u) = T(u \to v)$. Consequently, $f_T = X_T \nabla_G^\top = X_T \mathbb{I}_G$.*

*Proof.* $X_T(v) - X_T(u)$ is the sum of those arcs in $\overline{E}(T)$ which go towards $v$ and go opposite to $u$. ∎

In view of Lemma 65, we now obtain the promised canonical factorization of $P$:

$$P = \frac{\sum_T f_T}{k(G)} = X_G \mathbb{I}_G, \qquad (12)$$

where $X_G = \frac{\sum_T X_T}{k(G)}$.

**Corollary 66** *For any $c \in C^1(G)$, the set of all potentials of $c$ is $X_G^\top c + \mathbf{Ker}\nabla$.*

By Lemma 65, we have $X_T(v-u) = T(u \to v)$ and thus $\nabla_T^\top X_T(v-u) = v - u$ whenever $u$ and $v$ are in the same component of $G$.

We now assume $T$ is a tree with $V(T) = \{v_1, \cdots, v_n\}$ and $E(T) = \{e_1, \cdots, e_{n-1}\}$. Take

$$R = \begin{pmatrix} I_{n-1} & 0 \end{pmatrix}_{(n-1)\times n} \text{ and } N = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ -1 & -1 & \cdots & -1 \end{pmatrix}_{n\times(n-1)}.$$

Then the fact $\nabla_T^\top X_T(v_i - v_n) = v_i - v_n$, $i = 1, \cdots, n-1$, can be expressed as $\nabla_T^\top X_T N = N$ and hence we get $R\nabla_T^\top X_T N = RN$, which in turn implies $(R\nabla_T^\top)(X_T N) = I_{n-1}$. You can check that this is exactly what was asserted by Exercise 60.

The Dirichlet Problem is an extremely important problem in mathematical physics. In general, it asks if there is a harmonic function $F$ on a given region $\Omega$ which coincides with a given function $f$ on the boundary $\partial\Omega$ of the region, and if such a solution exists, if it is unique.

Correspondingly, many problems in the discrete world can be formulated as a discrete Dirichlet Problem: Given a graph $G$ and $x \in C^0(G)$, find $\phi \in C^0(G)$ such that $L\phi = x$. Since $\mathbf{Im}L = \mathbf{Im}\nabla^\top\nabla = \mathbf{Im}\nabla^\top$, we can restrict to the equation

$$L\phi = \nabla^\top c, \tag{13}$$

where $c \in C^1(G)$ and $\phi$ an unknown.

**Theorem 67** *$\phi$ is a solution to Eq.* (13) *if and only if $\phi$ is a potential induced by $c$.*

*Proof.* Eq. (13)$\Leftrightarrow \nabla^\top c - \nabla^\top Sc = \nabla^\top \nabla\phi \Leftrightarrow Pc - \nabla\phi \in \mathbf{Ker}\nabla^\top \Leftrightarrow$ $Pc = \nabla\phi$. ∎

We have obtained a tree solution for computing the potential and hence a tree solution for the Dirichlet problem Eq. (13). Let us now indicate briefly a matrix solution for Eq. (13).

Suppose $G$ is connected and hence $\mathbf{Ker}L$ has dimension 1. This means that Eq. (13) has a unique solution $\phi$ with $\phi(v_n) = 0$. Let $L'$ be the matrix obtained from $L$ by deleting the row and column labelled by $v_n$ and $\phi'$, $\nu$ the vector obtained from $\phi$, $\nabla^\top c$, respectively, by deleting the row labelled by $v_n$. Then $L'$ is nonsingular and thus $\phi' = L'^{-1}\nu$ provides us the required matrix solution.

**Exercise 68** *Compare the tree solution and the matrix solution, which must be equal, and try to deduce some interesting facts.*

Can we find a matrix solution for Eq. (4)? Yes, we can derive one very quickly once we recognize the defining linear equations.

Note that Eq. (4) can be rephrased as

$$\begin{cases} Bw = 0, \\ Cz = 0, \\ z = n + \mathbf{Diag}(r)w, \end{cases} \tag{14}$$

where $B$ is a matrix the columns of whose transpose form a basis of the cocycle space of $G$, and $C$ is a matrix the columns of whose transpose form a basis of the cycle space of $G$.

A choice of good bases for $Z(G)$ and $B(G)$ will make our computation comfortable.

Fix an orientation $A$ and a spanning forest $T$ of $G$ . Suppose $e_1, \cdots, e_{n-c} \in A$ are the arcs on $T$ and $e_{n-c+1}, \cdots, e_m \in A$, are the arcs outside of $T$, also called chords.

The fundamental circuits or simply the $f$-circuits of $G$ with respect to $T$ are the $m-n+c$ circuits $C_{e_{n-c+1}}, \cdots, C_{e_m}$, where

$$C_{e_i} = e_i - f_T(e_i). \tag{15}$$

The fundamental cutsets or simply the $f$-cutsets of $G$ with respect to $T$ are the $n-c$ cutsets $B_{e_1}, \cdots, B_{e_{n-c}}$, where

$$B_{e_j} = \sum_{e \in A} T^{e_j}_{e^- e^+} e. \tag{16}$$

**Lemma 69** *For* $1 \le i \le n-c < j \le m$, *we have* $-\langle C_{e_i}, e_j \rangle = \langle e_i, B_{e_j} \rangle = T^{e_j}_{e_i^- e_i^+}$.

Define the fundamental cutset matrix and the fundamental circuit matrix with respect to $T$ to be

$$B_{f,T} = \begin{pmatrix} B_{e_1}^\top \\ \vdots \\ B_{e_{n-c}}^\top \end{pmatrix}_{(n-c)\times m} \quad \text{and } C_{f,T} = \begin{pmatrix} C_{e_{n-c+1}}^\top \\ \vdots \\ C_{e_m}^\top \end{pmatrix}_{(m-n+c)\times m},$$

respectively. Since the first $n-c$ columns correspond to Tree edges and the latter columns correspond to Non-tree edges (chords), we write $B = B_{f,T} = \begin{pmatrix} B^T & B^N \end{pmatrix}$, $C = C_{f,T} = \begin{pmatrix} C^T & C^N \end{pmatrix}$. Observe that

$$B^T = I_{n-c}, C^N = I_{m-n+c}, B^N = -(C^T)^\top = Y^*. \tag{17}$$

*This can be seen from $BC^\top = 0$. It is also immediate from Lemma 69. Note that if $\begin{pmatrix} I_{n-c} & Y \end{pmatrix}$ is the generator matrix for an $[m, n-c]$ code $\mathcal{C}$, then $\begin{pmatrix} -Y^\top & I_{m-n+c} \end{pmatrix}$ is a parity check matrix for $\mathcal{C}$.

Comparing dimensions, we now find that the set of fundamental circuits form a basis of the cycle space and the set of fundamental cutsets is a basis of the cut space of $G$. We are ready to solve Eq. (14).

$$w = \begin{pmatrix} w_T \\ w_N \end{pmatrix},\ z = \begin{pmatrix} z_T \\ z_N \end{pmatrix};$$

$$Bw = 0 \Rightarrow w_T + Yw_N = 0 \Rightarrow w = C^\top w_N;$$

$$z = \mathbf{Diag}(r)w + n = \mathbf{Diag}(r)C^\top w + n \Rightarrow 0 = Cz = (C\mathbf{Diag}(r)C^\top)w + Cn \Rightarrow w = -(C\mathbf{Diag}(r)C^\top)^{-1}Cn.$$

∎

We have seen that solving Eq. (14) is to find an almost bicycle. Observe that $B(G) \cap Z(G) = \{0\}$ in $C^1(G;\mathbb{R})$, that is, a bicycle on $G$ with real coefficients can only be 0. But how about a bicycle with coefficients in a general ring $R$? Note that such an $R$-bicycle is nothing but an element $x \in C^1(G;R)$ satisfying $\mathcal{C}_T x = 0$, where

$$\mathcal{C}_T = \begin{pmatrix} B_{f,T} \\ C_{f,T} \end{pmatrix} = \begin{pmatrix} I_{n-c} & Y \\ -Y^\top & I_{m-n+c} \end{pmatrix}. \tag{18}$$

We mention that the structure of the module of $R$-bicycles on a graph is the key to understand many questions on graphs.

**Exercise 70** *There is a nonzero $R$-bicycle on $G$ if and only if there is $\beta \in C^0(G,R)$ such that $L\beta = 0$, but $\nabla\beta \neq 0$.*

**Exercise 71** *Prove that $\mathbf{det}\mathcal{C}_T = \sqrt{\mathbf{det}(\mathcal{C}_T\mathcal{C}_T^\top)} = \mathbf{det}(B_{f,T}B_{f,T}^\top) = \mathbf{det}(C_{f,T}C_{f,T}^\top) = k(G)$, where $\mathcal{C}$ is as specified in Eq.* (18).

**Exercise 72** *Let $G$ be a graph and $F$ a field of characteristic $p$. Prove that there is a nonzero $F$-bicycle on $G$ if and only if $p \mid k(G)$.*[*]

**Exercise 73** *There exists a nonzero $Z_d$-bicycle $f$ on a graph $G$ if and only if* $\gcd(d,k(G)) \neq 1$.

[*] H. Shank, Graph property recognition machines, *Math. Systems Theory* **5** (1971), 45–49.

If I can give an abstract proof of something, I'm reasonably happy. But if I can get a concrete, computational proof and actually produce numbers I'm much happier. I'm rather an addict of doing things on the computer, because that gives you an explicit criterion of what's going on. I have a visual way of thinking, and I'm happy if I can see a picture of what I'm working with. – John Milnor

End of Lesson Six 28/9/05

# Under Construction

We will address the Jacobian of a graph in this lesson. We will also indicate a beautiful proof of the Rayleigh's Monotonicity Law in the exercises to conclude this part on electrical network.

Consider the integer matrix $\mathcal{C}_T$ defined by Eq. (18). Suppose that its invariant factors are $n_1 \mid n_2 \mid \cdots \mid n_m$. The Jacobian of $G$ is the abelian group $\mathcal{J}(G) = \bigoplus_{i=1}^{m} \mathbb{Z}_{n_i}$.

Is it really an invariant of the graph $G$? Or should we indicate the dependence on the spanning forest $T$ in the notation for this object?

To justify that $\mathcal{J}(G)$ is independent of the chosen spanning forest $T$ and further associate it with some geometric interpretations, we have a warm-up of some basic facts on (point) lattice.

Let $M \in \mathbb{R}^{m\times n}$ be of full column rank and its column vectors are $M^1, \cdots, M^n$. The lattice $L$ in $\mathbb{R}^m$ with generator matrix $B$ is a discrete subgroup of $\mathbb{R}^m$ given by $L = \{\sum_{i=1}^n a_i M^i : \ a_i \in \mathbb{Z}\}$, namely $L = \mathbf{Im} M|_{\mathbb{Z}^n}$. We say that $\mathcal{B} = \{M^1, \cdots, M^n\}$ is a basis of $L$. The fundamental domain of $L$ with respect to the basis $\mathcal{B}$ is $F_{\mathcal{B}} = \{\sum_{i=1}^n r_i M^i : \ 0 \le r_i \le 1\}$. The rank of $L$ is $\mathbf{rank} L = n$ and the determinant of $L$ is $\mathbf{det} L = \sqrt{\mathbf{det}(M^\top M)}$. Note that both $\mathbf{rank} L$ and $\mathbf{det} L$ are independent of the choice of a basis of $L$. Further observe that $\mathbf{det} L$ is just the $n$-dimensional volume of a fundamental domain of $L$.

**Exercise 74** *Show that* $\mathbb{Z} + \sqrt{2}\mathbb{Z}$ *is a free abelian group of rank two but is not a lattice in the real line.*

**Exercise 75** *Prove that* $\mathbf{det} L = Vol(F_{\mathcal{B}})$ *where* $\mathcal{B}$ *is a basis of the lattice* $L$. *

*Hint: Expand $M$ to be $\mathcal{M} = \begin{pmatrix} M & \overline{M} \end{pmatrix}_{m\times m}$ such that $\mathcal{M}^\top \mathcal{M} = \begin{pmatrix} M^\top M & \\ & I_{m-n} \end{pmatrix}$. The $m$-volume of the fundamental domain of the lattice generated by $\mathcal{M}$ is $|\mathbf{det}\mathcal{M}|$ and is the product of the $n$-volume of the fundamental domain of $L$ and the $(m-n)$-volume of that of the lattice generated by $\overline{M}$.

Let $L_0$ and $L_1$ be two lattices in $\mathbb{R}^m$. We say that $L_1$ is a sublattice of $L_0$ provided we have $L_1 \leq L_0$ and $\mathbf{rank} L_1 = \mathbf{rank} L_0$. Recall that the index of a subgroup in a group is the number of cosets of it. Since $L_1$ is surely a subgroup of $L_0$, we can speak of its index in $L_0$, denoted $[L_0 : L_1] = |\frac{L_0}{L_1}|$.

**Exercise 76** *Let $L_1$ be a sublattice of $L_0$. Show that for any generator matrix $M_i$ of $L_i$, $i = 0, 1$, there is a nonsingular integer matrix $N$ such that $M_1 = M_0 N$. Prove that $[L_0 : L_1] = \frac{\mathbf{det}(L_1)}{\mathbf{det}(L_0)} = \mathbf{det} N$. Also illustrate that each fundamental domain of $L_1$ is the disjoint union of $[L_0 : L_1]$ translations of some fundamental domain of $L_0$.*

We refer to $\mathbb{Z}^m$ as the integer lattice in $\mathbb{R}^m$. Any sublattice of $\mathbb{Z}^m$ is called an integral lattice.

Let $L$ be a lattice in $\mathbb{R}^m$ of rank $n$ and with a generator matrix $M$. Its dual lattice $L^\sharp$, which is also called its reciprocal lattice or polar lattice, is defined to be $Hom(L, \mathbb{Z}) = \{y \in \mathbb{R}^m : \ \exists z \in \mathbb{R}^n, y = Mz, \text{and } \langle x, y\rangle \in \mathbb{Z}, \forall x \in L\}$. In group representation theory, the character group (dual group) of an abelian group $L$ is just $Hom(L, \mathbb{Z}) \cong L$. But we are now discussing a point lattice in Euclidean spaces which has richer structure than a mere abelian group.

For any $y \in L^\sharp$, say $y = Mz$, $z \in \mathbb{R}^n$, we have

$Mz \in L^\sharp \Leftrightarrow \forall x \in L, \langle x, Mz\rangle \in \mathbb{Z} \Leftrightarrow \forall w \in \mathbb{Z}^n, \langle w, M^\top Mz\rangle = \langle Mw, Mz\rangle \in \mathbb{Z} \Leftrightarrow M^\top Mz \in \mathbb{Z}^n \Leftrightarrow z \in (M^\top M)^{-1}\mathbb{Z}^n$.

Therefore, we arrive at

**Theorem 77** $L^\sharp = M(M^\top M)^{-1}\mathbb{Z}^n$.

**Theorem 78** *$N$ is the generator matrix of $L^\sharp$ if and only if $N = MH$ for some $H \in \mathbb{R}^{n\times n}$ such that $M^\top N = M^\top MH$ is an integer matrix with determinant $\pm 1$.*

*Proof.* $\mathbb{Z}^n = A\mathbb{Z}^n$ if and only if $A$ is an integer matrix with determinant $\pm 1$. ∎

When $M^\top N = I$ in Theorem 78, the basis of $L^\sharp$ corresponding to $N$ is known as the dual basis of the basis of $L$ consisting of the columns of $M$. That is, Theorem 77 just singles out the dual basis of a given basis. It is interesting to compare the expression of the dual basis and the orthogonal projection operator described in Exercise 61.

**Exercise 79** *The dual basis of the dual basis of a basis is the basis itself.*

**Exercise 80** *The dual lattice of a lattice is still a lattice.*

**Exercise 81** *Prove that* $\mathbf{rank}(L) = \mathbf{rank}(L^\sharp)$ *and* $\mathbf{det} L \mathbf{det} L^\sharp = 1$*. If* $L$ *is integral, then we have* $L$ *is a sublattice of* $L^\sharp$ *with index* $\mathbf{det}(M^\top M)$*.*

Lattice appears naturally in many parts of mathematics and computer science. Have you heard of the famous $L^3$ *lattice reduction algorithm? If not, have a look into the following:

L$\acute{a}$szlo Lov$\acute{a}$sz, An Algorithmic Theory of Numbers, Graphs, and Convexity, SIAM, 1986.

Jeffrey C. Lagarias, Point lattices, Ch. 19 of *Handbook of Combinatorics* **I**, (Eds., Ronald L. Graham, Martin Grötschel, Lászlo Lovász), The MIT Press and North Holland, 1995.

Hendrik W. Lenstra, Jr., Flags and lattice basis reduction, In: *European Congress of Mathematics*, Vol. I (Barcelona, 2000), volume **201** of Progr. Math., pages 37–51, Birkh$\breve{a}$user, Basel, 2001.

*A.K. Lenstra, H.L. Lenstra, L. Lov$\acute{a}$sz, Factoring polynomials with rational coefficients, *Math. Ann.* **261** (1982), 513–534.

According to quantum theory, the real electrical flow takes integer values rather than real values. We use a subscript $I$ to denote functions taking integer values. Thus, $C_I(G)$ stands for $C^1(G,\mathbb{Z})$, the integer lattice in $C^1(G,\mathbb{R})$. Similarly, we put $Z_I(G) = Z(G) \cap C_I(G)$ and $B_I(G) = B(G) \cap C_I(G)$. We know that $C^1(G,\mathbb{R}) = Z(G) \bigoplus B(G)$. However, when entering the quantum world, it turns out that $\frac{C_I(G)}{Z_I(G) \bigoplus B_I(G)}$ can be trivial only if $G$ is a forest. Let us demonstrate that this indicating object of the quantum effect is nothing but $\mathcal{J}(G)$ and hence the well-definedness of $\mathcal{J}(G)$ follows.

**Theorem 82** $\mathcal{J}(G) = \frac{C_I(G)}{Z_I(G) \bigoplus B_I(G)}$.

*Proof.* The appearance of $I_{n-c}$ in $B_{f,T}$ means that $B_I(G)$ is generated by $B_{f,T}^\top$; the appearance of $I_{m-n+c}$ in $C_{f,T}$ says that $Z_I(G)$ is generated by $C_{f,T}^\top$. We thus know that $Z_I(G) \bigoplus B_I(G)$ has $\mathcal{C}_T^\top$ as its generator matrix. But $\mathcal{C}_T$, and hence $\mathcal{C}_T^\top$ has $N = \begin{pmatrix} n_1 & & \\ & \ddots & \\ & & n_m \end{pmatrix}$ as its Smith normal form. It follows that $\mathcal{C}_T^\top = PNQ$ for $P, Q \in \mathbb{Z}^{m\times m}$ with the property that $P^{-1}, Q^{-1} \in \mathbb{Z}^{m\times m}$ *. Finally, we have $\frac{C_I(G)}{Z_I(G) \bigoplus B_I(G)} = \frac{\mathbb{Z}^m}{\mathcal{C}_T^\top \mathbb{Z}^m} = \frac{\mathbb{Z}^m}{PNQ\mathbb{Z}^m} = \frac{P^{-1}\mathbb{Z}^m}{N(Q\mathbb{Z}^m)} = \frac{\mathbb{Z}^m}{N\mathbb{Z}^m} = \bigoplus_{i=1}^m \mathbb{Z}_{n_i} = \mathcal{J}(G)$, concluding the proof. ∎

It turns out that $\mathcal{J}(G)$ has rich geometric representations. Indeed, $\mathcal{J}(G) = \frac{C_I}{Z_I \bigoplus B_I}$ has the following expressions:

*This amounts to saying that $\mathbf{det}P, \mathbf{det}Q \in \{1, -1\}$

$$\frac{P(C_I)}{B_I} = \frac{S(C_I)}{Z_I} = \frac{Z_I^\sharp}{Z_I} = \frac{S(Z_I)}{Z_I} = \frac{B_I^\sharp}{B_I} = \frac{P(B_I)}{B_I} = \frac{\nabla^\top(C_I)}{\nabla^\top(B_I)}. \quad (19)$$

Our task below is to convince you that they are all representations of the Jacobian of $G$.

**Theorem 83** $\frac{C_I(G)}{Z_I(G) \bigoplus B_I(G)} = \frac{P(C_I(G))}{B_I(G)}$.

*Proof.* By Theorem 58, $P$ restricted to $C_I(G)$ has $Z_I(G)$ as its kernel. ∎

The next step is to determine $(B_I(G))^\sharp$.

Fix an orientation $A$ and a spanning forest $T$ of $G$. Suppose $e_1, \cdots, e_{n-c} \in A$ are the arcs on $T$ and $e_{n-c+1}, \cdots, e_m \in A$, are the arcs outside of $T$. Write $P = \begin{pmatrix} P^T & P^N \end{pmatrix}$ corresponding to the partition of $E(G)$ into tree edges and non-tree edges.

Combining Eqs. (15) and (17), we see that the $i$th column of $B_{f,T}$ is just $f_T(e_i)$, namely

$$\begin{pmatrix} B_{f,T} \\ 0 \end{pmatrix} = \begin{pmatrix} I_{n-c} & Y \\ 0 & 0 \end{pmatrix} = f_T. \tag{20}$$

By Exercise 63, $f_T P = f_T$. Henceforth, Eq. (20) gives

$$B_{f,T} P^T = I_{n-c}. \tag{21}$$

Since the columns of $B_{f,T}^{\top}$ form a basis $\mathcal{B}$ of $B_I$ * and since the columns of $P$ must lie in $B(G)$, Eq. (21) demonstrates that the columns of $P^T$ is the dual basis of $\mathcal{B}$. Theorem 78 implies at this moment that

**Theorem 84** *$P^T$ is a generator matrix of $(B_I(G))^\sharp$.*

*See the proof of Theorem 82.

There are some immediate corollaries of Theorem 84.

**Corollary 85** *$B_I(G)$ is generated by $P^T((P^T)^\top P^T)^{-1}$ for any spanning forest $T$ of $G$.*

*Proof.* Theorem 77 and Exercise 79. ∎

**Corollary 86** *$(B_I(G))^\sharp = P(C_I(G)) = P(B_I(G))$.*

*Proof.* The first equality is due to Exercise 64 and the second one is by Theorem 58. ∎

**Exercise 87** *Let $T$ and $T'$ be two spanning forests of $G$. Let $B = B_{f,T'}$ be the fundamental cut matrix of $G$ with respect to $T'$. Then there is an integer matrix $C$ with $\mathbf{det} C = \pm 1$ such that $P^T = B^\top (BB^\top)^{-1} C$.*

We remark that we can see the weaker fact that $(B_I(G))^\sharp \supseteq P(C_I(G))$ by merely using the properties of $P$ given in Lemmas 56 and 57:

$$x \in P(C_I(G)) \Rightarrow \exists c \in C_I(G), x = Pc \Rightarrow \forall b \in B_I,\ \langle x, b\rangle = \langle Pc, b\rangle = \langle c, Pb\rangle\ ^{*} = \langle c, b\rangle\ ^{\dagger} \in \mathbb{Z} \Leftrightarrow x \in (B_I(G))^\sharp.$$

*Lemma 57

†Lemmas 56 and 57

Appealing to Theorem 83 and Corollary 86, we see that $P$ induces an isomorphism from $\frac{C_I}{Z_I \bigoplus B_I}$ to $\frac{B_I^\sharp}{B_I}$. Corresponding to the factorization $P_G = X_G \mathbb{I}_G$ given in Eq. (12), we mention that $\mathbb{I}_G$ induces an isomorphism from $\frac{C_I}{Z_I \bigoplus B_I}$ to $\frac{\mathbb{I}_G(C_I)}{\mathbb{I}_G(B_I)}$ and $X_G$ induces an isomorphism from $\frac{\mathbb{I}_G(C_I)}{\mathbb{I}_G(B_I)}$ to $\frac{B_I^\sharp}{B_I}$.

**Exercise 88** *Finish the proof of Eq.* (19).

**Exercise 89** *Prove that* $S_G$ *induces an isomorphism from the integral cohomology group* $H^1(G,\mathbb{Z}) = \frac{C_I(G)}{B_I(G)}$ *to* $(Z_I(G))^\sharp$.

**Exercise 90** *Determine the Jacobian of the following graphs: Complete graphs, line graphs of complete graphs, Petersen graph, hypercubes.*

**Exercise 91** *Let* $G$ *be a graph with* $c$ *components and let* $v_1, \cdots v_c$ *be* $c$ *vertices in pairwise different components. Show that* $\{\nabla v : v \in V(G) \setminus \{v_1, \cdots v_c\}\}$ *is a basis for* $B_I(G)$. *(Hint: Lemma 65)*

We remark that Exercise 71 asserts that $|\mathcal{J}(G)| = k(G)$ . Here is the outline of a solution to Exercise 71.

In light of Eq. (20), for any $S \subseteq A$ with $|S| = n-c$, $\mathbf{det} B_{f,T}(\cdot, S) \neq 0$ if and only if $f_T(e), e \in S$, constitute a basis of $C^1(T)$, which, by Theorem 55, is equivalent to the assertion that $G(S)$ is a spanning forest of $G$. If the spanning forest $G(S_1)$ is obtained from another spanning forest $G(S_2)$ by swapping a pair of edges, we have that $|\mathbf{det} B_{f,T}(\cdot, S_1)| = |\mathbf{det} B_{f,T}(\cdot, S_2)|$. By virtue of Exercise 43, we get that $\mathbf{det} B_{f,T}(\cdot, S) = \pm \mathbf{det} B_{f,T}(\cdot, T) = \pm 1$ for any spanning forest $G(S)$. An application of the Binet-Cauchy formula yields $\mathbf{det}(B_{f,T} B_{f,T}^\top) = k(G)$ and then the remaining claims in Exercise 71 follow easily. ∎

Since $\mathcal{J}(G)$ has many different representations and a lattice can possess many bases, we will have many formulae for $k(G)$ based on our knowledge on lattice presented earlier. These formulae are usually called Matrix-Tree Theorems.

**Exercise 92** *Write a survey on various proofs of Matrix-Tree theorems and their generalizations and variations.*

Kimmo Eriksson, Classroom note: An easy bijective proof of the matrix-forest theorem, *Australasian Journal of Combinatorics* **12** (1995), 301–303.

**Theorem 93** *Let $G$ be a graph with $\mathcal{J}(G) = \mathbb{Z}_{n_1} \bigoplus \cdots \bigoplus \mathbb{Z}_{n_r}$, where $n_1 \mid n_2 \mid \cdots \mid n_r$. There exists a nonzero $Z_d$-bicycle $f$ on a $G$ with* $\gcd\{f(e) : e \in A\} = 1$ *if and only if $d \mid n_r$.*

*Proof.* ∎

Please compare Theorem 93 with Exercise 72.

An Eulerian edge cut of a graph $G$ is an even spanning subgraph of it of the form $G(S, V - S)$.

**Corollary 94** *$G$ has an Eulerian edge cut if and only if $k(G)$ is even.*

`http://www.math.umn.edu/~reiner/REU/REU.html`

Research Experiences for Undergrads

Starting in the summer of 2000, I've been involved in mentoring summer REU projects in the School of Mathematics at the Univ. of Minnesota. Many of the projects involved spanning trees of graphs, Kirchhoff's Matrix-Tree theorem and its variants, graph Laplacians, chip-firing games and critical groups of graphs.

In particular, the critical group of a graph is an isomorphism invariant in the form of a finite abelian group. Its order is the number of spanning trees in the graph. Although there are many classes of graphs for which the spanning tree number is known, often through calculation of Laplacian eigenvalues, the structure of the critical group had been computed explicitly for very few examples prior to some of these REU's. – Vic Reiner unit resistance: Euclidean geometry

general resistance: Riemannian metric, general inner product on the vector bundle TM.

Energy minimization

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*Here we can add a remark by I.M. Gel'fand: there exists yet another phenomenon which is comparable in its inconceivability with the inconceivable effectiveness of mathematics in physics noted by Wigner - this is the equally inconceivable ineffectiveness of mathematics in biology.*

*In science, if you know what you are doing, you should not be doing it. In engineering, if you do not know what you are doing, you should not be doing it.*

*In a sense my boss was saying intellectual investment is like compound interest, the more you do the more you learn how to do, so the more you can do, etc..* – Richard Wesley Hamming (1915 - 1998), *Art of Doing Science and Engineering*, T&F STM, 1997.

End of Lesson Seven 12/10/05

The incidence matrix of an oriented graph $G$ is a matrix $A = (a_{ij})$ of order $n(G) \times e(G)$ such that $a_{ij} = 1$ if arc $e_j$ is incident at node $i$ and is directed away from node $i$.

The subgraph formed by the edges incident at a node of a graph is called an incidence cut of the graph.

bicycle. Compare Exercise 37......

We introduce here some results about counting spanning trees. You should collect (figure out) more interesting stuff when working on Exercise 92.

For a connected graph $G$, its complexity is defined to be $k(G)$, the number of spanning trees of $G$.

Kirchhoff found that if we think of a graph as an electrical network where each edge has a unit resistance, then the effective resistance between $u$ and $v$ is $\frac{k(G')}{k(G)}$, where $G'$ is the graph obtained from $G$ by identifying $u$ and $v$. It is no surprise as we know that we have the so-called tree-solution for the systems of electrical networks (Recall Theorem 51). Can you prove it?

An arborescence rooted at vertex 1 is a digraph $D$ satisfying $|V(D)| = |E(D)| + 1$ and for each vertex $i$ there is exactly one path starting at $i$ and ending at 1. For a graph $G$, we write Tree($G$) for the set of all spanning trees of $G$; for a digraph $D$ and $v \in V(D)$, we write $\text{Arbo}_v(D)$ for the set of spanning arborescences of $D$ rooted at $v$.

If we view a graph $G$ as a symmetric digraph $D$, then for any vertex $v$, there is a one-to-one correspondence between Tree($G$) and $\text{Arbo}_v(D)$.

Let $K_n$ be the complete graph on $n$ vertices (without loops)*.

*A graph is just a symmetric digraph and so you should not feel strange when we also call $K_n$ the complete digraph later.

There are $n^n$ functions $f$ from $[n]$ to $n$. For any such function $f$, we can construct a digraph $G^f$ with vertex set $[n]$ and arc set $\{if(i) : i \in [n]\}$. Clearly, each weak component of $G^f$ consisting of a cycle and several arborescences rooted at some vertex of the cycle. The vertices on these cycles are just the eventual image of the function $f$ and $f$ acts on them as a permutation. Let the eventual image of $f$ be $i_1 < i_2 < \cdots < i_t$. We can now build a spanning tree $T_f$ of $K_n$ with two special vertices $u = f(i_1)$ and $v = f(i_t)$ as follows: Add an edge between $f(i_\alpha)$ and $f(i_{\alpha+1})$ for $\alpha \in [t-1]$; For any $v \in [n]\setminus\{i_1, \cdots, i_t\}$, add an edge between $v$ and $f(v)$. For any spanning tree of $K_n$ and its two special vertices $u$ and $v$, which are not necessarily distinct, we can reverse the above procedure to get first $i_1 < i_2 < \cdots < i_t$ and then recover the function $f$. This gives a bijection between Tree$(K_n) \times [n] \times [n]$ and $[n]^{[n]}$, as was found by André Joyal, Une théorie combinatoire des séries formelles, *Advances in Mathematics* **42** (1981), 1–82. So we come to

**Cayley's Formula***: $k(K_n) = n^{n-2}$.

Several other interesting proofs of Cayley's Formula are presented nicely in J.H. van Lint and R.W. Wilson, A Course in Combinatorics, China Machine Press, 2004.

**Exercise 95** *A rooted tree is a tree with one of its vertices specified as the root. A rooted forest is the disjoint union of a set of rooted trees. Show that the number of spanning rooted forests of* $K_n$ *is* $(n+1)^{n-1}$. *Recall Exercise 52. Can you find any natural bijection between the set of spanning rooted forests and the set of parking functions?*

*A. Cayley, A theorem on trees, *Quart. J. Pure and App. Math.* **23** (1889), 376–378.

The term “tree” was first used in the current graph theoretical meaning by A. Cayley (1821–1895), one of the great mathematicians of the 19th century, who made important contributions to the theory of elliptic functions, analytic geometry and algebra. Cayley’s Formula was first stated explicitly by A. Cayley and he gave a vague idea of a combinatorial proof. However, Cayley pointed out an equivalent result had been proved by Borchardt earlier.

*Christmas trees were a long way into the future and are dated by some to 1855, when Frank Prüfer, a recent German immigrant, displayed a Christmas tree which attracted curiosity.* – `http://homepages.rootsweb.com/~george/johnsgermnotes/germhis6.html`

The best known proof of Cayley's Formula makes use of the so-called Prüfer code*.

For more on Prüfer code and other coding and decoding algorithms for trees, see:

Saverio Caminiti, Irene Finocchi, Rossella Petreschi, A unified approach to coding labeled trees, *Lecture Notes in Computer Science* **2976** (Editor, Martin Farach-Colton) pp. 339 – 348.

Sally Picciotto, How to Encode a Tree, Ph.D. thesis, University of California, San Diego, 1999.

*H. Prüfer, Neuer beweis eines Satzes uber Permutationen, *Arch. Math. Phys.* **27** (1918), 742–744.

Kirchhoff' celebrated Matrix-Tree Theorem gives a determinant counting spanning trees in a graph. It has at least three different well-known proofs: one via the Binet-Cauchy Theorem [*], one via a deletion-contraction induction [†], and one due to Chaiken [‡] via a sign-reversing involution. – S. Hirschman, V. Reiner, Note on the Pfaffian Matrix-Tree Theorem, *Graphs and Combinatorics* **20** (2004), 59–63.

[*] D.B. West, Introduction to Graph Theory, Prentice Hall, 1996.

[†] C. Godsil, G. Royle, Algebraic Graph Theory, Springer, 2001.

[‡] S. Chaiken, A combinatorial proof of the all minors matrix tree theorem, *SIAM J. Algebraic Discrete Methods* **3** (1982), 319–329.

The matrix tree theorem for oriented trees was stated without proof by J.J. Sylvester in 1857, then forgotten for many years until it was independently rediscovered by W.T. Tutte [*Proc. Cambridge Phil. Soc.* **44** (1948), 463–482.]. The first published proof in the special case of undirected graphs, when the matrix $A$ is symmetric, was given by C.W. Borchardt [*Crelle* **57** (1860), 111–121.]. Several authors have ascribed the theorem to Kirchhoff, but Kirchhoff proved a quite different (though related) result. – D. Knuth, The Art of Computer Programming, Vol. 1: Fundamental algorithms, (3rd Edition) p. 583, Addison-Wesley, 1997.

Let $A=(a_{ij})_{n\times n}$ and $D=\begin{pmatrix} \sum_{i=1}^n a_{1i} & & \\ & \ddots & \\ & & \sum_{i=1}^n a_{ni} \end{pmatrix}$. Denote by $\mathcal{D}$ the first-order diagonal minor of $D-A$. Here is the matrix tree theorem appeared in our citation of Knuth in the last slide. Putting all $x_i$ to be 1, Theorem 96 (ii) becomes Cayley's Formula.

**Theorem 96** (i) *$\mathcal{D}$ is the arborescence-generating determinant of $K_n$, meaning that*

$$\sum_{T\in Arbo_1(K_n)} \prod_{ij\in E(T)} a_{ij} = \mathcal{D}; \tag{22}$$

(ii) *For each tree $T\in Tree(K_n)$, put $x^{\mathbf{deg}(T)} = x_1^{\mathbf{deg}_T(1)}\cdots x_n^{\mathbf{deg}_T(n)}$. Then,*

$$\sum_{T\in Tree(K_n)} x^{\mathbf{deg}(T)} = (x_1+\cdots+x_n)^{n-2}x_1x_2\cdots x_n. \tag{23}$$

We will follow Temperley * to establish Theorem 96 (i).

The definition of a determinant means that $\mathcal{D}$ is a signed sum of all transversals of the involving matrix. The point of the proof of Theorem 96 (i) rests on a nice way of grouping together these terms. In application, we often get the value of $\mathcal{D}$ by algebraic manipulation and thus this double-counting helps us know more about the distribution of arborescences of a given digraph. It may also happen that a matrix can be represented as the minor of $D - A$ corresponding to some weighted digraph and so the knowledge on its arborescences may help us understand a determinant.

*H.N.V. Temperley, Graph Theory and Applications, pp. 24–25, John Wiley & Sons, 1981.

*Proof.* (of Theorem 96 (i)) For any $f: [n] \setminus \{1\} \to [n]$, we can associate a digraph $G_f$ on $[n]$ by adding an arc from $i$ to $f(i)$ for each $i \in [n] \setminus \{1\}$. You can easily satisfy yourselves that $G_f$ is a vertex-disjoint union of several cycles and an arborescence rooted at 1. We denote the number of cycles of $G_f$ by $t_f$ *. By expanding the determinant, we see that $\mathcal{D}$ is a signed sum of some terms $\pi_f = \prod_{i=2}^{n} a_{if(i)}$ for functions $f$ from $[n] \setminus \{1\}$ to $[n]$. Each $\pi_f$ arises in $2^{t_f}$ ways when evaluating the determinant and their total contribution to $\mathcal{D}$ is the signed sum $(1-1)^{t_f}\pi_f$. To see it, just note that each cycle of $G_f$ either comes from $A$ and hence has negative sign or from $D$ and hence of positive sign whereas an arborescence rooted at 1 must come from $D$ and has positive sign. ∎

*Note that $t_f + 1$ is just the number of (weak) components of $G_f$.

Theorem 96 (ii) gives a good expression for the enumerator of spanning trees according to their degree sequences. A well-known proof for it is to find a recurrence relation and then solve it *. Note that a proof with even more algebraic flavor is presented by Martin & Reiner †. In their proof, they make use of the following very useful algebraic lemma on identification of factors.

**Lemma** Let $R$ be a Noetherian integral domain (e.g., a polynomial or Laurent ring in finitely many variables over a field). Let $f \in R$ be a prime element, so that the quotient ring $R/(f)$ is an integral domain, and let $K$ denote the field of fractions of $R/(f)$. Let $A \in R^{n\times n}$ be a square matrix. If the reduction $\widetilde{A} \in (R/(f))^{n\times n}$ has $K$-nullspace of dimension at least $d$, then $f^d$ divides $\mathbf{det}(A)$ in $R$.

*Claude Berge, Graphs, pp. 41–42, North-Holland, 1985.

†Jeremy L. Martin, Victor Reiner, Factorizations of some weighted spanning tree enumerations, *J. Comb. Theory, Ser. A* **104** (2003), 287–300.

We deduce Theorem 96 (ii) from Theorem 96 (i) by using a simple determinant calculation.

*Proof.* (of Theorem 96 (ii)) Let $f = \sum_{i=1}^{n} x_i, T = \begin{pmatrix} x_2 & & & \\ & x_3 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$, and $X = \begin{pmatrix} x_2 & x_3 & \cdots & x_n \end{pmatrix}$. Setting $a_{ij} = x_i x_j$ in Eq. (22), we see immediately that the LHS of Eq. (23) is just $\mathbf{det}(fT - X^\top X) = \mathbf{det}\begin{pmatrix} fT & X^\top \\ X & 1 \end{pmatrix} = (1 - X(fT)^{-1}X^\top)\mathbf{det}(fT)$

$= (1 - X(fT)^{-1}X^\top) f^{n-1} \prod_{i=2}^{n} x_i = (f - XT^{-1}X^\top) f^{n-2} \prod_{i=2}^{n} x_i$

$= x_1 f^{n-2} \prod_{i=2}^{n} x_i =$ the RHS of Eq. (23), as required. ∎

From Theorem 96 (ii), it is easy to get the following generalization of Exercise 95: The number of spanning rooted forests of $K_n$ with $k$ components is $\binom{n-1}{k-1}n^{n-k}$. Can you find a bijection between these rooted forests and those parking functions of length $n$ with $k$ 1's?

For more stories of the history of counting trees, read R.P. Stanley, Enumerative Combinatorics, II, pp. 65–69, China Machine Press, 2004.

**Exercise 97** *For each arborescence* $T \in Arbo_1(K_n)$, *put* $x^{\mathbf{deg}^-(T)} = x_1^{\mathbf{deg}_T^-(1)} \cdots x_n^{\mathbf{deg}_T^-(n)}$. *Show that* $\sum_{T \in Arob_1(K_n)} x^{\mathbf{deg}^-(T)} = (x_1 + \cdots + x_n)^{n-2}x_1$. *(Hint: Take* $a_{ij} = x_j$ *in Theorem 96 (i).)*

**Exercise 98** *Let* $C_n$ *be the* $n$*-cube, namely the graph with vertex set* $\mathbb{Z}_2^n$ *and two vertices being adjacent if and only if the Hamming distance between them is one. Show that* $k(C_n) = 2^{2^n-n-1}\prod_{i=1}^n i^{\binom{n}{i}}$.

**Exercise 99** *Let* $G$ *be a connected plane graph and let* $G^*$ *be its dual. Prove that* $k(G) = k(G^*)$.

An Eulerian tour of a digraph $G$ is a closed walk which passes through every arc of $G$ exactly once. A digraph without isolated vertex and possessing an Eulerian tour is called an Eulerian digraph. Clearly, a digraph is Eulerian if and only if it is (weak) connected and balanced* and has at least one arc.

*A digraph $G$ is balanced provided $\mathbf{deg}_G^+(v) = \mathbf{deg}_G^-(v)$ holds for each $v \in V(G)$.

**Theorem 100 (BEST Theorem)** [*] *Let $G$ be an Eulerian digraph and let $e$ be an arc of $G$ starting at $v$. Then the number of Eulerian tours of $G$ starting at $e$ is $|Arbo_v(G)| \prod_{u \in V} (\mathbf{deg}_G^+(u) - 1)!$.*

In contrast to the existence of the BEST Theorem, we still do not know whether it is possible to count the number of Eulerian cycles of a graph[†].

[*] This theorem in the full generality first appeared in T. van Aardenne-Ehrenfest, N.G. de Bruijn, Circuits and trees in oriented linear graphs, *Simon Stevin* **28** (1951), 233–237. The degree four case can also be found in C.A.B. Smith, W.T. Tutte, On unicursal paths in a network of degree 4, *Amer. Math. Monthly* **48** (1941), 233–237.

[†] Mark Jerrum, Counting, Sampling and Integrating: Algorithms and Complexity, Birkhäuser, 2003.

*Proof.* (of Theorem 100) It suffices to construct a bijection between the set of all Eulerian tours beginning with $e$ and the set $\{(T, p_u)_{u \in V(G)} : T \in Arbo_v(G), p_u \in P_{u,T}\}$, where $P_{u,T}$ stands for the set of all mappings $p$ from the set of outgoing arcs at $u$ to $[\mathbf{deg}_G^+(u)]$ satisfying $p(f) = \mathbf{deg}_G^+(u)$ for the unique arc $f \in E(T) \cup \{e\}$. Indeed, given such a $(T, p_u)_{u \in V(G)}$, beginning with $e$, each time we come to a vertex $u$ we continue the trail by following the outgoing arc which we have not walked through and has the smallest $p_u$ value and we will finally complete an Eulerian trail exactly when we will again traverse $e$. ∎

**Exercise 101** *For any $f \in [m]^{[m]^k}$, define a sequence as follows: $X_1 = X_2 = \cdots = X_k = 0; X_{n+k+1} = f(X_{n+k}, X_{n+k-1}, \cdots, X_{n+1})$ when $n \geq 0$. For how many of the $m^{m^k}$ functions $f$ is this sequence periodic with a period of the maximum length $m^k$?*

Around 1960, the Dutch physicist Pieter Willem Kasteleyn (1924-1996) developed an elegant method for counting perfect matchings in a certain class of "Pfaffian orientable" graphs, which includes all planar graphs as a strict subclass.

Tutte * considered graphs drawn on the 2-dimensional sphere with the property that the antipodal map induces an order-reversing bijections between the faces of the 2-dimensional simplicial complex. He proved that the number of spanning trees of such a graph is the square of the number of its self-dual spanning trees.

*W.T. Tutte, On the spanning trees of self-dual maps, *Annals of the NY Academy of Sciences* **319** (1979), 540–548.

Richard W. Kenyon, James G. Propp, David B. Wilson, Trees and matchings, *The Electronic Journal of Combinatorics* **7** (2000), 34 pages.

R.B. Bapat, G. Constantine, An enumerating function for spanning forests with color restrictions, *Linear Algebra and its Applications* **173** (1992), 231–237.

Gregor Masbaum, Matrix-tree theorems and the Alexander-Conway polynomial, In: Invariants of knots and 3-manifolds (Kyoto 2001), *Geometry and Topology Monographs* **4** (2002), 201–214.

A. Abdesselam, The Grassmann-Berezin calculus and theorems of the matrix-tree type, *Adv. Appl. Math.* **33** (2004), 51–70.

Consider a ring $R$ and an $R$-module $M$. Form the direct sum $M^{E(G)} = \bigoplus_{e \in E(G)} M$, which consists of all $n$-tuples $(x_1, \cdots, x_n)^\top$, $x_i \in R$. We often work with the example that $R = M$ and thus $M^{E(G)}$ becomes a free $R$-module of rank $|E(G)|$ – Recall that both division ring and commutative ring have the Invariant Dimension Property. This includes the important example of a linear space over a field. We also often assume that $R = \mathbb{Z}$ and hence $M$ is an abelian group, the discussion of which will be very important for many graph theory applications.

The kernel of the coboundary mapping is those functions from $V(G)$ to $R$ taking constant values on each component of $G$, also referred to as harmonic 0-forms. The image of the coboundary mapping is just the cut space of $G$. The image of the boundary mapping is the function from $V(G)$ to $R$ the sum of whose values over each component of $G$ is zero. The kernel of the boundary mapping is the cycle space of $G$, also called the set of harmonic 1-forms.

homology and comology

Cocycle module and cycle module, boundary and coboundary

$L$ and the bicycle matrix $C$

Francesca Rapetti, Franois Dubois, Alain Bossavit, Discrete Vector Potentials for Nonsimply Connected Three-Dimensional Domains, *SIAM Journal on Numerical Analysis* **41** 1505–1527.

Proposition 7.36 in: Martin Aigner, Combinatorial Theory, Springer, 1979.

## Squaring the square

The first paper of Tutte, written jointly with R.L. Brooks, C.A.B. Smith, and A.H. Stone, was published in 1940 when he was still an undergraduate. It made ingenious use of graphs and electrical networks to solve the problem of squaring the square. The further study of this recreational problem motivates the main part of the work of Tutte which has grown into a large body of beautiful theorems, challenging problems, and wide-ranging applications.

For any graph $G$, with any 1-chain $f \in \mathbb{F}_2^{E(G)}$ we associate a 0-chain $\partial f \in \mathbb{F}_2^{V(G)}$ called the boundary of $f$, by requiring $\partial f(v) = \sum_{v \sim e} f(e)$. Note that if $e$ is a loop at $v$, then $v$ is incident with $e$ twice and thus the contribution of $f(e)$ to $\partial f(v)$ is $2f(e) = 0$. Those 1-chains from $\mathbf{Ker}(\partial)$ are called 1-cycles.

Dirk Vertigan, Bicycle Dimension and Special Points of the Tutte Polynomial, *Journal of Combinatorial Theory B* **74** (1998), 378–396.

It is a remarkable fact that the number of cosets of $Z_I \bigoplus B_I$ in $C_I$ is equal to $k(G)$, the number of spanning trees of $G$. This fact appears in numerous disguises throughout the literature, and it would be good to have a simple proof of it by means of a bijective correspondence. But for the time being we have only an algebraic proof, a version of which is given in the following sections. − Norman Biggs

Bijection:

$\frac{Z_I^\sharp}{Z_I} = \mathcal{J}(G) \leftrightarrow \{g_T : g_T(x) = \langle f_T(x), \sum_{e \in T} e \rangle, T$ is a spanning forest$\}$

## III. Matroid, duality and Tutte polynomial

Peter J. Cameron, Notes on matroids and codes, unpublished expository article, 1998, available at `http://www.maths.qmul.ac.uk/~pjc/comb/matroid.pdf`.

Thomas Brylawski, James Oxley, The Tutte polynomial and its applications, in: Matroid Applications (Ed., Neil White), pp. 123–225, Cambridge University Press, 1992.

D.J.A. Welsh, Complexity: Knots, Colourings and Counting, Cambridge University Press, 2000.

Matroid (also called combinatorial geometry) is a structure which underlies many combinatorial problems and a good example of Non-parametric Mathematics*. Many results of graph theory extend or simplify in the theory of matroids. Matroid theory provides a unified treatment for questions in the fields of graphs, linear codes, projective geometries, combinatorial optimization and many others.

Matroids were introduced Whitney to study planarity and algebraic aspects of graphs, by MacLane to study geometric lattices, and by Van der Waerden to study independence in vector spaces.

*Think of a set of cities connected by some roads of certain lengths. We abstract it to be a graph by omitting the continuous length parameters. In the same way, matroid is a combinatorial concept arising from the elimination of continuous parameters of linear dependence of vectors.

A matroid structure can be represented in various ways, namely it has various interconnected aspects.

Which set of axioms to use to define or describe the matroid structure will depend on the background of the user and the nature of the problem in which a matroid structure arises. It is just like the case that you choose to use the polar coordinates (the radial coordinate) or the Cartesian coordinates according to your judgement which one will facilitate your computation – a point has different algebraic representations in different coordinates systems but it is the same geometric point.

A parallel example is the definition of a topological space. You can define it starting from open sets, closed sets, closure operation, or neighborhood, which are different aspects of a topological space and each of them satisfies a set of axioms that characterize the topological space structure.

If we take the concept of a vector space as one of the most basic in mathematics we can regard matroid theory as having the same relationship to linear algebra as does point set topology to the theory of real variables. – D.J.A. Welsh

A hereditary family or ideal is a collection of sets such that every subset of a set in it is also contained in it. A hereditary system $M$ on a set $E$ is a nonempty ideal $\mathbf{I_M} \subseteq \mathbf{2^E}$. The various ways of specifying that ideal is called the aspects of $M$.

The elements of $\mathbf{I_M}$ are the independent sets of $M$. The elements in $\mathbf{D_M} = \mathbf{2^E} \setminus \mathbf{I_M}$ are called dependent. A base is an independent set which is maximal under inclusion and a circuit is a dependent set minimal under inclusion; $\mathbf{B_M}$ and $\mathbf{C_M}$ denote the families of bases and circuits, respectively. The rank of a $A \in E$ is the maximum size of an independent set in it and will be denoted by $\rho_M(A)$.

Clearly, a hereditary system on a finite set is determined by any of $\mathbf{I_M}, \mathbf{D_M}, \mathbf{B_M}, \mathbf{C_M}, \rho_\mathbf{M}$, etc., as each of them specifies the others.

A matroid is a hereditary system on a finite set whose independent sets fulfil the augmentation property: If $I_1$ and $I_2$ are independent and $|I_1| < |I_2|$, then there exists $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\}$ is independent.

**Exercise 102** *Show that when specifying the matroid structure on an infinite set by requiring the hereditary system has the augmentation property, it may happen that there is no basis, and thus such an object may lose the aspect of bases.* *

*There is a theory of infinite matroid. Just as you have seen the connection and the difference between linear analysis, which is the title Bollobas used for his textbook on linear functional analysis, and (finite dimensional) linear algebra, you can imagine the relationship between the infinite matroid theory and the matroid theory discussed here.

**Exercise 103** *Let $E$ be a finite set and $\mathbf{I}$ a nonempty hereditary collection of subsets of $E$. Prove that $\mathbf{I}$ is the set of independent sets of a matroid on $E$ if and only if for every $A \subseteq E$, any two maximal members of $\mathbf{I}$ contained in $A$ have equal size.*

**Example 104 (Vectorial Matroid)** *There is a mapping $f$ from $E$ to a linear space $V$ such that $I \in \mathbf{I}_{\mathrm{M}}$ if and only if $\{f(a) : \; a \in I\}$ is a system of linearly independent vectors in $V$.*

**Example 105 (Cycle Matroid)** *$E$ can be mapped bijectively to the edge set of a graph and the independent sets correspond exactly to those acyclic subsets under this mapping.*

**Example 106 (Cocycle Matroid)** *$E$ can be mapped bijectively to the edge set of a graph and the independent sets correspond exactly to those subsets whose deletion do not increase the number of connected components.*

Both cycle matroids and cocycle matroids are referred to as graphic matroids.

**Exercise 107** *Every graphic matroid is a vectorial matroid. (Hint: Theorem 55 and Remark 59.)*

Since a hereditary system, and hence a matroid, has many aspects, it is useful to be able to recognize the matroid structure from other aspects.

**Theorem 108** *A hereditary system on a finite set is a matroid if and only if it satisfies any of the following axioms: Base exchange axiom: If $B_1, B_2 \in \mathbf{B_M}$ and $x \in B_1 \setminus B_2$, then there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathbf{B_M}$. Circuit weak elimination axiom: If $C_1, C_2$ are distinct elements of $\mathbf{C_M}$ and $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathbf{C_M}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$. Circuit strong elimination axiom: If $C_1, C_2 \in \mathbf{C_M}$ and $e \in C_1 \cap C_2, f \in C_1 \setminus C_2$ then there exists $C_3 \in \mathbf{C_M}$ such that $f \in C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$. Submodular inequality: $\rho_M(A) + \rho_M(B) \geq \rho_M(A \cup B) + \rho_M(A \cap B)$.*

We will develop some more characterizations for the matroid structure later.

**Exercise 109** *Let $R$ be an integral domain and let $N \subseteq R^S$ be a module over $R$. Prove that the supports of elements of $N$ which are minimal under inclusion satisfy the circuit weak elimination axiom and hence they are the circuits of a matroid on $S$ as long as $S$ is finite. This matroid is called the matroid of the chain group $N$ and denoted $M(N)$.*

**Exercise 110** *Assume that $R$ is a field and $N$ is a vector subspace of $R^S$. Show that $M(N)$ is a vectorial matroid with the representation $f: S \to R^S/N$ given by $f(s) = \chi(s) + N$ for any $s \in S$. Prove that $U \subseteq S$ contains a basis of $M(N)$ if and only if $S \setminus U$ is an independent set in $M(N^\perp)$.* * †

*Hint: Span$(U + N) = R^S \Leftrightarrow R^{S\setminus U} \cap N^\perp = (R^U)^\perp \cap N^\perp = (R^U + N)^\perp = \{0\}$.

†It is not necessarily true that $N + N^\perp = R^S$. But as the inner product which we have in mind is the standard one, the resulting symmetric bilinear space $V = R^S$ is nonsingular, namely $V^\perp = \{0\}$, and so many familiar facts hold; see, W. Scharlau, Quardratic and Hermitian Forms, Springer, 1985.

**Exercise 111** *Assume that $R$ is a field and $N$ is a $d$-dimensional subspace of $R^S$. Let $Q \in R^{(n-d)\times n}$ be a generator matrix of the code $N^\perp$, where $n = |S|$. Show that $M(N)$ is a vectorial matroid with the representation $f : S \to R^{n-d}$ where $f(s)$ is the $s$th column of $Q$ for any $s \in S$. Deduce from this representation and the last claim of Exercise 110 that for any generator matrix $P \in R^{d\times n}$ and any $A \subseteq S$, the submatrix of $Q$ formed by taking columns labelled by $A$ is a nonsingular matrix if and only if the submatrix of $P$ formed by deleting those columns labelled by $A$ is a nonsingular matrix.*

**Exercise 112** *Suppose that a matroid $M$ on a finite set $S$ has a vectorial representation $f$ over a field $F$. Consider the matrix whose columns are labelled by $S$ and whose $s$th column is $f(s)$. Assume that the row space of this matrix is $N' \leq F^S$. Let $N = N'^\perp$. Prove that $M = M(N)$.*

**Exercise 113** *Let $E$ be a finite set and $f: 2^E \to \{0, 1, \cdots\}$ be a function satisfying the submodular inequality and the additional property that $f(\emptyset) = 0$*. Then the function $\rho$ defined by $\rho(A) = \min_{B \subseteq E}(f(B) + |A \setminus B|)$ is the rank function of some matroid on $E$.*

**Exercise 114** *Let $G$ be a finite graph with vertex set $V$ and edge set $E$. Define a function $f$ ($g$) from $2^E$ ($2^V$) to $\mathbb{Z}^+$ by putting $f(A)$ ($g(B)$), $A \in 2^E$ ($B \in 2^V$), to be the number of vertices (edges) which are incident to at least one edge (vertex) in $A$ ($B$). Prove that $f$ is a submodular function on $E$ and $g$ is a submodular function on $V$. Try to give some descriptions of the matroid whose rank function is generated by $f$ or $g$ as described in Exercise 113.*

*We call such a function a submodular function. Note that the definition of it may be different in other literature.

Consider the game of Bingo. Each player has a card with some numbers written on it. The caller announces in turn the numbers in a sequence. The first player all of whose numbers have been called is the winner. Note that the prize cannot be shared. What conditions should the sets of numbers on cards satisfy? Let $C_i$ be the set of numbers on the $i$th card.

If $C_i \subseteq C_j$ then the player holding the $j$th card can never win*, which is unsatisfactory. We want to avoid the situation in which two players complete their cards at the same time and the prize is disputed. Suppose that $C_1$ and $C_2$ are the sets of numbers on any two cards and $e \in C_1 \cap C_2$. If the numbers in $C_1 \cup C_2$ are called with $e$ last, then both players 1 and 2 would claim the prize (contrary to what we want), unless the prize has already been claimed by, say player 3, where $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

*When all his numbers are announced, then all the numbers of $i$ have been announced earlier or at the same time

In other words, the set of $C_i$'s should be a clutter for which the circuit weak elimination axiom holds. But a set family can be the circuits of a hereditary system if and only if it is a clutter. Hence, what we find now is that these $C_i$'s should be the set of circuits of a matroid!

The above analysis leads to another characterization of the matroid structure from the aspect of circuits.

**Theorem 115** *Let $\mathcal{C}$ be a family of subsets of $E$. Then $\mathcal{C}$ is the family of circuits of a matroid if and only if it has the following property: for any total ordering of $E$, there is a set $C \in \mathcal{C}$ whose greatest element is smaller than the greatest element of any other set in $\mathcal{C}$.*

The greedy algorithm is a most short-sighted way of trying to find an optimal solution for a question. It proceeds in steps and in each of which one moves to a local optimum. Matroid is precisely the structures in which the greedy algorithm works successfully.

More formally, suppose that we are given a weight function $w$ from $E$ to $\mathbb{R}^+$. The weight of a subset $A$ of $E$ is $w(A) = \sum_{x \in A} w(x)$. In order to find a maximum weight member from a set $\mathcal{B} \subseteq 2^E$, the greedy algorithm works inductively from the empty set on as follows: Assume that $\{x_1, x_2, \ldots, x_i\}$ has been chosen, the next point $x_{i+1}$ is chosen to have maximum weight among all those elements of $E$ such that $\{x_1, x_2, \ldots, x_i, x_{i+1}\}$ is a subset of some member of $\mathcal{B}$.

**Theorem 116** *Let* $\mathcal{B} \subseteq 2^E$ *be a nonempty clutter. If for any weight function* $w$ *on* $E$ *the greedy algorithm chooses a member of* $\mathcal{B}$ *of maximum weight, the set* $\mathcal{B}$ *must be* $\mathbf{B_M}$ *for some matroid* $M$ *on* $E$*.*

*Proof.* Let $\mathbf{I} = \{I : \exists B \in \mathcal{B}, s.t., I \subseteq B\}$. We only need to check that $\mathbf{I}$ has the augmentation property. Suppose $I_1, I_2 \in \mathbf{I}$ with $|I_1| = k$ and $|I_2| = k + 1$. Let $w(e) = k + 2$ for $e \in I_1$, $w(e) = k + 1$ for $e \in I_2 \setminus I_1$, and $w(e) = 0$ for $e \notin I_1 \cup I_2$. Clearly, the greedy will first choose the all the elements of $I_1$. Now $w(I_2) \geq (k + 1)^2 > k(k + 2) = w(I_1)$, so the greedy algorithm must continue after absorbing $I_1$ and adds an element $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathbf{I}$. ∎

Suppose that the set $E$ has been weighted by $w$. Now any element in $\binom{E}{k}$ can be written as $\{e_1, \cdots, e_k\}_{\geq 0}$ to indicate that $w(e_1) \geq \cdots \geq w(e_k)$. We say that $\{e_1, \cdots, e_k\}_{\geq 0}$ dominates $\{f_1, \cdots, f_k\}_{\geq 0}$ if $w(e_i) \geq w(f_i)$ for each $i \in [k]$.

**Theorem 117** *If a nonempty family* $\mathrm{B} \subseteq \mathbf{2^E}$ *is the family of bases of a matroid on the finite set* $E$*, then for any weight function on* $E$*, the greedy algorithm produces a member of* $\mathrm{B}$ *which dominates all others.*

*Proof.* Suppose the greedy algorithm chooses successively $e_1, \ldots, e_k$. If there is another basis $\{\ell_1, \ldots, \ell_k\}_{\geq 0}$ which is not dominated by $\{e_1, \ldots, e_k\}_{\geq 0}$, we look at the smallest index $i$ such that $w(e_i) < w(\ell_i)$.

According to the augmentation property, there is an element from $\{\ell_1, \ldots, \ell_i\}$ which can be added into $\{e_1, \ldots, e_{i-1}\}$ to form an independent set. But the weight of any of them is greater than $w(e_i)$, contradicting with the rule of the greedy algorithm.∎

**Exercise 118** *All heaviest bases of a matroid are possible to be obtained from the greedy algorithm.*

**Theorem 119** *Let* $\emptyset \neq B \subseteq \binom{E}{k}$. *Then* $\mathrm{B}$ *is the set of bases of some matroid if and only if for any ordering of* $E$ *there is a member of* $\mathrm{B}$ *which dominates all others.*

*Proof.* ($\Rightarrow$): Theorem 117.

($\Leftarrow$): Suppose $\mathbf{B}$ has the mentioned ordering property, we intend to prove that it has the base exchange property. Let $B_1, B_2 \in \mathbf{B}$ and $x \in B_1 \setminus B_2$. We order the elements of $E$ in such a way that for any $z \in B_1 \setminus \{x\}$, $y \in B_2 \setminus B_1, w \notin B_1 \cup B_2$ we have $z > y > x > w$. Any set dominating $B_1$ and $B_2$ has to be of the form $(B_1 \setminus \{x\}) \cup \{y\}$ for some $y \in B_2 \setminus B_1$. ∎

The concept of Coxeter matroid has both a geometric aspect, involving polytopes, and an algebraic aspect, involving Coxeter groups and Bruhat order.

A.V. Borovik, I.M. Gelfand, N. White, Coxeter Matroids, Birkhauser, 2003.

Matroids are a branch of combinatorics which can be viewed in one way as finite geometric configurations, and in another way as structures which allow the extremely efficient algorithmic solution of optimization problems. — N. White

The viewpoint of the subject of matroids, and related areas of lattice theory, has always been, in one way or another, abstraction of algebraic dependence or, equivalently, abstraction of the incidence relations in geometric representations of algebra. – Jack Edmonds

It turns out to be useful to regard "pure matroid theory", which is only incidently related to the aspects of algebra which it abstracts, as the study of certain classes of convex polyhedra. – Jack Edmonds

Observe that the complement of any spanning forest of a plane graph corresponds to a spanning forest of the dual of the plane graph. This observation suggests the concept of dual object in the general setting of hereditary systems.

The dual of a hereditary system $M$ on a finite $E$ is the hereditary system $M^*$ on $E$ whose bases are the complements of the bases of $M$. The aspects $\mathbf{B}^*(\mathbf{B}_{M^*})$, $\mathbf{C}^*$, $\mathbf{I}^*$, $\rho^*$ of $M^*$ are the cobases, cocircuits (also called cutsets), etc. of $M$.

The span function of a hereditary system $M$ on $E$ is the function $\sigma_M$ on $2^E$ defined by $\sigma_M(X) = X \cup \{e \in E : \ Y \cup \{e\} \in \mathbf{C_M}$ for some $Y \subseteq X\}$. $X$ is a spanning set (superbase) of $M$ provided we have $\sigma_M(X) = E$. Both spanning function $\sigma_M$ and the family of spanning sets, denoted $\mathbf{S}(M)$, are aspects of a hereditary system.

**Exercise 120** *$X$ is an independent set (subbase) of $M^*$ if and only if $E \setminus X$ is a spanning set (superbase) of $M$.*

**Exercise 121** *For any hereditary system $M$ we have $(M^*)^* = M$.*

**Exercise 122** *If $M$ is a matroid, then $M^*$ is a matroid.*

An important secret of elementary linear algebra is that the concepts "spanning set" and "independent set" are dual. Thus the fact that a minimal spanning set is a basis is dual to the assertion that a maximal independent set is a basis. This is a reflection of the result that if $M$ is a matroid on a set $\Omega$, then the complements of the bases of $M$ are the bases of a second matroid on $\Omega$. – C. Godsil, G. Royle, Algebraic Graph Theory, Springer, 2001.

Restriction: $\mathbf{I_{M|F} = \{X \subseteq F : \ X \in I_M\}}$

Contraction: $\mathbf{S_{M.F} = \{X \subseteq F : \ X \cup \overline{F} \in S_M\}}$

$M|F$ is obtained by deleting $E \setminus F$ while $M.F$ is obtained by contracting $E \setminus F$. So, we also often use another set of symbols for them:

$$M|F = M \setminus (E \setminus F), M.F = M/(E \setminus F).$$

For hereditary systems, restriction and contraction are dual operations. This duality is most intuitive for plane graphs. Exercise 99 is trivial from this viewpoint.

**Theorem 123** *For a hereditary system* $M$*, we have* $(M.F)^* = M^*|F, (M|F)^* = M^*.F$*.*

*Proof.* $\mathbf{I}_{(M.F)^*} = \{X \subseteq F : \; F - X \in \mathbf{S}_{M.F}\} = \{X \subseteq F : \; (F - X) \cup \overline{F} \in \mathbf{S}_M\} = \{X \subseteq F : \; \overline{X} \in \mathbf{S}_M\} = \{X \subseteq \mathbf{I_{M^*}}\} = \mathbf{I_{M^*|_F}}$. This proves the first statement. For the second one, apply the first to $M^*$ and take dual. ∎

**Exercise 124** *Let* $M$ *be a hereditary system on* $E$*. Prove that* $X \in \mathbf{C}_{M \backslash F}$ *if and only if* $X \in \mathbf{C}_M$ *and* $X \in E \setminus F$ *while* $X$ *is independent in* $M \backslash F$ *if and only if* $X \subseteq E \backslash F$ *and* $X$ *is independent in* $M$*.*

**Exercise 125** *Let $M$ be a matroid on $E$. Show that (i) $X \subseteq E \backslash F$ is a circuit of $M/F$ if and only if $X = Y \cap (E \setminus F)$ for a circuit $Y$ of $M$ and is minimal with respect to these two properties; (ii) $Y \subseteq E \backslash F$ is independent in $M/F$ if and only if there is a maximum independent subset $X \subset F$ such that $X \cup Y$ is independent in $M$.*

In a hereditary system, a loop (coloop, also called bridge) is an element forming a circuit (cocircuit) of size 1.

**Exercise 126** *If $e$ is either a loop or a coloop, then $M/e = M \setminus e$. (Hint: For $e$ being a loop, go to Exercises 124 and 125. Then use Theorem 123 to deduce the result for the coloop case.)*

**Exercise 127** **(i) For $L \le \mathbb{R}^n$, define $L \setminus n \subseteq \mathbb{R}^{n-1}$ to be $\{x = (x_1 \cdots x_{n-1}) : (x_1 \cdots x_{n-1}0) \in L\}$, and define $L/n \subseteq \mathbb{R}^{n-1}$ to be $\{x = (x_1 \cdots x_{n-1}) : \exists r \in \mathbb{R}$, such that $(x_1 \cdots x_{n-1}r) \in L$. Prove that $L^\perp/n = (L \setminus n)^\perp$ and $L^\perp \setminus n = (L/n)^\perp$. (ii) Prove Farkas' Lemma: For any subspace $L$ of $\mathbb{R}^n$, either there is $x \in L$ satisfying $x \ge 0$ and $x_1 > 0$, or there is $y \in L^\perp$ satisfying $y \ge 0$ and $y_1 > 0$, but not both. (Hint: Use (i) and do induction on $n$.)*

Farkas' Lemma has a lot of equivalent formulations and various proofs. The proof suggested by Exercise 127 is a good illustration of the power of understanding an object inductively by the deletion-contraction algorithm.

*A. Bachem, A. Dress, W. Wenzel, Five variations on a theme by Gyula Farkas, *Adv. Appl. Math.* **13** (1992), 160–185.

Winfried Hochstättler, Jaroslav Nešetřil, Linear programming duality and morphisms, *Comment. Math. Univ. Carolin.* **40** (1999), 577–592.

Jaroslav Nešetřil, Claude Tardif, Duality theorems for finite structures (characterising gaps and good characterisations), *J. Combin. Theory B* **80** (2000), 80–97.

The nullity function of a matroid $M$ is defined to be $n_M(A) = |A| - \rho_M(A)$. As usual, we write $n_M^*$ for $n_{M^*}$.

**Theorem 128** *If $M$ is a matroid on $E$ with rank function $\rho$, then $M^*$ is a matroid on $E$ with rank function $\rho^*$ such that for any $A \subseteq E$, it holds $\rho^*(E) - \rho^*(A^*) = n(A)$ and $n^*(A^*) = \rho(E) - \rho(A)$, where $A^* = E \setminus A$.*

*Proof.* Let $I$ be a maximal independent subset of $A$ in $M$ and $I \cup J$ a basis of $M$. Note that $J \subseteq A^*$. If $K = A^* \setminus J$, the $K$ is a independent subset of $A^*$ in $M^*$ since it is contained in $(I \cup J)^*$. We thus derive $\rho^*(A^*) \geq |K| = |A^*| - |J| = |A^*| - \rho(E) + \rho(A)$. Dualizing the arguments gives $\rho(A) \geq A - \rho^*(E) + \rho^*(A^*)$. But we have $|A| + |A^*| = |E| = \rho(E) + \rho^*(E)$, which implies that the two inequalities before are equalities and so the theorem follows. ∎

Duality principle for matroids: Every proposition about a matroid induces a 'co-proposition'.

$A \leftrightarrow E \backslash A$; deletion$\leftrightarrow$contraction; spanning set$\leftrightarrow$independent set; $\rho(E) - \rho(A) \leftrightarrow |A| - \rho(A) = n(A)$; circuit$\leftrightarrow$cut set; loop$\leftrightarrow$coloop (bridge)

Duality between a clutter $\mathcal{C}$ and its blocker $b(\mathcal{C})$: $b^2(\mathcal{C}) = \mathcal{C}$.

Let $\mathcal{R}, \mathcal{S} \in 2^E$. $(\mathcal{R}, \mathcal{S})$ is said to be a blocking system for $E$ provided it holds for all $f \in \mathbb{R}^E$ that

$$\underset{R \in \mathcal{R}}{Min} \ \underset{x \in R}{Max} f(x) = \underset{S \in \mathcal{S}}{Max} \ \underset{x \in S}{Min} f(x). \tag{24}$$

http://www.math.sjtu.edu.cn/teacher/wuyk/blocking.pdf

Cocircuits and bases: Cocircuits of a matroid are the minimal sets intersecting every base. Bases are the minimal sets intersecting every cocircuit. Theorem 115 and Theorem 119.

dual matroid, orthogonal complement of a subspace, dual code: Exercises: 109, 110, 111, 112

Example 105 and Example 106

The information of an object can be read from its mirror image!

We begin with an example on how to read the generating function of the cuts of a graph from the generating function of the cycles. Here we are consider objects over binary fields and hence, a cycle is just the set of edges across a bipartition of $V(G)$ and a cycle is just an even subgraph of $G$.

Let $G$ be a simple connected graph and so each edge with endpoints $i$ and $j$ can be marked by a variable $w_{ij}$. A subset $S$ of $E(G)$ thus corresponds to $w(S) = \sum_{ij \in S} w_{ij}$. The generating function of all cuts of $G$ is $\mathcal{C}(G, x) = \sum_S x^{w(S)}$, where $S$ runs over all cuts of $G$; similarly, the generating function of all cycles of $G$ is $\mathcal{E}(G, x) = \sum_S x^{w(S)}$, where $S$ runs over all cycles of $G$.

Recall the definition of hyperbolic functions:

$$\text{cosh}(x,y) = \frac{x^y + x^{-y}}{2}, \text{sinh}(x,y) = \frac{x^y - x^{-y}}{2}, \text{tanh}(x,y) = \frac{\text{sinh}(x,y)}{\text{cosh}(x,y)}.$$

**Theorem 129 (Van der Waerden 1941)**

$$\mathcal{C}(G,x) = (2^{n-1}x^{\frac{w(E)}{2}} \prod_{ij\in E} \text{cosh}(x, -\frac{1}{2}w_{ij}))\mathcal{E}(G, \text{tanh}(x, -\frac{w}{2})),$$

*where* $\mathcal{E}(G, \text{tanh}(x, -\frac{w}{2}))$ *is obtained from* $\mathcal{E}(G,x)$ *by replacing each* $x^{w(U)} = \prod_{ij\in U} x^{w_{ij}}$, *$U$ Eulerian, with* $\prod_{ij\in U} \text{tanh}(x, -\frac{w_{ij}}{2})$.

*Proof.* Let $W = w(E)$. Since $G$ is connected, a cut $(A,B)$ corresponds to two functions $\sigma_1, \sigma_2 \in \{1,-1\}^E$ such that $\sigma_1^{-1}(1) = \sigma_2^{-1}(-1) = A$ and $\sigma_2^{-1}(1) = \sigma_1^{-1}(-1) = B$. This allows us write

$$\mathcal{C}(G,x) = \frac{1}{2}x^{\frac{W}{2}} \sum_{\sigma} \prod_{ij\in E} x^{-\frac{1}{2}w_{ij}\sigma_i\sigma_j}.$$

We now substitute hyperbolic functions for the exponential terms: $x^y = \cosh(x,y) + \sinh(x,y) = \cosh(x,y)(1+\tanh(x,y))$ and obtain $\mathcal{C}(G,x) = \frac{1}{2}x^{\frac{W}{2}} \sum_{\sigma}\prod_{ij\in E} \cosh(x, -\frac{1}{2}w_{ij}\sigma_i\sigma_j)(1+\tanh(x,-\frac{1}{2}w_{ij}\sigma_i\sigma_j))$.

We make use of $\cosh(x,y) = \cosh(x,-y)$ and $-\tanh(x,y) = \tanh(x,-y)$ and expand the product of those terms of $1+ \tanh(x, -\frac{1}{2}w_{ij}\sigma_i\sigma_j)$, yielding $\mathcal{C}(G,x) = \frac{1}{2}x^{\frac{W}{2}} \sum_{\sigma}\prod_{ij\in E} \cosh(x,-\frac{1}{2}w_{ij})(1 + \sigma_i\sigma_j \tanh(x,-\frac{1}{2}w_{ij})) = \frac{1}{2}x^{\frac{W}{2}} \prod_{ij\in E} \cosh(x,-\frac{1}{2}w_{ij}) \sum_{\sigma}\prod_{ij\in E}(1+\sigma_i\sigma_j \tanh(x,-\frac{1}{2}w_{ij}))$

$$= \frac{1}{2} x^{\frac{W}{2}} \prod_{ij \in E} \text{cosh}(x, -\frac{1}{2} w_{ij}) \sum_{\sigma} \sum_{U \subseteq E} \prod_{ij \in U} \sigma_i \sigma_j \, \text{tanh}(x, -\frac{1}{2} w_{ij})$$

$$= \frac{1}{2} x^{\frac{W}{2}} \prod_{ij \in E} \text{cosh}(x, -\frac{1}{2} w_{ij}) \sum_{U \subseteq E} \sum_{\sigma} \prod_{ij \in U} \sigma_i \sigma_j \, \text{tanh}(x, -\frac{1}{2} w_{ij})$$

$$= \frac{1}{2} x^{\frac{W}{2}} \prod_{ij \in E} \text{cosh}(x, -\frac{1}{2} w_{ij}) \sum_{U \subseteq E} (\prod_{ij \in U} \text{tanh}(x, -\frac{1}{2} w_{ij})) \sum_{\sigma} \prod_{ij \in U} \sigma_i \sigma_j$$

$$= \frac{1}{2} x^{\frac{W}{2}} \prod_{ij \in E} \text{cosh}(x, -\frac{1}{2} w_{ij}) \sum_{U \subseteq E} (\prod_{ij \in U} \text{tanh}(x, -\frac{1}{2} w_{ij})) \sum_{\sigma} \prod_{i \in V(G)} \sigma_i^{d_U(i)},$$

where $d_U(i)$ means the number of edges in $U$ incident with the vertex $i$.

Note that for any fixed $U \subseteq E(G)$ for which there is a vertex $i$ with $d_U(i)$ odd, $\sum_\sigma \prod_{i \in V(G)} \sigma_i^{d_U(i)}$ must be zero, as the terms arising from the assignment $\sigma$ where $\sigma_i = 1$ cancel out the corresponding terms with $\sigma_i = -1$. This says that we only need to consider the contribution of those Eulerian $U$, namely cycles. But, if $U$ is Eulerian, we have $\sum_\sigma \prod_{i \in V(G)} \sigma_i^{d_U(i)} = 2^n$, where $n = |V(G)|$. It then follows

$$\mathcal{C}(G,x) = \frac{1}{2} x^{\frac{W}{2}} \prod_{ij \in E} \cosh(x, -\frac{1}{2} w_{ij}) \sum_{U \text{ Eulerian}} 2^n \prod_{ij \in U} \tanh(x, -\frac{1}{2} w_{ij})$$

$$= (2^{n-1} x^{\frac{W}{2}} \prod_{ij \in E} \cosh(x, -\frac{1}{2} w_{ij})) \sum_{U \text{ Eulerian}} \prod_{ij \in U} \tanh(x, -\frac{1}{2} w_{ij})$$

$$= (2^{n-1} x^{\frac{W}{2}} \prod_{ij \in E} \cosh(x, -\frac{1}{2} w_{ij})) \mathcal{E}(G, \tanh(x, -\frac{w}{2})),$$

as desired. ∎

Mathematics is an incredibly exciting and creative field of endeavor. Yet most people never see it that way. Nonmathematicians too often assume that we mathematicians sit around what Newton did three hundred years ago or calculating a couple of extra million digits of $\pi$. They do not realize that more new mathematics is being created now than at any other time in the history of humankind. – Colin C. Adams, The Knot Book, AMS, 2004.

For any natural number $x$, we write $p_G(x)$ for the number of vertex colourings of $G$ with colors $\{1, \ldots, x\}$. Note that the chromatic number of $G$ is just the least $x$ such that $p_G(x) \geq 1$.

The function $p_G(x)$ can be represented as a polynomial. Indeed, $p_G(x) = \sum_{i=1}^{n} \pi_r(G)(x)_r$, where $(x)_r = x(x-1)\cdots(x-r+1)$ is the falling factorial and $\pi_r(G)$ is the number of partitions of $V(G)$ into $r$ noon-empty independent sets. Note that there can be no two polynomials taking the same values on all nonnegative integers. In light of it, $p_G(x) = \sum_{i=1}^{n} \pi_r(G)(x)_r$ is called the chromatic polynomial of the graph $G$ in the indeterminant $x$. The chromatic polynomial was introduced by Birkhoff in his course of attacking the Four Color Conjecture.

Inclusion-exclusion: $1-\sum \chi_{A_i}+\sum_{i,j}\chi_{A_i\cap A_j}-\cdots=\prod_i(1-\chi_{A_i})=1-\chi_{\cup_i A_i}$.

The Inclusion-exclusion principle also follows from the Möbius inversion applied to the Boolean algebra and thus it reflects some kind of duality relation!

Möbius inversion for Boolean algebra: Set $M=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, O=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $S=OM$. Then for $U=\bigotimes^n S$, it holds $U^2=\bigotimes^n(S^2)=\bigotimes^n(I_2)=I_{2^n}$. Since $B_n$, the Boolean algebra of rank $n$, is just $\prod_{i=1}^n B_1$, $\bigotimes^n M$ is nothing but the zeta function of $B_n$.

The secret of Möbius inversion for general posets: $(I+N)^{-1}=\sum_{i=0}^{\infty}(-1)^i N^i=\sum_{i=0}^{k-1}(-1)^i N^i$ provided $N^k=0$.

Another way to see that $p_G$ has a polynomial representation is to count colorings with the inclusion-exclusion principle. This is the original method by which Whitney * deduces the chromatic polynomial. It will be easy to see from this deduction method that the Tutte polynomial is a generalization of the chromatic polynomial.

You can find here an expression of the chromatic polynomial in terms of Möbius function (following G.C. Rota):

J.H. van Lint and R.M. Wilson, A Course in Combinatorics, China Machine Press, 2004, pp. 340–341.

*Hassler Whitney (1907–1989) is a pioneer in topology and a Wolf Prize winner. He had a degree in music from Yale and got his PhD from Harvard with the thesis The Coloring of Graphs. His thesis advisor at Harvard is George Birkhoff.

Suppose we have a fixed number $z$ of colors at our disposal. Any way of assigning one of these colors to each vertex of the graph in such a way that any two vertices which are joined by an arc are of different colors, will be called admissible coloring, using $z$ or fewer colors. We wish to find the number $M(z)$ of admissible colorings, using $z$ or fewer colors. ... – H. Whitney, A logical expansion in mathematics, *Bull. Amer. Math. Soc.* **38** (1932), 572–579.

For any edge $e$ of $G^*$, let $G'$ be obtained from $G$ by deleting (cutting) $e$, and $G''$ be obtained by contracting (fusing) $e$. Here is the additive character of the chromatic polynomial:

$$p_{G'}(x) = p_G(x) + p_{G''}(x). \tag{25}$$

As a corollary, we know that $\chi(G') = \min(\chi(G), \chi(G''))$. As another corollary, we can again verify that $p_G$ has a polynomial representation. Equations having a form similar to Eq. (25) is important in graph theory and, more general, matroid theory. In other words, the deletion or contraction operation will play quite a role in many places.

*It may be a loop or a parallel edge!

Let $G$ be a loopless graph with $V(G) = [n]$.

Define $\mathcal{A}_G$ to be the set of hyperplanes in $\mathbb{R}^n$, $x_i = x_j$, $ij \in E(G)$ and put $r(\mathcal{A}_G)$ to be the number of connected components * of the complement of the union of these hyperplanes in $\mathcal{A}_G$.

Denote by $AO(G)$ the set of acyclic orientations of $G$.

By checking that $|AO(G)|$, $r(\mathcal{A}_G)$ and $(-1)^n p_G(-1)$ all satisfy the same deletion/contraction relation and has the same boundary values, one can verify that $|AO(G)| = r(\mathcal{A}_G) = (-1)^n p_G(-1)$.

You can also establish a bijection between the regions of $\mathcal{A}_G$ and $AO(G)$ to get $|AO(G)| = r(\mathcal{A}_G)$.

*They are called the regions of the hyperplane arrangement $\mathcal{A}_G$.

**Theorem 130** *Let $G$ be a graph with $n$ vertices and $m$ edges $e_1, \cdots, e_m$ in that order. Call a subset of $E(G)$ a broken circuit if it is obtained from the edge set of a circuit of $G$ by deleting the edge of highest index. Then $p_G(x) = \sum_{i=0}^{n-c}(-1)^i a_i x^{n-i}$,* where $a_i$ is the number of $i$-subsets of $E(G)$ containing no broken cycle, and $c$ is the number of components of $G$.*

*Proof.* Do induction on $|E(G)|$ by Equation (25). ∎

**Exercise 131** *Let $G = (V, E)$ be a graph with $V = V_1 \cup V_2$ such that $G\langle V_1 \cap V_2 \rangle$ is a complete graph without loops and there is no edge connecting $V_1 \setminus V_2$ to $V_2 \setminus V_1$. Prove that $p_G(x) = \frac{p_{G\langle V_1 \rangle}(x) p_{G\langle V_2 \rangle}(x)}{p_{G\langle V_1 \cap V_2 \rangle}(x)}$.*

*Every edge subset of size bigger than $n - c$ must contain a broken circuit.

Let $M$ be a matroid on $E$. If there is $T \subseteq E$ with $0 < |T| < |E|$ such that $\rho(A) = \rho(A \cap T) + \rho(A \cap T^*)$ *, then we say that $M$ is the direct sum of $M|_T$ and $M|_{T^*} = M \setminus T$ and record it as $M = M|_T \bigoplus M \setminus T$. A matroid is connected if it is not a direct sum. If $e$ is a loop or a coloop of $M$, we clearly have

$$M = M|_{\{e\}} \bigoplus M \setminus e = M|_{\{e\}} \bigoplus M/e; \tag{26}$$

Compare with Exercise 126.

*Recall that $T^*$ denotes $E \setminus T$.

A matroid invariant is a function from the set of matroids to a commutative ring which takes the same value on isomorphic matroids. A matroid invariant $f$ is a Tutte-Grothendieck invariant if it satisfies

$$\begin{cases} f(M) = f(M \setminus e) + f(M/e), & \text{if } e \text{ is neither a loop nor a coloop;} \\ f(M_1 \oplus M_2) = f(M_1)f(M_2). \end{cases} \tag{27}$$

If we replace the condition of $f(M) = f(M \setminus e) + f(M/e)$ by $f(M) = \sigma f(M \setminus e) + \tau f(M/e)$ for two fixed parameters $\sigma$ and $\tau$ in the definition of a Tutte-Grothendieck invariant, we come to the so-called generalized Tutte-Grothendieck invariant.

Let $M_0$ be the loop matroid and $M_0^*$ the coloop matroid. $M_0$ and $M_0^*$ both have the one element set as the ground set and $|\mathbf{I}_{M_0}| = 1$ and $|\mathbf{I}_{M_0^*}| = 2$.

Eq. (27) says that a Tutte-Grothendieck invariant $f$ is determined by $f(M_0) = x$ and $f(M_0^*) = y$. But does such an invariant exist? Eq. (27) indicates a way to calculate $f(M)$ by repeatedly using deletion/contraction operations. But can we guarantee that we will get the same answer if we walk along two ways of cutting and fusing? The Tutte polynomial of a matroid $M$ on $E$ is $T(M; x, y) = \sum_{A \subseteq E} (x-1)^{\rho(E)-\rho(A)} (y-1)^{|A|-\rho(A)}$. The unique existence of the Tutte-Grothendieck invariant with specified initial values $x$ and $y$ is immediate from the following remarkable properties of the Tutte polynomial. Recall from Exercise 126 that $T(M/e; x, y) = T(M \setminus e; x, y)$ when $e$ is either a loop or a coloop. Also remember Eq. (26).

**Theorem 132** *(i)* $T(\emptyset; x, y) = 1$*; (ii) If* $e$ *is a loop, then* $T(M; x, y) = yT(M\backslash e; x, y)$*; (iii) If* $e$ *is a coloop, then* $T(M; x, y) = xT(M/e; x, y)$*; (iv) If* $e$ *is neither a loop nor a coloop, then* $T(M; x, y) = T(M \backslash e; x, y) + T(M/e; x, y)$.

*Proof.* ∎

The next result indicates the robustness of the Tutte-Grothendieck invariant, that is, every generalized Tutte-Grothendieck invariant is easily obtained from a Tutte-Grothendieck invariant.

**Exercise 133** *Let $F$ be a ring. For any matroid $M$ there is a unique function $f(M;x,y,\sigma,\tau)$ belonging to the polynomial ring $F[x,y,\sigma,\tau]$ and having the following properties: (i) $f(\emptyset;x,y,\sigma,\tau) =$ 1; (ii) If $e$ is a loop, then $f(M;x,y,\sigma,\tau) = yf(M \setminus e;x,y,\sigma,\tau)$; (iii) If $e$ is a coloop, then $f(M;x,y,\sigma,\tau) = xf(M/e;x,y,\sigma,\tau)$; (iv) If $e$ is neither a loop nor a coloop, then $f(M;x,y,\sigma,\tau) = \sigma f(M \setminus e;x,y,\sigma,\tau) + \tau f(M/e;x,y,\sigma,\tau)$. Furthermore, if $E$ is the underlying set of $M$, the function $f$ is given by* *

$$T'(M;x,y,\sigma,\tau) = \sigma^{|E|-\rho(E)}\tau^{\rho(E)}T(M;\frac{x}{\tau},\frac{y}{\sigma}) = \sigma^{n(E)}\tau^{\rho(E)}T(M;\frac{x}{\tau},\frac{y}{\sigma}).$$

*When evaluating $T'(M;x,y,\sigma,\tau)$ by assigning values in $F$ to $x,y,\sigma,\tau$, we usually cannot evaluate the three factors one by one, as $\sigma$ and $\tau$ may fail to be invertible and thus the factors may have no definition.

By Theorem 128, we know that

$$T(M;x,y) = \sum_{A\subseteq E} (x-1)^{n_{M^*}(A)}(y-1)^{n_M(A)} = T(M^*;y,x) \quad (28)$$

and that

$$T'(M;x,y,\sigma,\tau) = T'(M^*;y,x,\tau,\sigma). \quad (29)$$

Eqs. (28) and (29) reflect the intrinsic symmetry of the Tutte polynomial.

**Exercise 134** *Let* $M$ *be a matroid. Show that: (i)* $T(M;1,1)$ *is the number of bases of* $M$*; (ii)* $T(M;2,1)$ *is the number of independent sets of* $M$; *(iii)* $T(M;1,2)$ *is the number of*

**Exercise 135** *(i) For a binary matroid* $M, |T(M,-1,-1)| = |T(M,1,1)|$ *equals the number of bicycles in* $M$. *(ii) A binary matroid has an odd number of bases if and only if it has no nontrivial bicycle.*

For any graph $G$, we directly use $\rho_G$, $n_G$ for the rank function and nullity function of the corresponding cycle matroid (Example 105) and so on. A graph parameter which only depends on the cycle matroid of the graph is a matroid invariant. There are graph invariants*, say the chromatic polynomial, which are not a matroid invariant. But many of them turn out to satisfy some deletion/contraction relation and hence is almost a matroid invariant, as we shall demonstrate later. It is thus no surprise that the Tutte polynomial subsumes many important graph invariants, including the chromatic polynomial and the tree counting polynomial, and it has applications in knot theory, statistical mechanics and elsewhere.

*Tutte-Grothendieck invariants and 4-invariants are two classes of most important graph invariants. See, say, Chapter 6. Algebraic structures associated with embedded graphs, in: S. Lando, A. Zvonkin, Graphs on Surfaces and Their Applications, Springer, 2004.

Let $\mathcal{G}$ denote the set of all graphs. We write $E_n$ for the graph consisting of $n$ isolated vertices. For any graph $G$, $k(G)$ stands for the number of its components.

**Theorem 136** *Let $F$ be a ring and $F[x, y, \alpha, \sigma, \tau]$ the polynomial ring with five variables $x, y, \alpha, \sigma, \tau$. There is a unique map $U$ : $\mathcal{G} \to F[x, y, \alpha, \sigma, \tau]$ such that $U(E_n) = \alpha^n$ for every $n \geq 1$ and for every $e \in E(G)$ we have $U(G; x, y, \alpha, \sigma, \tau) =$ **

$$\begin{cases} xU(G-e) & \text{if } e \text{ is a bridge,} \\ yU(G-e) & \text{if } e \text{ is a loop,} \\ \sigma U(G-e) + \tau U(G/e) & \text{otherwise.} \end{cases} \tag{30}$$

*Furthermore, $U(G; x, y, \alpha, \sigma, \tau) = \alpha^{k(G)} \sigma^{n(G)} \tau^{\rho(G)} T(G; \frac{\alpha x}{\tau}, \frac{y}{\sigma})$†.*

*In this recurrence relation the roles of $G-e = G \backslash e$ and $G/e$ are not symmetric.

†The graph invariant $U(G)$ is not a matroid invariant as can be seen from the fact that $k(G)$ is not a matroid invariant.

For a nice introduction to Tutte's polynomial and other stories of Tutte, see

Arthur M. Hobbs and James G. Oxley, William T. Tutte (1917–2002), *Notices Amer. Math. Soc.* **51** (2004), 320–330.

This polynomial, a considerable generalization of the chromatic polynomial, was constructed by Tutte in 1954, building on his work seven years earlier. ... Similar to the chromatic polynomial, the Tutte polynomial can be defined recursively by the cut and fuse operations. The main virtue of the Tutte polynomial is that during the process much less information is lost about the graph than in the case of the chromatic polynomial. – B. Bollobas, *Modern Graph Theory*, Springer, 2002.

The Tutte polynomial is polynomial time to compute for planar graphs when $q = 2$ (Ising model).

The Tutte polynomial is also polynomial time to compute for all graphs on the curve and 6 isolated points:

But else where the Tutte polynomial is NP hard to compute (Jaeger, Vertigan, Welsh, Provan, 1990's).

The density of water varies as a function of temperature, and generally as a continuous function. Of course the variation is not continuous in the neighbourhood of the boiling point, nor at the freezing point. Although we are accustomed to such behaviour, it is paradoxical. The forces acting between the individual molecules vary continuously as the temperature varies. Why then should there be a change of state at certain temperatures? Statistical physics is devoted to the attempt to understand this behaviour. – C.D. Godsil, M. Grötschel, D.J.A. Welsh, Combinatorics in Statistical Physics, in: Handbook of Combinatorics (Eds. R.L. Graham, M. Grötschel, L. Lovász) Vol II, pp. 1925–1954, The MIT Press, 1995.

Graph theory is generally believed to be elegant and easy; whereas statistical mechanics has the undeserved reputation of being obscure and difficult. ... The book tries to show how graph theory and statistical physics have cross-fertilised one another. – H.N.V. Temperley, Graph Theory and Applications, John Wiley, 1981.

The general theory of statistical mechanics states that all the equilibrium properties of an assembly of interacting molecules, atomic magnets etc. are known if we can calculate the partition function. This is a generating function, one term for each of the permissible configurations of the assembly, each term being given a weight related to the energy of the corresponding configuration. – H.N.V. Temperley, Graph Theory and Applications, John Wiley, 1981.

The pivot that is essential for the model to at least have mathematical meaning is a function called the partition function

$$Z = \sum_{\sigma} \exp(\frac{-E(\sigma)}{kT}),$$

in which we define $\sigma$ to be a state of the particular model, $E(\sigma)$ to be the total energy of this state, $T$ to be the absolute temperature, and $k$ to be Boltzmann's constant. The sum itself is taken over all the states of the particular model. If the partition function of a model can be derived exactly, then this model is said to be exactly solvable. Numerous models have been shown to be exactly solvable, especially since the advent of Drinfeld's quantum group. Due to this idea of a quantum group, and also by independent work in statistical mechanics, the partition function has shown to be closely related to invariants of knots (and links). – K. Murasugi, *Knot Theory and Its Applications*, Birkhauser, 1996.

The canonical problem of classical statistical thermodynamics is the calculation (either analytically or numerically) of the partition function $Z$. For a system in thermodynamic equilibrium, if the partition function is known, one can obtain exact results for all thermodynamic quantities such as the magnetization, susceptibility and specific heat. – Daniel A. Lidar, On the quantum computational complexity of the Ising spin glass partition function and of knot invariants, *New Journal of Physics* **6** (2004) 167. `http://www.iop.org/EJ/article/1367-2630/6/1/167/njp4_1_167.html`

The Potts model of a physical system is a graph $G$ whose vertices represent particles and edges describe interactions between pairs of particles. For a $q$-state Potts model, each state of the system corresponds to a function $\omega : V(G) \to [q]$ and the value $\omega_a$ of $\omega$ at a vertex $a$ is the state of $a$ or the spin at $a$ *.

The Hamiltonian (or the energy function) of a system (without external field) in state $\omega$ is the sum of the energies on edges with endpoints having the same spins, namely

$$H(\omega) = \sum_{ab \in E} (1 - \delta(\omega_a, \omega_b)) J_{ab}, \qquad (31)$$

*The spin can have the value of one of the $q$ equally spaced angles.

where $J_{ab}$ represents the interaction between $a$ and $b$ along the edge $ab$ and can be thought of as a variable associated with the edge $ab$*. For simplicity, we now take $J_{ab} = 1$ for all edges $ab$†.

Put $\beta = \frac{1}{k_B T}$, where $k_B$ is the Boltzmann constant and $T$ is the temperature of the system, and call $\beta$ the inverse temperature of the system‡.

*Compare with the definitions of the generating functions of cycles and cuts of a graph preceding Theorem 129.

†There is no loss of generality as we can add multiple edges to return to the general case. This is also the case when we simplify the discussion of an electrical network with general resistance distribution to the discussion of a network with unit resistance everywhere.

‡When discussing electrical network, we have conductance and resistance.

The partition function of the $q$-state Potts model on $G$ is $P_G(q,\beta) = \sum_{\omega\in[q]^V} e^{-\beta H(\omega)} =$

$$\sum_{\omega\in[q]^V} e^{-\beta\sum_{ab\in E}(1-\delta(\omega_a,\omega_b))} = e^{-\beta|E|}\sum_{\omega\in[q]^V} e^{\beta\sum_{ab\in E}\delta(\omega_a,\omega_b)}. \quad (32)$$

$e^{-\beta H(\omega)}$ is the Potts measure of the state $\omega$.

The probability that the physical system is in state $\omega$ is

$$\frac{e^{-\beta H(\omega)}}{P_G(q,\beta)}. \quad (33)$$

To analyze the behavior of the physical system, the quantity displayed in Eq. (33) should be investigated. Let us discuss its numerator and denominator, respectively.

Note that at high temperature all states have about the same probability, while at low temperature the system is far more likely to be in the higher energy state (and lower Potts measure). This suggests that Potts model may help us understand (model) the very important phase transition behavior.

What is the state possessing the lowest possible energy of a system (and hence have the biggest possibility to exist)? To determine this state is a hard problem in a theoretical sense. Indeed, for the 2-state Potts model (Ising model) this question is equivalent to determining a cut of maximum size in a graph. Making use of a celebrating result called the PCP (Probabilistically Checkable Proofs) Theorem*, it is proved that there is no PTAS (polynomial-time approximation scheme) for the max-cut problem unless $P = NP$.

*See: G. Ausiello et al., Complexity and Approximations: Combinatorial Optimization Problems and Their Approximability Properties, Springer, 1999.

We now turn to the denominator of Eq. (33). What is $P_G(q, \beta)$ like? Is it a polynomial in $q$?

Given $\omega \in [q]^{V(G)}$ and $e \in E(G)$. Let us see how the contribution of $\omega$ to $P_G(q, \beta)$ corresponds to some term in $P_{G-e}(q, \beta)$ or $P_{G \setminus e}(q, \beta)$. Note that if the end points of $e$ have different spins, it contributes nothing to the RHS of Eq. (32), so the weight of the configuration $\omega$ naturally equals to the weight of the same configuration on $G - e$. On the other hand, if the spins are the same, the edge contributes something, but the action is local, so the weight equals that of a configuration on $G \setminus e$ naturally induced by $\omega$, with perhaps some weighting factor.

Thus, the Potts model partition function has a deletion-contraction reduction, and hence by Theorem 30, must be an evaluation of the Tutte polynomial. In particular, it is really a polynomial in $q$. Note that this argument was used to discuss the chromatic polynomial before.

**Exercise 137** *Write down explicitly the recurrence relation for the* $q$*-state Potts model partition function and then use it to prove that* $P_G(q,\beta) = e^{-\beta|E(G)|}Z_G(q,v)$*, where* $v = e^{\beta} - 1$ *and*

$$Z_G(q,v) = q^{k(G)}v^{\rho(G)}T(G; 1 + \frac{q}{v}, 1 + v). \tag{34}$$

Exercise 137 says that $P_G(q,\beta)$ can be expressed via the Tutte polynomial. We next try to work out a useful expression for it directly. Letting $v = e^{\beta} - 1$, then we have

$P_G(q,\beta) = \sum_{\omega\in[q]^V} e^{-\beta H(\omega)} = \sum_{\omega\in[q]^V} \prod_{ab\in E(G)} e^{-\beta(1-\delta(\omega_a,\omega_b))} =$
$e^{-\beta|E|} \sum_{\omega\in[q]^V} \prod_{ab\in E(G)} e^{\beta\delta(\omega_a,\omega_b)} = e^{-\beta|E|} \sum_{\omega\in[q]^V} \prod_{ab\in E(G)} (1+v\delta(\omega_a,\omega_b))$

$= e^{-\beta|E|} \sum_{\omega\in[q]^V} \sum_{A\subseteq E_\omega} v^{|A|}$ *

$$= e^{-\beta|E|} \sum_{A\subseteq E} q^{k(A)} v^{|A|} \quad \text{by double counting!} \qquad (35)$$

*Here $E_\omega$ is the set of edges whose endpoints have the same spin in the configuration $\omega$.

Comparing Eq. (35) and Eq. (34), we find that

$$Z_G(q,v) = \sum_{A\subseteq E} q^{k(A)} v^{|A|}. \tag{36}$$

We call $Z_G(q,v)$ the dichromatic polynomial of the graph $G$.

**Exercise 138** *(i) $Z_G(q,v)$ is the unique polynomial such that $Z_{E_n}(q,v) = q^n$ for every integer $n \geq 1$ and*

$$Z_G = Z_{G-e} + vZ_{G/e} \tag{37}$$

*for every $e \in E(G)$. (ii) $Z_G(q,v)$ is the polynomial $U(G;x,y,\alpha,\sigma,\tau)$ evaluated at $\alpha = q, \sigma = 1, \tau = v, x = \frac{v}{q} + 1$ and $y = v + 1$.*

**Exercise 139** *Make use of Eqs. (25) and (37) to show that*

$$p_G(x) = Z_G(x,-1) = x^{k(G)}(-1)^{|V(G)|-k(G)} T(G; 1-x, 0). \tag{38}$$

Another way to see Eq. (38) is as follows.

Recall that at zero temperature, high energy states prevail, i.e. we really need to consider states where the endpoints on every edge are different. Such a state just corresponds to a proper coloring of a graph!

B.A. Cipra, An introduction to the Ising model, *Amer. Math. Monthly* **94** (1987), 937–959.

D.J.A. Welsh, The computational complexity of some classical problems from statistical physics, In: Disorder in Physical Systems (G. Grimmett and D. Welsh eds.), Clarendon Press, Oxford, 1990, pp. 307–321.

After that, I like to tell graduate students that, if they launch into a field and get frustrated because they didn't understand it, that is not necessarily time wasted. In this particular example – my launching into the Onsager solution – the reason I could understand the key idea during the 15 minutes ride with Luttinger was that I had "prelearned" the whole subject very well, I was able to appreciate the whole strategy. – Chen Ning Yang

Both the partition function of the Potts model and the cut set generating function encode the state/energy distribution in a similar way and so their close relationship is not surprising.

**Exercise 140** *Deduce Theorem 129 from the symmetry of the Tutte polynomial as indicated in Eqs.* (28) *and* (29).

The weight enumerator of a code $C$ of length $n$ is the polynomial $W_C(x,y) = \sum_{c\in C} x^{n-wt(c)} y^{wt(c)} = \sum_{i=0}^{n} A_i x^{n-i} y^i$, where $A_i$ is the number of words of weight $i$ in $C$.

**Theorem 141 (Curtis Greene 1976)** *Let $C$ be a code over a field with $q$ elements and $M$ the corresponding vector matroid. Then $W_c(x,y) = y^{n-dim(C)}(x-y)^{dim(C)} T(M; \frac{x+(q-1)y}{x-y}, \frac{x}{y})$.*

*Proof.* ∎

Theorem 141 together with the symmetry of the Tutte polynomial leads to

**Theorem 142 (MacWilliams identity)** $W_{C^{\perp}}(x,y) = \frac{1}{|C|}W_C(x+(q-1)y, x-y)$.

For another beautiful proof of the MacWilliams identity using some kind of inversion formula*, see Marshall Hall, Jr., *Combinatorial Theory*, Second Edition, John Wiley & Sons, Inc., 1986.

*A duality relation again!

Note that Theorem 130 implies that the number of $i$-subsets of $E(G)$ containing no broken cycle is independent of the way of indexing $E(G)$. A similar phenomenon holds for the general Tutte polynomial.

But I recalled that Hassler Whitney, giving the chromatic polynomial in terms of broken circuits, had encountered a similar phenomenon. – W.T. Tutte, Graph polynomials, *Adv. Appl. Math.* **32** (2004), 5–9.

Observe that for any plane graph $G$, there is a natural bijection between the edge-cuts of $G$ and the even subgraph of its dual graph $G^*$. Thus, MacWilliams identity simply expresses the generating function of edge-cuts of any graph $G$ using the generating function of even subgraph of the same graph $G$.

Thomas Britz, Higher support matroids, preprint.

Alexander Barg, On some polynomials related to weight enumerators of linear codes, *SIAM Journal on Discrete Mathematics*, **15** (2002), 155–164.

Dirk Vertigan, Latroids and their representation by codes over modules, *Trans. Amer. Math. Soc.* **356** (2004), 3841–3868.

Abstract: It has been known for some time that there is a connection between linear codes over fields and matroids represented over fields. In fact a generator matrix for a linear code over a field is also a representation of a matroid over that field. There are intimately related operations of deletion, contraction, minors and duality on both the code and the matroid. The weight enumerator of the code is an evaluation of the Tutte polynomial of the matroid, and a standard identity relating the Tutte polynomials of dual matroids gives rise to a MacWilliams identity relating the weight enumerators of dual codes. More recently, codes over rings and modules have been considered, and MacWilliams type identities have been found in certain cases.

In this paper we consider codes over rings and modules with code duality based on a Morita duality of categories of modules. To these we associate latroids, defined here. We generalize notions of deletion, contraction, minors and duality, on both codes and latroids, and examine all natural relations among these.

We define generating functions associated with codes and latroids, and prove identities relating them, generalizing abovementioned generating functions and identities.

Let $X = \{x_1, \cdots, x_n\}$ and $Y = \{y_1, \cdots, y_n\}$ be two bases of a linear space $V$. Is there a bijection $f$ from $[n]$ to itself such that $(X \setminus \{x_i\}) \cup \{y_{f(i)}\}$ is a basis for each $i \in [n]$?

Another formulation: For any nonsingular $n \times n$ matrix $A$, is there a bijection $f$ such that $A(i, f(i)) \neq 0$ for each $i \in [n]$?

From this formulation, it is easy to see that the answer to the previous question is yes.

**Exercise 143** *Let $B$ and $B'$ be two bases of a matroid $M$. Prove that there is a bijection $f$ from $B$ to $B'$ such that $(B \setminus \{e\}) \cup \{f(e)\}$ is a basis for each $e \in B$.* *

*Our proof for the matrix case makes use of the continuous parameters when evaluating a determinant. You have to search for a non-parametric proof for this general fact. It is helpful to first try Exercise 144 if you have difficulty in working out this exercise.

**Exercise 144** *Let $B$ be a basis of a matroid $M$ and suppose $y \in M \setminus B$, $x \in B$. Prove that*

(i) $B \cup \{y\}$ *contains a unique circuit $C_y$, which is known as a fundamental circuit of $B$;*

(ii) $y \in C_y$;

(iii) $(B \cup \{y\}) \setminus \{x'\}$ *is a basis of $M$ if and only if* $x' \in C_y$;

(iv) $(B \setminus \{x\}) \cup \{y'\}$ *is a basis of $M$ if and only if there is a circuit $C$ containing $\{x, y'\}$ such that there is no circuit $C'$ satisfying $x \in C'$ and $C' \setminus B \subsetneq C \setminus B$.*

**Exercise 145** *Let $B$ and $B'$ be two bases of a matroid $M$. For any $x \in B$, let $Sym(x, B, B')$ be the set of elements $y \in B'$ such that both $(B \cup \{y\}) \setminus \{x\}$ and $(B' \cup \{x\}) \setminus \{y\}$ are bases of $M$. Prove that $Sym(x, B, B') \neq \emptyset$. (Hint: Make use of Exercise 144.)*

**Exercise 146** [*] *For a matroid $M$ and an integer $k \geq 2$, there are bases $B, B'$ and $x \in B$ with $|Sym(x, B, B')| = k$ if and only if there is a circuit $C$ and a cocircuit $C^*$ of $M$ such that $|C \cap C^*| = k + 1$.*

**Exercise 147** [†]*Let $B$ and $B'$ be two bases of a matroid $M$ and suppose $X \subseteq B$. Prove that there exists $X' \subseteq B'$ such that both $(B \cup X') \setminus X$ and $(B' \cup X) \setminus X'$ are bases of $M$.*

[*] Joseph E. Bonin, On basis-exchange properties for matroids, *Discrete Mathematics* **187** (1998), 265–268.

[†] C. Greene, A multiple exchange property for bases, *Proc. Amer. Math. Soc.* **39** (1973), 45–50.

A matroid $M$ is base orderable if given any two bases $X$ and $Y$ there is a bijection $f$ from $X$ to $Y$ such that for each $x \in X$ both $(X \setminus \{x\}) \cup \{f(x)\}$ and $(Y \setminus \{f(x)\}) \cup \{x\}$ are bases of $M$. For a vectoral matroid, to determine whether or not it has the base orderable property amounts to checking that for some specified nonsingular matrix $A$ whether or not there exists a permutation matrix $P$ such that both $PA$ and $A^{-1}P^{\top}$ have nonzero diagonal elements everywhere.

A matroid $M$ is strongly base orderable if given any two bases $X$ and $Y$ there is a bijection $f$ from $X$ to $Y$ such that for each $A \subseteq X$ both $(X \setminus A) \cup f(A)$ and $(Y \setminus f(A)) \cup A$ are bases of $M$.

**Example 148** *The cycle matroid of the complete graph $K_4$ is not base orderable. Note that each cycle matroid is a vectoral matroid and hence the set of nonsingular matrices for which the permutation matrix $P$ described above exist is a proper subset of the general linear group. Further discussion of such nonsingular matrices seems interesting.*

It is known that all transversal matroids are strongly base orderable *. It is also known that a base orderable matroid need not be strongly base orderable†.

*R.A. Brualdi, E.B. Scrimger, Exchange systems, matchings, and transversals, *J. Comb. Theory* **5** (1968), 244-257.

†A.W. Ingleton, Conditions for representability and transversality of matroids, *Proc. Fr. Br. Conf.* Springer Lec. Notes, **211** (1970), 62–67.

In 1948 Claude Shannon published a landmark paper "A mathematical theory of communication" that signified the beginning of both information theory and coding theory. – W.C. Huffman, V. Pless, Fundamentals of Error-Correcting Codes, Cambridge, 2003.

Bela Bollobas, Oliver Riordan, A Tutte polynomial for colored graphs, *Combinatorics, Probability and Computing* **8** (1999) 45–93.

# Matroid union and matroid intersection

Grassmann-Plücker relation

A.W.M. Dress, W. Wenzel, Valuated matroids: A new look at the greedy algorithm, *Applied Mathematics Letters* **3** (1990), 33–35.

A. Vince, The greedy algorithm and Coxeter matroids, *Journal of Algebraic Combinatorics* **11** (2000), 155–178.

## IV. Marriage problems

Many of the most beautiful theorems of mathematics are of the form: Such and such a necessary condition is also sufficient. The necessity is frequently obvious or at least easy to see, but to establish the sufficiency is the real trick. – George Minty (Cited in: Andrew Lenard, An application of the marriage lemma, *Mathematics Magazine* **74** (2001), 234–238.)

Richard Brualdi, Chap. 4. Introduction to matching theory, and Chap. 5. Transversal matroids, in: Neil White, (Ed.), Combinatorial Geometries, *Encyclopedia of Mathematics and its Applications* **29**, Cambridge University Press, 1987, pp. 53–97.

Tamás Fleiner, A fixed-point approach to stable matchings and some applications, *Mathematics of Operations Research* **28** (2003), 103–126.

Andreas Dress, The theorem of the $k-1$ happy divorces, *Ann. Comb.* **4** (2000), 183–194.

Donald E. Knuth, Stable Marriage and Its Relation to Other Combinatorial Problems: An Introduction to the Mathematical Analysis of Algorithms, American Mathematical Society, 1996.

**Exercise 149 (Bollobas, p. 94, Exercise 20)** [*]*Prove the following form of the Schröder-Bernstein theorem. Let $G$ be a bipartite graph with vertex classes $X$ and $Y$ having arbitrary cardinalities. Let $A \subseteq X$ and $B \subseteq Y$. Suppose there are complete matchings from $A$ into $Y$ and from $B$ into $X$. Prove that $G$ contains a set of independent edges covering all the vertices of $A \cup B$.*

[*]This is known as the Mendelsohn-Dulmage Theorem. See, N.S. Mendelsohn, A.L. Dulmage, Some generalizations of the problem of distinct representatives, *canad. J. Math.* **10** (1958), 230–241. Also see, David Gale, Alan J. Hoffman, Two remarks on the Mendelsohn-Dulmage theorem, *Annals of Discrete Mathematics* **15** (1982), 171–177.

Ising model and Pfaffian

A legend says that Jack Edmonds shouted Eureka – you shrink! when he found a good characterization for matching (and the matching algorithm) in 1963, the day before his talk at a summer workshop at RAND Corporation with celebrities

stable marriage, edge list coloring of bipartite graph

## Slither Game*

Consider the following game played on a graph. Two players choose edges in turn subject to the condition that at each stage the set of edges that are chosen forms a path. The first player with no legal move loses.

On any given graph, either the first player has a winning strategy, or the second player does. How to characterize these two types of graphs and what are the respectively winning strategies?

*Based on a graph theory notes of Chris Godsil

## V. Flows

Theorem 1: A graph has a nowhere-zero 2-flow if and only if each of its components is Eulerian.

chain group

nowhere-zero flow

Cun-Quan Zhang, Integer Flows and Cycle Covers of Graphs, Monographs and Textbooks in Pure and Applied Mathematics 205, Marcel Dekker, Inc. 1997.

The important ideas of combinatorics do not usually appear in the form of precisely stated theorems but more often as general principles of wide applicability. − William T. Gowers, The two cultures of mathematics, in: *Mathematics: frontiers and perspectives*, 65–78, Amer. Math. Soc., Providence, RI, 2000.

## VI. Perfect graph and intersection representation

Gérard Cornuéjols, Combinatorial Optimization: Packing and Covering, SIAM, 2001.

M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, Second Edition, Elsevier, 2004.

Andreas Brandstädt, Van Bang Le, Jeremy P. Spinrad, Graph Classes - A Survey, *SIAM Monographs on Discrete Mathematics and Applications* **3**, Philadelphia, PA, 1999.

Terry A. McKee, F.R. McMorris, Topics in Intersection Graph Theory, *SIAM Monographs on Discrete Mathematics and Applications* **2**, Philadelphia, PA, 1999.

We will only consider graphs without loops and multiple edges in the discussion below (even when no such assumption is clearly made.) For each graph $G$, it is easy to see that $\chi(G) \geq \omega(G)$. A graph is perfect if each of its (vertex-)induced subgraph has equal chromatic number and clique number.

In 1960, Berge came out of the following two conjectures.

Strong Perfect Graph Conjecture (SPGC): A graph $G$ is perfect if and only if $G$ and its complement $\overline{G}$ has no induced subgraph that is an odd cycle of length at least 5.

Weak Perfect Graph Conjecture (WPGC): A graph $G$ is perfect if and only if $\overline{G}$ is perfect.

WPGC is first proved by Lovász and later by Fulkerson.

Lovász * stunned the world of combinatorics by proving this important and well-known conjecture at the age of 22. Fulkerson also studied it, reducing it to a statement he thought was too strong to be true. When Berge told him that Lovász had proved it, within hours he proved the missing lemma †, thus illustrating that a theorem becomes easier to prove when known to be true. – D.B. West, Introduction to Graph Theory, China Machine Press, Second Edition, 2004.

* L. Lovász, Normal hypergraphs and the perfect graph conjecture, *Disc. Math.* **2** (1972), 253–267.

†D.R. Fulkerson, Blocking and anti-blocking pairs of polyhedra, *Math. Programming* **1** (1971), 168–194.

As with the SPGC, which is clearly stronger than WPGC, a proof can be found in

Maria Chudnovsky, Paul Seymour, Neil Robertson, Robin Thomas, The strong perfect graph theorem, *Annals of Mathematics*, in press.

Here are some interesting web materials.

V. Chvátal, Claude Berge: 5.6.1926 – 30.6.2002, available from `http://www.cs.concordia.ca/~chvatal/perfect/claude2.pdf`. P. Seymour, How the proof of the strong perfect graph conjecture was found, available from `http://www.cs.concordia.ca/~chvatal/perfect/pds.pdf`. Open problems on perfect graphs, available from `http://www.cs.concordia.ca/~chvatal/perfect/problems.html`

We present a proof of the WPGC below, following

G.S. Gasparian, Minimal imperfect graphs: a simple approach, *Combinatorica* **16** (1996), 209–212.

Noting that $\alpha(\overline{H}) = \omega(H)$ and that the induced subgraph operation and the complementing operation are commutative operations, we know that the WPGC follows from

**Lemma 150** *A graph $G$ is imperfect if and only if there is an induced subgraph $H$ of $G$ such that $n(H) > \alpha(H)\omega(H)$.*

The backword direction of Lemma 150 is easy: If $n(H) > \alpha(H)\omega(H)$, the considering a partition of $V(H)$ into $\chi(H)$ parts of disjoint independent sets, because each part can have at most $\omega(H)$ vertices and $n > \alpha\omega$, we arrive at the conclusion that $\chi(H) > \omega(H)$, and so $G$ is imperfect, as claimed.

A minimally imperfect graph * is an imperfect graph whose induced subgraphs other than itself are all perfect. In view of the fact that each imperfect graph contains a minimally imperfect graph, to establish the forward direction of Lemma 150, it suffices to verify

**Lemma 151** *If* $H$ *is minimally imperfect, then* $n(H) \geq \alpha(H)\omega(H)+1$.

We prove Lemma 151 in four steps.

*Shifting attention to this concept is a successful use of the standard technique of looking at the extremal case. We can often detect some singularity, namely source of information, in such extremality consideration. Partitionalable graphs, a class of graphs including all minimally imperfect graphs, has a high regularity and can be characterized in terms of an extremely succinct matrix equation.

**Lemma 152** *Let $H$ be minimally imperfect. Then each independnet set $S$ of $H$ is disjoint from some $\omega(H)$-clique of $H$.*

*Proof.* We need only consider the case that $S \neq \emptyset$. If such an $S$ intersect with every $\omega(H)$-clique, we have $\chi(H) \leq 1 + \chi(H - S) = 1 + \omega(H - S) \leq \omega(H)$, contracting with $\chi(H) > \omega(H)$. ∎

**Lemma 153** *Let $H$ be minimally imperfect and $v \in V(H)$. Then $\chi(H) = \chi(H - v) + 1 = \omega(H - v) + 1 = \omega(H) + 1$.*

*Proof.* As $H$ is minimally imperfect, we have $\chi(H) > \omega(H)$ and $\chi(H - v) = \omega(H - v)$. But it is clear that $\chi(H)$ can only take value either $\chi(H - v)$ or $\chi(H - v) + 1$ and $\omega(H)$ can only take value either $\omega(H - v)$ or $\omega(H - v) + 1$. ∎

Let $H$ be minimally imperfect and $S_0 = \{v_1, \cdots, v_\alpha\}$ be one of its $\alpha(H)$-independent sets. For each $i \in [\alpha]$, Lemma 153 implies that $G - v_i$ can be partitioned into $\omega(H)$ independent sets, denoted $S_{i1}, \cdots, S_{i\omega}$. Write $\mathcal{S}$ for the multiset $\{S_0\} \cup \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_\alpha$, where $\mathcal{S}_1 = \{S_{11}, \cdots, S_{1\omega}\}, \cdots, \mathcal{S}_\alpha = \{S_{\alpha 1}, \cdots, S_{\alpha\omega}\}$ *. $\mathcal{S}$ consists of $1 + \alpha\omega$ members.

**Lemma 154** *Each vertex $v$ of $H$ lies in exactly $\alpha(H)$ members of $\mathcal{S}$, and any $\omega(H)$-clique of $H$ is disjoint from exactly one member of $\mathcal{S}$.*

*It turns out that this multiset is indeed a set and possesses some remarkable pattern.

*Proof.* If $v = v_i$, then no member from $\mathcal{S}_i$ can include $v$; If $v \neq v_i$, then exactly one member from $\mathcal{S}_i$ include $v$. Thus, this allows us to distinguish two possibilities, either $v \in S_0$ or $v \notin S_0$, and come to the same result that $v$ can be found in exactly $\alpha(H)$ members of $\mathcal{S}$.

Take an $\omega(H)$-clique $K$. Observe that the intersection of a clique and an independent set can have no more than one element. Thus, by Pigeon's hold principle, for any $i \in [\alpha]$, if $v_i \notin K$, each member of $\mathcal{S}_I$ is not disjoint from $K$ while if $v_i \in K$, we can infer that there is exactly one member from $\mathcal{S}_i$ that is disjoint from $K$. Finally, applying the above fact to address the two cases, $|K \cap S_0| = 0$ or $|K \cap S_0| = 1$, respectively, yields the remaining claim and completes the proof. ∎

We are ready to offer a proof of WPGC.

*Proof.* (of Lemma 151) By Lemma 152, for each member $S$ of $\mathcal{S}$, we can get an $\omega(H)$-clique $\overline{S}$ which is disjoint from $S$.

Construct an $(\alpha(H)\omega(H) + 1) \times n(H)$ matrix $A$ whose rows are those characteristic vectors of the members of $\mathcal{S}$. Correspondingly, we construct an $(\alpha(H)\omega(H) + 1) \times n(H)$ matrix $B$ by replacing any row corresponding to an $S \in \mathcal{S}$ by the characteristic vector of $\overline{S}$.

By Lemma 154 we obtain $AB^\top = J_{1+\alpha\omega} - I_{1+\alpha\omega}$. Since $J - I$ is nonsingular, we deduce that $\alpha(H)\omega(H) + 1 \leq n(H)$, as was to be shown. ∎

**Exercise 155** *Let* $H$ *be minimally imperfect. Then, we have the followings: (i)* $n(H) = 1 + \alpha(H)\omega(H)$*. (ii)* $H$ *has exactly* $n(H)$ $\alpha$*-independent sets* $Q_i, i \in [n]$*, and exactly* $n(H)$ $\omega$*-cliques* $S_i, i \in [n]$*, such that* $|S_i \cap Q_j| = 1 - \delta(i, j)$*.*

**Exercise 156** *An irreflexive and transitive relation on a set is called a partial ordering relation. A set together with a partial ordering relation on it is called a poset. The width of a poset is the largest size of any antichain of the poset and the dimension of a poset is the minimum linear orderings whose intersection is the poset. (i) Show that the comparability graph of a poset is perfect; (ii) Show that the incomparability graph of a poset is perfect; (iii) Prove that the dimension of any finite poset is less than or equal to its width.*

Do Exercises 5.3.28, 5.3.29, 5.3.30 in [West].

P.C. Fishburn, Interval Orders and Interval Graphs, Wiley, 1985.

A graph is an interval graph if each vertex $v$ can be associated with an interval $I_v$ in $\mathbb{R}$ such that there is an edge between $u \neq v$ if and only if $I_u \cap I_v \neq \emptyset$.

A digraph is an interval digraph if each vertex $v$ can be associated with an ordered pair of intervals $\{S_v, T_v\}$ in $\mathbb{R}$ such that there is an arc from $u$ to $v$, which are not necessarily distinct, if and only if $T_u \cap S_v \neq \emptyset$.

There are many characterization results on interval graphs and interval digraphs. But we now intend to introduce the so-called Weiner digraph * and give a characterization of it.

*In the literature it is simply named as interval digraphs or, more precisely, the comparability digraph of the interval order. To avoid confusion with the concept of interval digraph presented above, we adopt the current usage.

He received his Ph.D. from Harvard at the age of 18 with a dissertation on mathematical logic supervised by Karl Schmidt. From Harvard Wiener went to Cambridge, England, to study under Russell who told him that in order to study the philosophy of mathematics he needed to know more mathematics so he attended courses by G.H. Hardy. In 1914 he went to Göttingen to study differential equations under Hilbert, and also attended a group theory course by Edmund Landau. He was influenced by Hilbert, Landau and Russell but also, perhaps to an even greater degree, by Hardy. At Göttingen he learned that:

... mathematics was not only a subject to be done in the study but one to be discussed and lived with.

From: `http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Wiener_Norbert.html`

A digraph is said to be a Weiner digraph if for each vertex $v$ there corresponds an interval $[\ell_v, r_v]$ such that there is an arc going from $u$ to $v$ if and only if $r_u < \ell_v$. Note that the underlying graph of a Weiner digraph is just the complement of an interval graph.

**Exercise 157** *Let $\mathcal{S}$ be the set of all interval digraphs and $\mathcal{T}$ the set of all Weiner digraphs. Prove that $\mathcal{T} \subsetneq \mathcal{S}$.*

When working with Betrand Russell, Norbert Weiner (1894-1964) published a paper * in which the concept of Weiner digraph first came to the surface. His paper makes extensive use of Russell's notation for symbolic logic and is hard to understand. But it is said that the following Fishburn's characterization of the Weiner digraphs (Theorem 158) is already anticipated by Weiner in his papers in that period (at his early 20's!) †.

We use the symbol **i+j** to represent the digraph which is a disjoint union of a path of length $i-1$ and a path of length $j-1$. It is easy to see that no Weiner digraph can contain a **2+2** as a vertex induced subdigraph. Another simple observation is that each Weiner digraph much be acyclic and transitive.

*Proceedings of the Cambridge Philosophical Society 17, (1914), 441–449.

†P.C. Fishburn, B. Monjardet, Norbert Weiner on the theory of measurement (1914, 1915, 1921), *J. Math. Psych.* **36** (1992), 165–184.

**Theorem 158** [*] *A transitive acyclic simple digraph is a Weiner digraph if and only if it does not contain a* **2+2** *as a vertex induced subdigraph.*

We will prove Theorem 158 by appealing to Farkas' Lemma. This is surely not the way Weiner found it when he was younger than 20 years. You are encouraged to produce a short proof yourself[†]. In addition, we remark that every nonshellabel poset contains **2+2** as an induced subposet. An elementary proof of this latter fact is given by L.J. Billera and A.N. Myers [‡].

[*] P.C. Fishburn, Intransitive indifference with unequal indifferent intervals, *J. Math. Psychology* **7** (1970), 144–149.

[†] Please compare your proof with the proof presented in: K.P. Bogart, An obvious proof of Fishburn's interval order theorem, *Discrete Math.* **118** (1993), 239–242.

[‡] Shellability of interval orders, *Order* **15** (1999), 113–117.

We have introduced the famous Farkas' Lemma in Exercise 127. Here is one of its very useful equivalent forms.

**Lemma 159** *For any* $A \in \mathbb{R}^{n\times m}$ *and* $b \in \mathbb{R}^n$, *either there is* $\alpha \in \mathbb{R}^m$ *such that* $A\alpha^\top \leq b^\top$ *or there is* $\beta \in \mathbb{R}^n$ *satisfying* $\beta A = 0$, $\beta b^\top < 0$ *and* $\beta \geq 0$, *but not both.*

*Proof.* Consider $L = \{\begin{pmatrix} -\beta b^\top & \beta \end{pmatrix} : \beta A = 0\} \leq \mathbb{R}^{n+1}$ and $L^\perp = \{\begin{pmatrix} \lambda & \lambda b - \alpha A^\top \end{pmatrix} : \lambda \in \mathbb{R}, \alpha \in \mathbb{R}^m\} \leq \mathbb{R}^{n+1}$. To see that this result follows from Exercise 127, it suffices to check the assertions below:

$$\exists x \in L, x \geq 0 \text{ and } x_1 > 0 \Leftrightarrow \exists \beta \geq 0, \beta A = 0, \beta b^\top < 0;$$

$$\exists y \in L^\perp, y \geq 0 \text{ and } y_1 = 1 \Leftrightarrow \exists \alpha, A\alpha^\top \leq b^\top.$$

∎

We shall proceed with some applications of Lemma 159 * and postpone a proof of Theorem 158 to the last moment so that your effort to proving it will not be hampered by our illustrations.

Given a digraph $G$ along with an interval $[\ell(e), u(e)]$ associated with each $e \in E(G)$, a circulation is an assignment of flow $f(e) \in [\ell(e), u(e)]$ such that the flow into each vertex equals the flow out of the same vertex.

*We follow Garth Isaak, Examples of Combinatorial Duality, unpublished lecture notes.

**Lemma 160** *A digraph with upper and lower bounds $u$ and $\ell$ for flow values has a circulation if and only if $\sum_{e\in[S,V-S]} u(e) \geq \sum_{e\in[V-S,S]} \ell(e)$.*

*Proof.* Use Lemma 159. Consider a certificate of inconsistency that maximizes the number of equations and inequalities with multiplier 0. (More details to fill in.) ∎

**Lemma 161** *If a digraph has both integral upper bounds and integral lower bounds on each arc, then it possesses an integral circulation whenever it has a circulation.*[*]

[*]Lemma 161 tells us that an integral version of Lemma 160 holds. But does it mean that we can find an integral version of Farkas' Lemma? In some sense, this is not possible unless $P = NP$. Our luck in getting the integral version of Lemma 160 relies on the nice structure of the constraint matrix appeared in this special context.

*Proof.* (of Lemma 161) If the circulation is not integral, we can find a set of arcs with non-integer flow values which form a cycle of the underlying graph after ignoring the directions on them. Walk around this cycle in one direction and increase flow for ares traversed in the forward direction and decrease flow for arcs traversed in a backward direction by an equal amount so that some flow becomes integral and all flows are still in the intervals posed for them. Repeating this operation we will finally arrive at an integral circulation. ∎

In a round-robin tournament there is exactly one game between each pair of players and there is exactly one winner for each game. A sequence $s_1, s_2, \cdots, s_n$ of nonnegative integers is a score sequence if $s_i$ records the number of wins for player $i$ in an $n$-person round-robin tournament.

The number of distinct score sequences of length $1, 2, \cdots 15$ are $1, 1, 2, 4, 9, 22, 59, 167, 490, 1486, 4639, 14805, 48107, 158808, 531469$.

**Exercise 162** *Construct an infinite family of score sequences each of which does not uniquely determine a tournament.*

**Exercise 163** *Design a way of arranging a Round Robin Tournament with $n$ players with the fewest possible rounds. (Hint: You can reduce the case of $n$ odd to the case of $n$ even.)*

**Theorem 164** **A sequence of nonnegative integers* $s_1, \cdots, s_n$ *is a score sequence if and only if*

$$\sum_{x \in S} s_x \leq \binom{n}{2} - \binom{n - |S|}{2} \qquad (39)$$

*for each* $S \subseteq [n]$ *with equality holding for* $S = [n]$ *and if and only if*

$$\binom{|S|}{2} \leq \sum_{x \in S} s_x \qquad (40)$$

*for each* $S \subseteq [n]$ *with equality holding for* $S = [n]$.

*H.G. Landau, On dominance relations and the structure of animal societies: III, the condition for a score sequence, *Bull. Math. Biophysics* **15** (1953), 143–148. I do not know if this Landau has any relation with the Nobel prize winner Lev Davidovic Landau (1908–1968) or the analytic number theorist Edmund Landau (1877–1938) who taught Weiner group theory in Göttingen.

*Proof.* * $s_1, \cdots, s_n$ is a score sequence if and only if there is a sequence of integers $f_{ij}, 1 \leq i < j \leq n$, which only take values either 1 or 0 and satisfy $\sum_{x<y}(1-f_{xy}) + \sum_{y<z} f_{yz} = s_y$ for $y \in [n]$ †.

Rewriting this, we get

$$-(s_y - y + 1) + \sum_{x<y}(-f_{xy}) + \sum_{y<z} f_{yz} = 0 \qquad (41)$$

for $y \in [n]$.

*Each game between players in $S$ can only contribute to $\sum_{x \in S} s_x$ and each game between players in $[n]-S$ can only contribute to $\sum_{x \in [n]-S} s_x$. This says that Eq. (39) and Eq. (40) are obviously necessary and the nontrivial part of the proof is the converse direction.

†For $x < y$, $f_{xy} = 1$ corresponds to the case that player $x$ wins player $y$.

Consider a digraph $G$ on vertex set $V = \{0\} \cup [n]$ with arcs $xy$ for each pair of vertices $x < y$. Let

$$\ell(xy) = \begin{cases} 0 & \text{if } x \neq 0; \\ s_y - y + 1 & \text{otherwise;} \end{cases} \quad u(xy) = \begin{cases} 1 & \text{if } x \neq 0; \\ s_y - y + 1 & \text{otherwise.} \end{cases}$$

Since $f(0y)$ is forced to be $s_y - y + 1$, what Eq. (41) really tells us is that $f$, if exists, is a circulation for the digraph $G$ with its lower bound and upper bound functions $\ell$ and $u$ as constructed above.

We know that $s_1, \cdots, s_n$ is a score sequence if and only if the digraph $G$ has an integral circulation, and thus, by combining Lemmas 160 and 161, if and only if for any subset $S \subseteq V$ we have

$$\sum_{e \in [S, V-S]} u(e) \geq \sum_{e \in [V-S, S]} \ell(e). \tag{42}$$

Therefore, our final task is to demonstrate that this system of inequalities in (42) is just equivalent to the system given in (39) as well as the system given in (40).

We show that the subsystem of (42) consisting of all those inequalities with $0 \notin S$ is equivalent to (39). Suppose $0 \notin S$. Then, $\sum_{e \in [S, V-S]} u(e) = \sum_{0<x<y, x \in S, y \in [n]-S} 1$ and $\sum_{e \in [V-S, S]} \ell(e) = \sum_{x \in S} (s_x - x + 1)$.

$\sum_{e\in[V-S,S]} \ell(e) \leq \sum_{e\in[S,V-S]} u(e) \Leftrightarrow$

$\sum_{x\in S}(s_x - x + 1) \leq \sum_{0<x<y, x\in S, y\in[n]-S} 1 \Leftrightarrow$

$\sum_{x\in S} s_x \leq \sum_{y\in S}(y-1) + \sum_{0<x<y, x\in S, y\in[n]-S} 1 =^*$

$= \binom{n}{2} - \binom{n-|S|}{2}$, which is just what (39) says.

In a similar tack, we will derive the equivalence of (40) with the subsystem of (42) for those $S$ with $0 \in S$.

*To see this equality, note that the number of pairs of $x < y$, which is $\binom{n}{2}$, is the sum of the number of those pairs with both $x$ and $y$ outside of $S$, which is $\binom{n-|S|}{2}$, the number of those with $y \in S$ and those with $x \in S$ and $y \in [n] - S$.

Assume now $0 \in S$. First, we check that $\sum_{e \in [S,V-S]} u(e) = \sum_{0<x<y, x\in S, y\in[n]-S} 1 + \sum_{y\in[n]-S}(s_y - y + 1)$ and $\sum_{e\in[V-S,S]} \ell(e) = 0$. By dint of it, we know that the relation $\sum_{e\in[V-S,S]} \ell(e) \leq \sum_{e\in[S,V-S]} u(e)$ becomes $\sum_{0<x<y,x\in S,y\in[n]-S} 1 + \sum_{y\in[n]-S}(s_y - y + 1) \geq 0$, or in a still better form,

$$\sum_{y\in\overline{S}} s_y \geq \sum_{y\in\overline{S}} (y-1) - \sum_{\substack{1\leq x<y \\ x\in S, y\in\overline{S}}} 1, \tag{43}$$

where $\overline{S} = [n] - S$. Since $\sum_{y\in\overline{S}}(y-1)$ counts the number of arcs entering some vertex in $\overline{S}$ from $[n]$ and $\sum_{\substack{1\leq x<y \\ x\in S, y\in\overline{S}}} 1$ counts the number of arcs entering a vertex in $\overline{S}$ from $S - \{0\}$, we conclude that the RHS of (43) is nothing but $\binom{|\overline{S}|}{2}$, and hence verifying our claim.

To complete the proof, we only need to demonstrate that (39) and (40) are equivalent. This is left as an exercise. ∎

**Exercise 165** *Show that* (39) *is equivalent to* (40)*.*

**Exercise 166** *Show that we can delete a set of* $2^n - n$ *inequalities among all those* $2^n$ *inequalities in* (39) *(*(40)*) and still guarantee the truth of Theorem 164.*

Richard Brualdi, Jian Shen, Landau's inequalities for tournament scores and a short proof of a theorem on transitive subtournaments, *J. Graph Theory* **38** (2001), 244–254.

Given a digraph $G$ along with a weight function $w \in \mathbb{R}^{E(G)}$, a negative flow for $(G, w)$ is a function $f \in \mathbb{R}_+^{E(G)}$ such that $\langle f, w\rangle = \sum_{e\in E(G)} f(e)w(e) < 0$ and for each vertex $v \in V(G)$, $\sum_{e\in\delta^+(v)} f(e) = \sum_{e\in\delta^-(v)} f(e)$, where $\delta^+(v)$ is the set of outgoing arcs at $v$ and $\delta^-(v)$ the set of incoming arcs at $v$; a negative circuit for $(G, w)$ is a negative flow with minimum support (sometimes we just identify it with the support); and a potential function $p$ for $(G, w)$ is a function $p \in \mathbb{R}^{V(G)}$ such that for each $e \in E(G)$ it holds $p(y) - p(x) \leq w(e)$, where $x$ is the initial vertex of $e$ and $y$ the terminal vertex of $e$.

**Lemma 167** *(i)* $(G, w)$ *possesses either a potential function or a negative flow, but not both;*

*(ii)* $(G, w)$ *has a potential function if and only if it has no negative circuit.*

*Proof.* Let $A = \nabla_G$ be the transpose of the incidence matrix of the digraph $G$ (see 0). Farkas' Lemma (Lemma 159) reads that: For any $w \in \mathbb{R}^n$, either

$$\exists p \in \mathbb{R}^m, Ap \leq w$$

or

$$\exists f \in \mathbb{R}^n_+, f^\top A = 0, f^\top w < 0$$

but not both.

$(G, w)$ has a negative flow if and only if it has a negative circuit– Recall that a cycle is a sum of circuits. ∎

*Proof.* (of Theorem 158) Let $G$ be a (finite) digraph. For each $v \in V(G)$, we associate two variables $r_v$ and $\ell_v$. Note that $G$ is a Weiner digraph if and only if the following system of linear inequalities has a solution:

$$\begin{cases} r_u - \ell_v \leq -1 & \text{for each pair of vertices } u \neq v \text{ with} \\ & \text{an arc } e \in E(G) \text{ going from } u \text{ to } v, \\ -r_u + \ell_v \leq 0 & \text{for each pair of vertices } u \neq v \text{ without} \\ & \text{any arc } e \in E(G) \text{ going from } u \text{ to } v, \\ -r_u + \ell_u \leq 0 & \text{for each vertex } u \in V(G). \end{cases} \tag{44}$$

It happens that the solutions $r_v$'s and $\ell_v$'s for (44) are in one-to-one correspondence with the potential functions of the following weighted digraph $D$ with $V(D)$ given by

$$\{R_v, L_v : \; v \in V(G)\}$$

and $E(D)$ along with its weighting function enumerated as

$$\begin{cases} L_v R_u \text{ with weight } -1 & \text{for each } uv \in E(G), \\ R_u L_v \text{ with weight } 0 & \text{for each } uv \notin E(G), \\ R_u L_u \text{ with weight } 0 & \text{for each } u \in V(G). \end{cases}$$

It now follows from Lemma 167 that $G$ is a Weiner digraph if and only if $D$ has no negative cycle. To finish the proof, we show that each negative cycle of shortest length in $D$ corresponds to an induced subgraph **2+2** of $G$. ∎

A Weiner digraph whose interval representation only uses unit intervals is called a semiorder digraph. Semiorder digraphs are comparability digraphs of semiorders.

**Exercise 168** [*]*A digraph is a semiorder digraph if and only if it has no induced subdigraph isomorphic to either* **2+2** *or* **1+3**.

[*]Kenneth P. Bogart, Douglas B. West, A short proof that "Proper = Unit", *Discrete Math.* **201** (1999), 21–23.

Interval order, semiorder and hyperplane arrangement

Richard P. Stanley, An Introduction to Hyperplane Arrangements, `http://www-math.mit.edu/~rstan/arrangements/`

Let $u_n$ denote the number of nonisomorphic $n$-element semiorders and let $v_n$ denote the number of semiorders on $[n]$.

$u_n$ is just the number of labelled semiorder digraphs and $v_n$ the number of (nonisomorphic) semiorder digraphs.

$v_n$ is the number of regions of the hyperplane arrangement $\mathcal{J}_{1^n}$ of $\mathbb{R}^n$: $x_i = x_j \pm 1$, $1 \le i < j \le n$.* $u_n$ is the number of regions of $\mathcal{J}_{1^n}$ intersecting the region $x_1 < x_2 < \cdots < x_n$.

*Consider the set of intervals $I_i = [x_i - 1, x_i]$ and note that $I_i$ is totally on the left of $I_j$ if and only if $x_i < x_j - 1$.

**Theorem 169** $\frac{v(n)}{n!} = u(n) = \binom{2n}{n}/(n+1)$ *.

*Proof.* Given a unit interval representation model $M$, say, $I_i = [x_i - 1, x_i]$, $i \in [n]$, where $x_1 < x_2 < \cdots < x_n$, the corresponding semiorder digraph $G_M$ is completely determined by the set $A_M$ of maximal intervals $[i, j]$ such that $x_i - x_j < 1$ – they are just all the maximal independent sets of $G_M$. A set of intervals comes from a unit interval representation if and only if its intervals are pairwise incomparable and cover $[n]$. (Think of the 'if' direction!)

$A_M$ can be drawn on the diagonal $x_1 = x_2$. Each interval $[i, j]$ on the diagonal corresponds to the points $(j, i-1)$.† These points below the diagonal form the set $C_M$.

*This number is often denoted by $C_n$ and called the Catalan number. Up to Oct. 2005, Richard Stanley already provides 135 combinatorial interpretations of it. See `http://www-math.mit.edu/~rstan/ec/catadd.pdf`.

†The point $(j, i-1)$ 'covers' the interval $[i-1, j]$ on the diagonal.

$A_M$ is incomparable if and only if there is a lattice path from $(0,0)$ to $(n,n)$ never rising above the diagonal whose outer corners coincides with the set $C_M$.

Finally, we can use the the reflection principle of D. Andr$\acute{e}$ to count such lattice paths. It can also be counted by the Chung-Feller Theorem (Exercise 170). ∎

**Exercise 170** *Let $X_n$ be the set of all $\binom{2n}{n}$ lattice paths from $(0,0)$ to $(n,n)$ with steps $(0,1)$ and $(1,0)$. Define the excedance of a path $P \in X_n$ to be the number of $i$ such that at least one point $(i,i')$ of $P$ lies above the diagonal $x_1 = x_2$. Show that the number of paths in $X_n$ with excedance $j$ is independent of $j$.*

**Exercise 171** [*]*Let* $\mathcal{S}_n$ *denote the set of* $n(n-1)$ *hyperplanes* $x_i - x_j = 0, 1, 1 \le i < j \le n$ *in* $\mathbb{R}^n$. *Show that the number of connected regions of* $\mathbb{R}^n - \cup_{H \in \mathcal{S}} H$ *is the same as the number of spanning trees of* $K_{n+1}$. *(Hint: Exercise 52, Exercise 95)*

[*]Jianyi Shi, The Kazhdan-Lusztig cells in certain affine Weyl groups, *Lecture Notes in Mathematics* **1179**, Springer, Berlin, 1986.

Richard W. Hamming, The Art of Probability for Scientists and Engineers, Addison-Welsey, 1991.

*I believe a life in which you do not try to extend yourself regularly is not worth living – but it is up to you to pick the goals you believe are worth striving for.*

*If you do not work on important problems, then it is obvious you have little chance of doing important things.*

*The courage to continue is essential since great research often has long periods with no success and many discouragement.*

For a given set $M$ of objects (for which intersection makes sense), the intersection graph $G_M$ of this intersection model $M$ has $M$ as vertex set and two vertices are adjacent if the intersection of the corresponding objects is nonempty. Note that an intersection graph must be a simple graph, that is, loopless and having no multiple edges.

**Theorem 172** *Every simple graph has an intersection representation, namely is the intersection graph of some intersection model.*

*Proof.* Each vertex corresponds to the set of its incident edges.∎

Interval graphs are those simple graphs having an interval model. Generally, define chordal graphs to be those graphs having a subtree model. More precisely, a graph $G$ is chordal provided $G = G_M$ where $M$ is a set of subtrees of a given tree.

**Exercise 173** *Each interval graph is chordal.*

Not all graphs have a subtree intersection representation, not mentioning interval intersection representation. Another generalization of the interval model is the box model. A $d$-box is a set of the form $I_1 \times I_2 \times \cdots \times I_d$, $I_i \in \mathbb{R}$. For a graph $G$, the boxity of $G$ is the minimum $d$ such that $G$ is the intersection graph of some $d$-boxes.

**Exercise 174** *Each simple graph has a finite boxity.*

**Theorem 175** *The following statements are equivalent:*

*(i) $G$ is chordal;*

*(ii) Every cycle of $G$ with length no less than 3 has a chord;*

*(iii) $G$ has a perfect elimination ordering;*

*(iv) For each $v \in V(G)$, let $C_v$ be the set of maximal cliques of $G$ containing $v$. The intersection graph of the set of maximal cliques of $G$ is a tree and $C_v$ induces a subtree of this tree. Moreover, these $C_v$'s give a subtree model of $G$;*

*(v) Every minimal vertex separator induces a complete subgraph of $G$.*

Chordal graphs is one of the first classes of graphs to be recognized as being perfect. Its perfectness follows directly from Theorem 175 (iii).

Note that if a graph is not chordal, we can always make it chordal by adding some edges, in view of Theorem 175 (ii).

The treewidth of a graph $G = (V, E)$, denoted $tw(G)$, is defined to be $\min\{\omega(H) - 1 : \ H = (V, E') \text{ is chordal}, \ E \subseteq E'\}$.

**Exercise 176** *A simple graph has treewidth at most one if and only if it is a forest.*

**Exercise 177** *For any graph $G$ it holds $tw(G) \leq |V(G)| - 1$.*

A tree decomposition of a graph $G$ is a pair $(T, \mathcal{X})$ consisting of a tree $T$ and $\mathcal{X} = \{X_t : X_t \subseteq V(G), t \in V(T)\}$ * for which 1) $\cup_{t \in V(T)} X_t = V(G)$; 2) for each edge $uv \in E(G)$, there is $t \in V(T)$ such that $u, v \in X_t$; 3) For any $v \in V(G)$, the vertices $t \in V(T)$ satisfying $v \in X_t$ form a subtree of $T$.

Note that Theorem 175 (iv) says that chordal graphs are exactly those graphs $G$ possessing a special kind of tree decompositions such that each $X_t$ forms a clique of $G$.

The width of a tree decomposition is $tw(G, (T, \mathcal{X})) = \max\{|X_t| - 1 : t \in V(T)\}$.

**Theorem 178** $tw(G) = \min\{tw(G, (T, \mathcal{X})) : (T, \mathcal{X})$ *is a tree decomposition of* $G\}$.

*For each $t \in V(T)$, you can think of $X_t$ as a bag containing some vertices of $G$.

The tree decomposition concept and its measure of value treewidth was first introduced by Halin [*]. But its importance has been even more clear after Robertson and Seymour [†] reintroduced them independently in the context of graph minor research.

Tree-decomposition has played an important role in structural graph theory, in complexity theory and in practical computation. Tree-decomposition transfers the separation properties of its tree to the graph decomposed. An important reason for the growing interest in the concept of treewidth is that many NP-hard problems can be solved in linear time when restricted on input graphs with bounded treewidth – they are done often with a Dynamic Programming arising from the tree decompositions.

[*]R. Halin, $S$-functions for graphs, *J. Geometry* **8** (1976), 171–186.

[†]Neil Robertson, Paul D. Seymour, Graph minors II: Algorithmic aspects of tree-width, *Journal of Algorithms* **7** (1986), 309–322.

Trees are graphs with some very distinctive and fundamental properties. It is therefore legitimate to ask to what extent those properties can be transformed to more general tree-like graphs. Tree decomposition and treewidth are just the right concept for this tree-likeness and they play crucial role in the proof of the Graph Minor Theorem of Robertson and Seymour.

Our goal in this last chapter is a single theorem, one which dwarfs any other result in graph theory and may doubtless be counted among the deepest theorems that mathematics has to offer: *in every infinite set of graphs there are two such that one is a minor of the other*. This graph minor theorem, inconspicuous though it may look at first glance, has made a fundamental impact both outside graph theory and within. Its proof, due to Neil Robertson and Paul Seymour, takes over 500 pages. – Reinhard Diestel, *Graph Theory*, Springer, Second Edition, 1999.

R.G. Downey, M.R. Fellows, Parameterized Complexity, Springer-Verlag, New York, 1999.

Ton Kloks, Treewidth: Computations and Approximations, *Lecture Notes in Computer Science* **842** Springer-Verlag, Berlin, Heidelberg, 1994.

Klaus Truemper, Matroid Decomposition, Revised Edition, Academic Press, 1992. *

*Available at `http://www.emis.de/monographs/md/`

Some facts:

$\delta(G) \leq tw(G)$.

Graphs of treewidth at most $k$ are closed under taking minors.

The above two items gives $\chi(G) \leq tw(G) + 1$ *.

If $W$ is a clique in $G$, then a tree decomposition of $G$ has a bag $X_t$ with $W \subseteq X_t$. This is a consequence of the Helly property for trees †.

*A greedy procedure leads to the well-known relation $\chi(G) \leq \max_H \delta(H) + 1$.

†Each family of pairwise intersecting subtrees of a tree have a common vertex.

Cops-and-robbers game:

Fix a graph $G$. A cop can move slowly in helicopter and a robber can move infinitely fast along cop-free edges of the graph. The cop and the robber are always aware of the position of each other. If we have $tw(G) + 1$ cops in the graph, there is a procedure to capture the visible robber.

Haven, escape strategy and treewidth:

P.D. Seymour, R. Thomas, Graph searching, and a min-max theorem for tree-width, *J. Combin. Theory B* **58** (1993), 22–33.

Charles Semple, Mike Steel, *Phylogenetics*, Oxford Lecture Series in Mathematics and its Applications **24**, Oxford University Press, 2003.

## Graphs in Biology

A food web graph is an acyclic digraph $D = (V, A)$, where the sets $V$ and $A$ has the biological interpretation that $V$ is a set of species and $(v, w) \in A$ if $v$ feeds or preys on species $w$. The competition graph associated with a food web $D = (V, A)$, denoted $C(D)$, is the graph $(V, E)$, where $\{v, v'\} \in E$ if and only if there exists some $w \in V$ such that both $(v, w)$ and $(v', w)$ are elements of $A$. The edges of the competition graph displays which pairs of species are in competition for some shared food resource.

**Lemma 179** *Every competition graph has at least one isolated vertex; For any given graph $G$, we can obtain a competition graph from $G$ by adding no more that $|E(G)|$ isolated vertices.*

The minimum number of isolated vertices needed to add to make a graph into a competition graph is called the competition number of the graph. To determine the competition number of a general graph is NP hard.

**Exercise 180** *A chordal graph with at least one isolated vertex is a competition graph.*

Curiously, competition graphs that arise in biology are typically interval graphs, even though a randomly generated graph is unlikely to be an interval graph. This is of interest to biologists as it suggests that competition can often be modelled by a one-dimensional 'niche space'. It also raises an interesting question, namely to characterize exactly those acyclic digraphs whose competition graphs are interval graphs. Although there has been much work on this problem, it is still unsolved.

A central task in computational biology is to reconstruct genes based on information about short overlapping fragments of DNA. Here we think of DNA sequences as comprising of a string of letters chosen from the four letter alphabets $\{A, C, G, T\}$. Mathematically, to test whether the set of data obtained from the overlapping information about DNA is consistent with the hypothesis that the gene is a linear arrangement of letters is tantamount to asking whether a given graph is an interval graph.

A generalization of interval graphs for the problem of 'physical mapping' of DNA are probe interval graphs.

Two vertices are joined only when at least one of them is a probe and these two intervals intersect.

A pedigree is an acyclic digraph in which the vertex set is partitioned into two subsets $M$ and $F$ so that each vertex either has indegree zero or has exactly one incoming arc from a vertex in $M$ and exactly one incoming arc from a vertex in $F$. For a pedigree with bipartition $\{M, F\}$ of the vertex set, an augmented pedigree graph $G$ is a graph that can be obtained from its underlying graph by adding some edges joining a vertex in $M$ and one in $F$.

**Lemma 181** *For a pedigree with bipartition $\{M, F\}$ of the vertex set. The underlying graphs induced by $M$ and induced by $G$ are both forests.*

**Lemma 182** *The chromatic number of any augmented pedigree graph is at most* 4; *If an augmented pedigree graph is chordal, then its clique number is at most* 4 * *and so its treewidth is at most* 3.

*It is equal to its chromatic number, as chordal graphs are perfect.

These laws, taken in the largest sense, being Growth with Reproduction; Inheritance which is almost implied by reproduction; Variability from the indirect and direct action of the external conditions of life, and from use and disuse; a Ratio of Increase so high as to lead to a Struggle for Life, and as a consequence to Natural Selection, entailing Divergence of Character and the Extinction of less-improved forms. – Charles Darwin, *On the origin of species*, London, John Murray, 1859.

We possess no pedigrees or armorial bearings; and we have to discover and trace the many diverging lines of descent in our natural genealogies, by characters of any kind which have long been inherited. – Charles Darwin (1809-1882) gentleman naturalist http://www.newton.cam.ac.uk/programmes/PLG/

Isaac Newton Institute for Mathematical Sciences

Phylogenetics is the reconstruction and analysis of trees and networks to describe and understand the evolution of species, populations and individuals. It is widely used in molecular biology and other areas of classification (such as linguistics), and has both led to and benefited from the development of new mathematical, statistical and computational techniques. Although the foundations of phylogenetics were laid down many decades ago, it is currently experiencing an exciting renaissance due to the wealth and types of biological data that are now becoming available. This programme will bring together key researchers in phylogenetics and related areas to further develop this important area of mathematical biology.

The main themes that will be worked on during this programme are new data types in phylogenetics; modelling reticulate evolution; constructing large trees; probabilistic models of evolution; and phylogenetic combinatorics. These themes provide a rich source of mathematical problems in areas such as combinatorics, graph theory, probability theory, topology, and algebraic geometry. Solutions to these problems will provide new insights to questions that are central to contemporary evolutionary biology.

P. Buneman, The recovery of trees from measure of dissimilarity, in: *Mathematics in the Archaeological and Historical Sciences* (Eds. F.R. Hodson, D.G. Kendall, P. Tautu), Edinburgh University Press, Edinburgh, 1971, pp. 387–395.

H.-J. Bandelt, A.W.M. Dress, Reconstructing the shape of a tree from observed dissimilarity data, *Advances in Applied Mathematics* **7** (1986), 309–343.

J.-P. Barthélemy, From copair hypergraphs to median graphs with latent vertices, *Discrete Mathematics* **76** (1989), 9–28.

D. Gusfield, Efficient algorithms for inferring evolutionary trees, *Networks* **21** (1991), 19–28.

An $X$-tree $\mathcal{T}$ is an ordered pair $(T; \phi)$, where $T$ is a tree and $\phi$ is a map from $X$ to $V(T)$ such that each $v \in V(T)$ of degree at most two is in $\phi(X)$. An $X$-split is a partition of $X$ into two non-empty sets. The $X$-split with blocks $A$ and $B$ is denoted by $A|B$, or equivalently, by $B|A$. For each edge $e$ of an $X$-tree $\mathcal{T}$, $T - e$ contains two components whose vertex sets are, say $V_1$ and $V_2$, respectively, and we call $\phi^{-1}(V_1)|\phi^{-1}(V_2)$ the $X$-split of $\mathcal{T}$ corresponding to $e$. The requirement that all vertices having degree not greater than two lie in the image of $\phi$ is equivalent to the requirement that there are different splits corresponding to different edges. We refer to the set of these $|E(T)|$ $X$-splits of $\mathcal{T}$ as $\Sigma(\mathcal{T})$.

A pair of $X$-splits $A_1|A_2$ and $B_1|B_2$ are compatible provided that not all four sets $A_i \cap B_j$ are nonempty (and hence exactly one of them is empty!).

**Theorem 183 (Splits-Equivalence Theorem)** *Let $\Sigma$ be a collection of $X$-splits. Then, there is an $X$-tree $\mathcal{T}$ such that $\Sigma = \Sigma(\mathcal{T})$ if and only if the splits in $\Sigma$ are pairwise compatible. Moreover, if such an $X$-tree exists, then, up to isomorphism, $\mathcal{T}$ is unique.*

*Proof.* The necessity is obvious. We now suppose that $\Sigma$ is a pairwise compatible collection of $X$-splits and we use induction on $|\Sigma|$ to simultaneously prove the existence and the uniqueness of $\mathcal{T}$ such that $\Sigma = \Sigma(\mathcal{T})$. Our inductive proof for the existence of $\mathcal{T}$ indicates the so-called tree popping algorithm * of reconstructing an $X$-tree from its split system.

*C.A. Meacham, Theoretical and computational considerations of the compatibility of qualitative taxonomic characters, in: *Numerical Taxonomy*, NATO ASI Series, Vol. G1 (Ed. J. Felsenstein), Springer-Verlag, Berlin, 1983, pp. 304–314.

When $|\Sigma| = 0$, $\mathcal{T}$ can only be the unique one vertex $X$-tree.

Now assume $|\Sigma| > 0$ and the result holds for any smaller size split system. Choose arbitrarily $A|B \in \Sigma$. Let $\mathcal{T}' = (T', \phi')$ be the unique $X$-tree with $\Sigma(\mathcal{T}') = \Sigma - \{A|B\}$.

We say that a subgraph $(V', E')$ of $T$ is monochromatic if it holds either $\phi'^{-1}(V') \subseteq A$ or $\phi'^{-1}(V') \subseteq B$. Since each edge $e$ corresponds to a split of $\mathcal{T}'$, the compatibility assumption tells us that exactly one component of $T' - e$ is monochromatic. We orient the edges of $T'$ so that they directs towards the corresponding monochromatic components. Since $T'$ is a tree, there is a vertex all of whose incident edges are leaving it with the assigned orientation. This means that $v$ is a vertex such that each component of $T' - v$ is monochromatic.

We further show that such a vertex $v$ is indeed unique. Otherwise, we have another vertex $v'$ such that each component of $T' - v'$ is monochromatic. Then choose any edge $e$ on the path connecting $v$ and $v'$. We know that exactly one component of $T' - e$ is monochromatic, say the component containing $v$. This implies that the component of $T' - e$ including $v'$ is nonmonochromatic, and hence at least one component of $T' - v'$ is nonmonochromatic, a contradiction.

After getting the vertex $v$ as claimed above, the tree popping algorithm goes as follows. Replace $v$ with two new vertices $v_A$ and $v_B$ and attach the components of $T' - v$ that were incident with $v$ to the new vertices in such a way that those subtrees containing vertices in $\phi'(A)$ and $\phi'(B)$ are attached to $v_A$ and $v_B$, respectively.

Take the map $\phi : X \to V(T)$ satisfying

$$\phi(x) = \begin{cases} \phi'(x) & \text{if } \phi'(x) \neq v', \\ v_A & \text{if } \phi'(x) = v' \text{ and } x \in A \\ v_B & \text{if } \phi'(x) = v' \text{ and } x \in B. \end{cases}$$

Note that $A|B \notin \Sigma(\mathcal{T}')$ implies that if $v_A$ ($v_B$) has degree two or less in $T$ then $v_A \in \phi(A)$ ($v_B \in \phi(B)$). This says that $\mathcal{T} = (T, \phi)$ is an $X$-tree. Furthermore, it is easy to check that $\mathcal{T}$ is the unique $X$-tree with $\Sigma(\mathcal{T}) = \Sigma$,[*†] concluding the proof. ∎

[*] $\mathcal{T}$ is obtained from $\mathcal{T}'$ by the expansion of the edge $v_A v_B$ and $\mathcal{T}'$ is obtained from $\mathcal{T}$ by contracting the edge corresponding to the split $A|B$, namely $v_A v_B$.

[†] It is also easy to see that $\mathcal{T}$ is an $X$-tree from the fact that $|\Sigma(\mathcal{T})| = |\Sigma| = |E(T)|$.

Let $\Sigma = \{S_1, \cdots, S_n\}$ be a nonempty set of $X$-splits. Define $V(\Sigma)$ to be the sets of $n$-tuples $(A_1, \cdots, A_n)$ where $A_i$ is a block of the $X$-split $S_i$ and $A_i \cap A_j \neq \emptyset$ for $i, j = 1, \cdots, n$. We say that $(A_1, \cdots, A_n)$ and $(B_1, \cdots, B_n)$ disagree on $S_i$ if $A_i \neq B_i$. Let $E(\Sigma)$ consist of all pairs of elements in $V(\Sigma)$ that disagree on exactly one element of $\Sigma$. The Buneman graph on $\Sigma$, denoted $G(\Sigma)$, is the graph with $V(\Sigma)$ as its vertex set and with $E(\Sigma)$ as its edge set. Let $\phi_\Sigma$ be the canonical mapping from $X$ to $V(\Sigma)$ such that $\phi_\Sigma(x)$ is the unique element of $V(\Sigma)$ each of whose component contains $x$.

**Theorem 184** *The following are equivalent: (i) $\Sigma$ is pairwise compatible; (ii) $G(\Sigma)$ is a tree; (iii) $|V(\Sigma)| = |\Sigma| + 1$; (iv) $|E(\Sigma)| = |\Sigma|$.*

**Exercise 185** *If $\Sigma$ is pairwise compatible, then $\Sigma = \Sigma(\mathcal{T})$, where $\mathcal{T} = (G(\Sigma), \phi_{\Sigma})$. Compare this result with Theorem 175 (iv).*

**Exercise 186** * *Let $\Sigma$ be a nonempty set of $X$-splits. Denote by $\mathcal{I}(\Sigma)$ the collection of subsets of $\Sigma$ that are either pairwise incomparable or have cardinality at most one. Prove that $|V(\Sigma)| = |\mathcal{I}(\Sigma)|$ and $|E(\Sigma)| = \sum_{I \in \mathcal{I}(\Sigma)} |I|$.*

* A. Dress, M. Hendy, K. Huber, V. Moulton, On the number of vertices and edges of the Buneman graph, *Annals of Combinatorics* **1** (1997), 329–337.

For each pair of $X$-trees $\mathcal{T}$ and $\mathcal{T}'$, define $d(\mathcal{T}, \mathcal{T}') = |\Sigma(\mathcal{T}) \triangle \Sigma(\mathcal{T}')|$. We know from Theorem 183 that $d$ is a metric on the set of all $X$-trees, which we will call the splits metric.

Suppose that $\mathcal{T} = (T, \phi)$ is an $X$-tree. Let $e \in E(T)$ and let $u$ and $v$ be its two endpoints. Let $T/e$ be the tree obtained by contracting the edge $e$ and denote the new vertex arising from the identification of $u$ and $v$ as $v_e$. Define

$$\phi_e(x) = \begin{cases} \phi(x) & \text{if } \phi(x) \neq u, v, \\ v_e & \text{otherwise.} \end{cases}$$

Denote $(T/e, \phi_e)$ by $\mathcal{T}_e$. We say that $\mathcal{T}/e$ is obtained from $\mathcal{T}$ by contracting $e$ and $\mathcal{T}$ is obtained from $\mathcal{T}/e$ by an expansion of $e$.

**Exercise 187** [*]*Let $\mathcal{T}$ and $\mathcal{T}'$ be two $X$-trees. Then $d(\mathcal{T},\mathcal{T}')$ is equal to the minimum number $k$ for which there is a sequence $\mathcal{T}_0, \cdots, \mathcal{T}_k$ of $X$-trees such that $\mathcal{T}_0 = \mathcal{T}$, $\mathcal{T}_k = \mathcal{T}'$, and $\mathcal{T}_i$ is obtained from $\mathcal{T}_{i-1}$ be either a contraction or an expansion of an edge.*

[*]D.F. Robinson, L.R. Foulds, Comparison of phylogenetic trees, *Mathematical Biosciences* **53** (1981), 131–147.

## VII. Probabilistic method

Probabilistically thinking helps you do very complex double counting and extract important invariants, without using such language it will sometimes gets very messy and makes you lost. Probabilistic method is a powerful tool and it is a great loss of you if you only use it in your probability final exam.

**Exercise 188** *Use probabilistic method to prove Theorem 27.*

It is important for him who wants to discover not to confine himself to one chapter of science, but to keep in touch with various others. – Jacques Hadamard

Shannon identified information with surprise. He chose the negative of the $\log$ of the probability of an event as the amount of information you get when the event of probability $p$ happens.

Linearity of expectation: Balancing vector, Tur$\acute{a}$n's Theorem (Exercise 4)

Lov$\acute{a}$sz's sieve: Suppose that an experiment can fail if any one of $n$ bad events occurs. We want to know if there is a non-zero probability that the experiment will succeed. The Lovasz local lemma guarantees that an experiment will succeed with nonzero probability when the events are "almost independent". There exists a satisfying truth assignment for any instance of $k$-SAT for $k \geq 10$ in which each variable is contained in at most $2^{\frac{k}{2}}$ clauses.

Shannon's Theorem, Graph Entropy

Capacity of binary symmetric channel, encoding, decoding, Chernoff bound

Phase transition

Cristian S. Claude, Information and Randomness: An Algorithmic Perspective, 2nd Edition, Springer, 2002.

Randomness is a mathematical concept, not a physical one. – Richard W. Hamming, The Art of Probability for Scientists and Engineers, Addison-Welsey, 1991.

Every human activity, EXCEPT Mathematics, must come to an end. – Paul Erdös