10: 30 – 12:30 am, January 10, 2006, Shang Yuan 108

ACM Class Graph Theory Final Exam

Name: ____________ Student ID: ____________ Score: ____________

1. (10 marks) Prove that a saturated hydrocarbon with k carbon atoms has $2k+2$ hydrogen atoms.

2. (15 marks) Two distinct leaves of a tree are said to form a cherry if they are adjacent to a common vertex. Let T be a tree such that no degree two vertex of which is adjacent to a leaf. Show that if T has at least four leaves, then it has at least two disjoint pairs of cherries.

3. (20 marks) The De Bruijn digraph $B(d,n)$ has $\{a_1 \cdots a_n : \ a_i = 1, 2, \cdots, d\}$ as the vertex set and there is an arc from $a_1 \cdots a_n$ to $b_1 \cdots b_n$ if and only if $a_i = b_{i-1}$ for $i = 2, \cdots, n$. Let A be the adjacency matrix of $B(d,n)$.

   (i) Show that $A^n = J$, where J is the matrix of all ones;

   (ii) Compute the number of spanning arborescences of $B(2,6)$.

4. (15 marks) An irreflexive and transitive relation on a set is called a partial ordering relation. A linear ordering is a partial ordering in which every two elements are comparable. Show that every partial ordering is contained in a linear ordering.

5. (20 marks) A poset is a set together with a partial ordering relation on it.

   (i) Show that the comparability graph of a poset is perfect;

   (ii) Show that the incomparability graph of a poset is perfect.

6. (20 marks) Let G be a graph with n vertices. Denote by $AO(G)$ the set of acyclic orientations of G and $p_G(x)$ the chromatic polynomial of G. Please verify that $|AO(G)| = (-1)^n p_G(-1)$.

7. (20 marks) We have a one-way street with n parking spaces $1, 2, \cdots, n$. Cars $C_1, C_2, \cdots, C_n$ arrive in some order such that no two cars arrives simultaneously. Each car C_i has a favorite parking spot $f(i) \in \{1, 2, \cdots, n\}$ and uses the following rule for parking: Drive to the first empty space after the $(f(i) - 1)$th one to park and if no such position is available then leave the street to find another street to park. Show that whether or not all n cars can successfully park in the street is independent of the arriving order of these n cars.

8. (20 marks) Prove Farkas' Lemma: For any subspace L of $\mathbb{R}^n$, either there is $x \in L$ satisfying $x \geq 0$ and $x_1 > 0$, or there is $y \in L^{\perp}$ satisfying $y \geq 0$ and $y_1 > 0$, but not both.