2 – 4 pm, April 22, 2005, Room 1106, Mathematics Building

F071Z522 Algebraic Topology Final Exam

Open Book

Name: ____________ Student ID: ____________ Score: ____________

ONLY answer those questions (or subquestions) which you are sure your answer will be both clear and complete. A not-well-organized argument does no good to your final grades; Even worse, they may earn negative points for you. Note that there are a total of 140 points on the exam.

0. (10 marks) Let $X_n = \{A \in GL_n(\mathbb{C}) : \ AN = NA\}$, where

$$N = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{bmatrix}_{n\times n}.$$

Equip X_n with the subspace topology inherited from $\mathbb{C}^{n^2}$. Compute $\pi_1(X_n)$. (hint: Show that X_n is homotopic to the n-torus by diagonalizing the matrices in X_n.)

1. (10 marks) Prove that $(I \setminus \{0\}, \{\frac{1}{n} : \ n = 1, 2, \cdots\})$ has the homotopy extension property.

2. (20 marks) Determine the category of path connected covering spaces over S^1. (hint: Consider the orbit category of $\mathbb{Z}$.)

3. (10 marks) Let f be a continuous map from S^n to S^m. For $x \in S^n$, define $g(x) = f(-x)$ and $h(x) = -f(x)$. Show that $g \simeq h$.

4. (10 marks) Let A be a subspace of X. Suppose there is a homotopy $f_t : \ X \longrightarrow X$ such that $f_0 = Id_X$, $f_1(X) \subseteq A$, and $f_t(A) \subseteq A$ for all t. Show that the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

5. (10 marks) Let $f, g : \ S^1 \to S^1$ be continuous maps such that for all $x \in S^1$, $f(x)$ and $g(x)$ are not antipodal. Show $f \simeq g$. If in addition there is $x_0 \in X$ such that $f(x_0) = g(x_0)$, show that $f \simeq g \ rel \ x_0$.

6. (30 marks) Let $f : \ S^1 \to S^1$ be a continuous map. f is said to be odd (even) if $f(-x) = -f(x)$ ($f(-x) = f(x)$) for each $x \in S^1$.

   (i) Since $\pi_1(S^1) = \mathbb{Z}$, f induces a group homomorphism f_* from $\mathbb{Z}$ to $\mathbb{Z}$ under the fundamental group functor, say $f_*(t) = nt$ for $t \in \mathbb{Z}$. We call n the degree of f and use the notation $\deg f$ for it. Write p for the map from $\mathbb{R}$ to S^1 with $p(s) = \exp(2\pi i s)$ for $s \in \mathbb{R}$ and ϵ the restriction of p to I. Show that $g = f \circ \epsilon$ can be lifted to a continuous map $\widetilde{g} : \ I \to \mathbb{R}$ such that $\widetilde{g}(1) - \widetilde{g}(0) = \deg(f)$ and $p \circ \widetilde{g} = g$. Further prove that any two continuous maps $f_1, f_2 : \ S^1 \to S^1$ are homotopic if and only if $\deg(f_1) = \deg(f_2)$[1]. (hint: If $\widetilde{g_1}$ and $\widetilde{g_2}$ are homotopic with endpoints fixed, then f_1 and f_2 are homotopic.)

[1] This is a very special case of Hopf's Theorem which classifies maps from an n-manifold to the n-sphere up to homotopy in terms of their Brouwer degrees.

(ii) Prove that every even map has even degree. (hint: $\deg(f_1 f_2) = \deg(f_1)\deg(f_2)$.)

(iii) Let $p_2 : S^1 \to S^1$ be given be $p_2(x) = x^2$ for every $x \in S^1$. If it holds $f(x)^2 = f(-x)^2$ for each $x \in S^1$, then there is a continuous map $\underline{f} : S^1 \to S^1$ such that $p_2 \circ \underline{f} = \underline{f} \circ p_2$.

(iv) If $\deg f$ is even, then f lifts to a continuous map $\widetilde{f} : S^1 \to S^1$ such that $f = p_2 \circ \widetilde{f}$. (hint: Use the lifting criterion.)

(v) If f is an odd map with even degree, then both $\underline{f}$ and $\underline{\widetilde{f}}$ are well-defined and both f and $\underline{\widetilde{f}} \circ p_2$ are lifts of $\underline{f} \circ p_2$.

(vi) Deduce from (v) that every odd map has odd degree[2]. In particular, every odd map is not null-homotopic. (hint: f and $\underline{\widetilde{f}} \circ p_2$ are of opposite parity. Use the unique lifting theorem.)

(vii) Use (vi) to prove that there is no continuous antipodal map from S^2 to S^1. [3]

7. (10 marks) Let F be a continuous map from $I \times I$ to a space X. Set $\alpha(t) = F(0,t), \beta(t) = F(1,t), \gamma(s) = F(s,0)$ and $\delta(s) = F(s,1)$ for $s, t \in I$. Prove that $\delta \simeq \alpha^{-1}\gamma\beta \ rel\ \{0,1\}$.

8. (30 marks) For each part, circle `T` for True or `F` for False. No explanation required.

`T` `F` a. (I, A) has the homotopy extension property for $A = \{0\} \cup \{\frac{1}{n} : n = 1, 2, \cdots\}$.

`T` `F` b. If X is a path-connected and locally path-connected space with a finite fundamental group, then every continuous map $X \to S^1$ is null-homotopic.

`T` `F` c. The composition of two covering projections is a covering projection.

`T` `F` d. There exists a continuous map f from $S^1 \subseteq \mathbb{C}$ to itself satisfying $\mathrm{Card}(f^{-1}(1)) = 5$ and $\mathrm{Card}(f^{-1}(-1)) = 4$.

`T` `F` e. A continuous map from S^1 to S^1 is of odd degree if and only if it is homotopic to the odd map $z \to z^k$ for some odd k.

`T` `F` f. Given a set X, a collection of spaces $\{X_j\}_{j\in J}$ and maps $g_j : X_j \to X$ for $j \in J$, the topology $\mathcal{T}$ coinduced on X by $\{g_j\}_{j\in J}$ is characterized by the property that for any space Y, any $j \in J$ and any map $f : X \to Y$, the topology on Y is coinduced by f if and only if it is coinduced by $f \circ g_j$.

`T` `F` g. Any two paths in a path-connected space are homotopic to each other.

`T` `F` h. If $f : X \to Y$ is a quotient map and Id is the identity map from Z to Z, then $f \times Id$ is a quotient map from $X \times Z$ to $Y \times Z$.

`T` `F` i. A bijective quotient map is a homeomorphism.

`T` `F` j. For each odd map $f : S^1 \to S^1$, there exists $x \in S^1$ such that $\mathrm{Card}(f^{-1}(x))$ is an odd number.

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[2] A shorter proof of this assertion can be adapted from the proof of Theorem 1.10 in "A. Hatcher, Algebraic Topology, Cambridge University Press, 2002". But you are now required to give a proof along the indicated approach. For a generalization of this result, see Proposition 2B.6 in the book of Hatcher.

[3] This is equivalent to Theorem 1.10 of Hatcher's book. The general version of it is the so-called Borsuk-Ulam Theorem, for which various proofs and unexpected applications can be found in "J. Matoušek, Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry, Springer, 2003".