A simple multibody system on a discrete circle

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二〇一八年五月三日，叶卡捷琳堡

The Ekaterinburg seminar

The seminar indicated in the title of the article started its work in 1966. By that time several younger researchers had been grouped around the present writer at Ural State University.
– Lev N. Shevrin, The Ekaterinburg seminar "Algebraic Systems" : 50 years of activities, URAL MATHEMATICAL JOURNAL, Vol. 3, No. 1, (2017) 3–26.

What is this digraph?

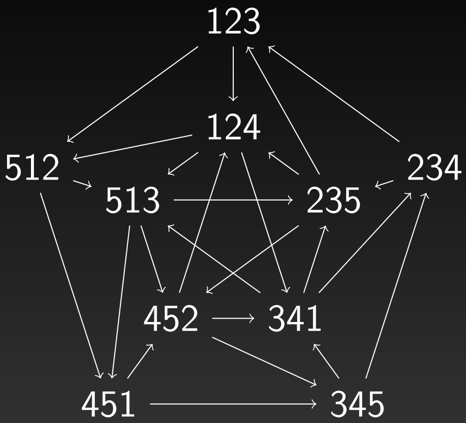
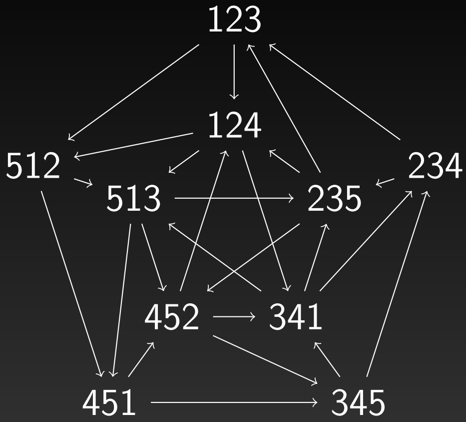


图: A Paranormal Day Gift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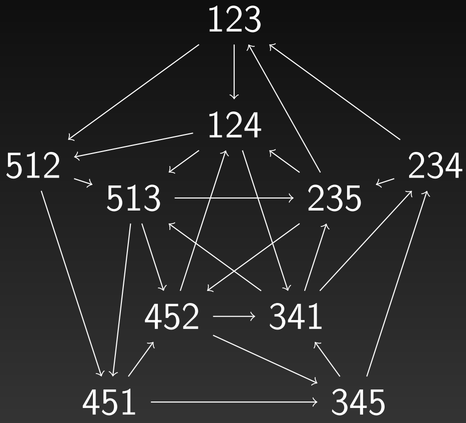
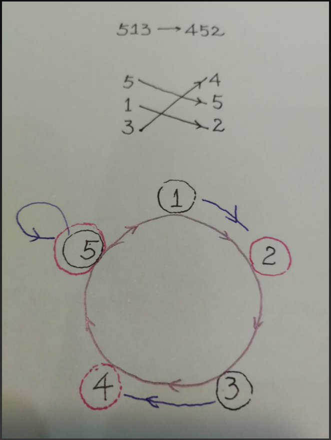
A roulette game





Let n be a positive integer and let $\mathbb{Z}_n$ denote the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of residue classes of integers modulo n, and let the integers $i \in \mathbb{Z}$ also denote their residue classes $i + n\mathbb{Z} \in \mathbb{Z}_n$ as long as no confusion can result. For each subset A of $\mathbb{Z}_n$, we call $i \in \mathbb{Z}_n$ a head of A provided $i \in A$ and $i - 1 \notin A$, and we call $i \in \mathbb{Z}_n$ a tail of A provided $i \in A$ and $i + 1 \notin A$. We call a subset of $\mathbb{Z}_n$ a proper interval if it has a unique head i and a unique tail j and we designate this proper interval by $[i, j]_n$. We often use $[j]_n$ for $[1, j]_n$.

The roulette game

We play a roulette game on a discrete circle with n slots. The n slots placed circularly can be naturally identified with $\mathbb{Z}_n$. We choose a positive integer k satisfying $1 < k < n$ and put k indistinguishable balls on k slots of the circle. Suppose that the slots of the k balls are read as $a_1, \ldots, a_k \in \mathbb{Z}_n$ such that the interval with head a_i and tail a_{i+1} contains exactly two balls, one at slot a_i and one at slot a_{i+1}, for $i = 1, \ldots, k-1$. In one step movement, the ball at slot a_i can move to any slot from the interval $[a_i, a_{i+1} - 1]_n$ with equal probability.

The k-body problem on an n-circle

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The matrix of the game

We study the linear algebra around the above simple multibody problem on a discrete circle.

We use $\mathrm{R}_{n,k}$ for the linear map from $\mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$ to itself that sends $A \in \binom{\mathbb{Z}_n}{k}$ to $\sum B$ where B runs through all out-neighbors of A in the digraph $\mathcal{R}_{n,k}$; we also use $\mathrm{R}_{n,k}$ for the matrix of the corresponding linear map with respect to the basis $\binom{\mathbb{Z}_n}{k}$.

$\mathrm{R}_{n,k}(A, B) = 1$ if and only if A can move to B in one step in the roulette game.

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More details to demonstrate

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- When k is even, we can determine all facets of the cone generated by the rows of $\mathrm{R}_{n,k}$.
- We show that $\mathrm{R}_{n,k}$ has rank $\binom{n}{k}$ when k is even and $\binom{n-1}{k-1}$ when k is odd.

Assume that $1 < k < n$. For $A \in \binom{\mathbb{Z}_n}{k-1}$ and $B \in \binom{[n-1]_n}{k}$, put

$$f_n(A) := \sum_{j \in \mathbb{Z}_n \setminus A} (-1)^{|[j]_n \cap A|} (A \cup \{j\}) \in \mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$$

and

$$g_n(B) := B + \sum_{j \in B} (-1)^{|[j]_n \cap B|} ((B \setminus \{j\}) \cup \{n\}) \in \mathbb{R}^{\binom{\mathbb{Z}_n}{k}}.$$

For $C = \{c_1, \ldots, c_k\} \in \binom{\mathbb{Z}_n}{k}$ and $\epsilon \in (\mathbb{Z}_n)^C$, let

$$C_\epsilon := \{c_1 - \epsilon_{c_1}, \ldots, c_k - \epsilon_{c_k}\} \in \left(\!\!\binom{\mathbb{Z}_n}{k}\!\!\right)$$

and

$$h_n(C) := \sum_{\epsilon \in \{0,1\}^C, C_\epsilon \in \binom{\mathbb{Z}_n}{k}} (-1)^{\sum_{i \in C} \epsilon_i} C_\epsilon \in \mathbb{R}^{\binom{\mathbb{Z}_n}{k}}.$$

We view $\mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$ as an Euclidean space with $\binom{\mathbb{Z}_n}{k}$ as a standard normal basis and we write $\langle \cdot, \cdot \rangle$ for the corresponding inner product. The next result claims that $\mathrm{R}_{n,k}$ is nonsingular when k is even; the columns of the inverse matrix of $\mathrm{R}_{n,k}$ can be read from h_n when k is even.

Theorem. When k is even and $1 < k < n$, $\langle \mathrm{R}_{n,k}(A), h_n(C) \rangle = 2\delta_{A,C}$ for all $A, C \in \binom{\mathbb{Z}_n}{k}$.

For any $k > 1$, the gap of a state $C \in \binom{\mathbb{Z}_n}{k}$, denoted by $\|C\|$, is given by

$$\|C\| := \min_{(a,b)\in C\times C, a\neq b} |[a,b]_n| - 1.$$

Let $\mathrm{R}_{n,k,t}$ denote the submatrix of $\mathrm{R}_{n,k}$ obtained by removing all rows indexed by $\{C : \|C\| < t\}$. Note that $\mathrm{R}_{n,k,1} = \mathrm{R}_{n,k}$.

Corollary. When k is even and $1 < k < n$, $\varphi \in \mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$ is a nonnegative linear combination of the rows of $\mathrm{R}_{n,k,t}$ if and only if its inner product with $h_n(C)$ is nonnegative when $\|C\| \geq t$ and is 0 when $\|C\| < t$ for all $C \in \binom{\mathbb{Z}_n}{k}$.

When $(k,t) = (2,1)$, the previous Corollary reduces to the characterization of Kalmanson matrix (CFT96, Theorem 33); when $(k,t) = (2,2)$, Corollary 1 reduces to the characterization of Kalmanson metrics (CFT96, Theorem 31) (DR97, Theorem 2.6) (CF98, Theorem) (KHP13, Theorem 5.2) (T18,Corollary 9).

Corollary. A function $d \in \mathbb{R}^{\left(\!\binom{\mathbb{Z}_n}{2}\!\right)}$ is a nonnegative combination of split metrics if and only if it holds

$$d(a_1,a_2)+d(a_1+1,a_2+1) \geq d(a_1+1,a_2)+d(a_1,a_2+1)$$

for all $\{a_1,a_2\} \in \binom{\mathbb{Z}_n}{2}$, and $d(A) = 0$ for all $A \in \left(\!\binom{\mathbb{Z}_n}{2}\!\right) \setminus \binom{\mathbb{Z}_n}{2}$.

CFT96 George Christopher, Martin Farach and Michael Trick. The structure of circular decomposable metrics. In *Algorithms—ESA '96 (Barcelona)*, volume 1136 of *Lecture Notes in Comput. Sci.*, pages 486–500. Springer, Berlin, 1996.

DR97 V. M. Demidenko and R. Rudolf, A note on Kalmanson matrices, *Optimization*, 40:285–294, 1997.

CF98 Victor Chepoi and Bernard Fichet. A note on circular decomposable metrics. *Geom. Dedicata*, 69(3):237–240, 1998.

KHP13 Aaron Kleinman, Matan Harel and Lior Pachter. Affine and projective tree metric theorems. *Annals of Combinatorics*, 17(1):205–228, 2013.

T18 Jonathan Terhorst. The Kalmanson complex. arXiv:1102.3177v3. January 9, 2018.

Theorem. Assume that k is odd and $1 < k < n$. Then,

$$\left\{ f_n(A) : \; A \in \binom{[n-1]_n}{k-1} \right\}$$

forms a basis of $\operatorname{Im} \mathrm{R}_{n,k}$, while both

$$\left\{ g_n(B) : \; B \in \binom{[n-1]_n}{k} \right\}$$

and

$$\left\{ h_n(C) : \; C \in \binom{[n-1]_n}{k} \right\}$$

are bases of $\operatorname{Ker} \mathrm{R}_{n,k}$. Moreover, $\operatorname{Im} \mathrm{R}_{n,k}$ and $\operatorname{Ker} \mathrm{R}_{n,k}$ are orthogonal complements of each other in $\mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$.

A lift of $\binom{\mathbb{Z}_n}{k}$

Let $\sigma_k(i) = i + 1$ for all $i \in \mathbb{Z}_k$. For any $\sigma \in \mathbb{Z}_k^{\mathbb{Z}_k}$ and $a \in (\mathbb{Z}_n)^{\mathbb{Z}_k}$, $\sigma^*(a)$ is the map in $(\mathbb{Z}_n)^{\mathbb{Z}_k}$ which sends $i \in \mathbb{Z}_k$ to $a(\sigma(i))$.

Let

$$X_{n,k} := \{a \in (\mathbb{Z}_n)^{\mathbb{Z}_k} : \sum_{i=1}^{k} |[a(i), a(i+1)-1]_n| = n\}.$$

$X_{n,k}$ has $k\binom{n}{k}$ elements which can be partitioned into $\binom{n}{k}$ orbits of σ_k^*. These orbits of σ_k^* can be naturally identified with $\binom{\mathbb{Z}_n}{k}$.

The roulette game with distinct balls

Assume that we color the k balls with colors from $\mathbb{Z}_n$ so that the interval whose head contains the ball with color i and whose tail contains the ball with color $i+1$ just contains these two balls. The corresponding Markov chain now has $X_{n,k}$ as its state set.

The roulette game with distinct balls, Contd.

Let $\mathrm{M}_{n,k}$ be the linear map from $\mathbb{R}^{X_{n,k}}$ to itself such that $\mathrm{M}_{n,k}(a) = \sum b$ where b runs through all elements of $X_{n,k}$ such that $b(i) \in [a(i), a(i+1)-1]_n$ for $i \in \mathbb{Z}_k$.

Theorem. $\dim \mathrm{Im}(\mathrm{M}_{n,k}) = (k-1)\binom{n}{k}$ and $\dim \mathrm{Ker}(\mathrm{M}_{n,k}) = \binom{n}{k}$.

For $a \in X_{n,k}$ and $\epsilon \in (\mathbb{Z}_n)^{\mathbb{Z}_k}$, let $a_\epsilon \in (\mathbb{Z}_n)^{\mathbb{Z}_k}$ be the map such that

$$a_\epsilon(i) = a(i) - \epsilon_i$$

for all $i \in \mathbb{Z}_k$, and let

$$h'_n(a) := \sum_{\epsilon \in \{0,1\}^{\mathbb{Z}_k}, a_\epsilon \in X_{n,k}} (-1)^{\sum_{1 \leq i \leq k} \epsilon_i} a_\epsilon \in \mathbb{R}^{X_{n,k}}.$$

Theorem.

- We have

$$\{a+(-1)^k\sigma_k^*(a) : \; a \in X_{n,k}, n \notin [a(1), a(2)-1]_n\}$$

as a basis of $\mathrm{Im}(\mathrm{M}_{n,k})$.

- We have

$$\Big\{ \sum_{j=1}^{k} (-1)^{(k+1)(j-1)} h_n'((\sigma_k^j)^*(a)) : \\ a \in X_{n,k}, n \in [a(k), a(1)-1]_n \Big\}$$

as a basis of $\mathrm{Ker}(\mathrm{M}_{n,k})$.

An endomorphism of $\wedge^k(\mathbb{R}^{\mathbb{Z}_n})$

Let us consider the kth exterior power $\wedge^k(\mathbb{R}^{\mathbb{Z}_n})$.

Suppose that $a_1, \ldots, a_k \in \mathbb{Z}_n$ are k different elements such that the interval with head a_i and tail a_{i+1} contains exactly two of these k elements, namely a_i and a_{i+1} for $i = 1, \ldots, k-1$.

Define $\mathrm{E}_{n,k}(a_1 \wedge \cdots \wedge a_k) = (\sum_{a \in [a_1, a_2 - 1]} a) \wedge \cdots \wedge (\sum_{a \in [a_{k-1}, a_k - 1]} a) \wedge (\sum_{a \in [a_k, a_1 - 1]} a)$.

This induces a well-defined linear map $\mathrm{E}_{n,k}$ from $\wedge^k(\mathbb{R}^{\mathbb{Z}_n})$ to itself.

Let $\Theta_{n,k}$ be the linear map from $\mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$ to $\bigwedge^k \mathbb{R}^{\mathbb{Z}_n}$ such that for all $C \in \binom{\mathbb{Z}_n}{k}$, $\Theta_{n,k}(C) := c_1 \wedge \cdots \wedge c_k$, where $C = \{c_1, \ldots, c_k\}$ and $c_i \in [c_{i+1} - 1]_n$ for all $i = 1, \ldots, k-1$.

Given $C \in \binom{\mathbb{Z}_n}{k}$, define $h''_n(C)$ to be

$$\sum_{\epsilon \in \{0,1\}^C, C_\epsilon \in \binom{\mathbb{Z}_n}{k}} (-1)^{(k-1)\mathbb{1}_{n \in C, \epsilon_n = 1} + \sum_{i \in C} \epsilon_i} \Theta_{n,k}(C_\epsilon) \in \wedge^k \mathbb{R}^{\mathbb{Z}_n}.$$

Recall that for all $A \in \binom{[n-1]_n}{k-1}$ and $B \in \binom{[n-1]_n}{k}$, it holds

$$f_n(A) := \sum_{j \in \mathbb{Z}_n \setminus A} (-1)^{|[j]_n \cap A|}(A \cup \{j\}) \in \mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$$

and

$$g_n(B) := B + \sum_{j \in B} (-1)^{|[j]_n \cap B|}((B \setminus \{j\}) \cup \{n\}) \in \mathbb{R}^{\binom{\mathbb{Z}_n}{k}}.$$

Grassmann's roulette

Theorem. Both $\{\Theta_{n,k} \circ f_n(A) : \ A \in \binom{[n-1]_n}{k-1}\}$ and $\{\mathrm{E}_{n,k} \circ \Theta_{n,k}(A \cup \{n\}) : \ A \in \binom{[n-1]_n}{k-1}\}$ are bases of $\operatorname{Im} \mathrm{E}_{n,k}$; while both $\{\Theta_{n,k} \circ g_n(B) : \ B \in \binom{[n-1]_n}{k}\}$ and $\{h''_n(C) : \ C \in \binom{[n-1]_n}{k}\}$ are bases of $\operatorname{Ker} \mathrm{E}_{n,k}$.

Define an inner product on $\bigwedge^k \mathbb{R}^n$ so that $\{\Theta_{n,k}(C) : \ C \in \binom{\mathbb{Z}_n}{k}\}$ becomes an orthonormal basis of $\bigwedge^k \mathbb{R}^n$. For all $C, C' \in \binom{\mathbb{Z}_n}{k}$, it holds that

$$\langle \mathrm{E}_{n,k}(\Theta_{n,k}(C)), h_n''(C') \rangle = 0.$$

This implies that $\operatorname{Im} \mathrm{E}_{n,k}$ and $\operatorname{Ker} \mathrm{E}_{n,k}$ are orthogonal complements of each other.

h, h' and h''

Define two linear maps $\Xi_R : \mathbb{R}^{X_{n,k}} \to \mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$ and $\Xi_E : \mathbb{R}^{X_{n,k}} \to \bigwedge^k \mathbb{R}^{\mathbb{Z}_n}$ such that

$$\begin{cases} \Xi_R(a) & := \{a(1), \ldots, a(k)\}, \\ \Xi_E(a) & := a(1) \wedge \cdots \wedge a(k), \end{cases}$$

for $a \in X_{n,k}$.

Then for all $a \in X_{n,k}$, it holds

$$h_n \circ \Xi_R(a) = \Xi_R \circ h'_n(a),$$
$$h''_n \circ \Xi_R(a) = \Xi_E \circ h'_n(a).$$

The key identity

Lemma. For any $a \in X_{n,k}$, it holds

$$(1 + (-1)^k \sigma_k^*)(a_{\{1\}^{\mathbb{Z}_k}}) = \mathrm{M}_{n,k} \circ h_n'(a).$$

Applying Ξ_R and Ξ_E to the equality above, we obtain two immediate consequences.

Corollary. For all $C \in \binom{\mathbb{Z}_n}{k}$, it holds

$$\begin{aligned} (1 + (-1)^k) C_{\{1\}^C} &= \mathrm{R}_{n,k} \circ h_n(C); \\ 0 &= \mathrm{E}_{n,k} \circ h_n''(C). \end{aligned}$$

Inner product

Let us consider an inner product (positive semi-definite symmetric bilinear form) $\lfloor , \rfloor$ in $\mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$ such that $\lfloor A, A' \rfloor := |A \cap A'|$ for $A, A' \in \binom{\mathbb{Z}_n}{k}$. This induces a new inner product $\langle , \rangle_R$ such that $\langle A, A' \rangle_R := \lfloor \mathrm{R}_{n,k}(A), \mathrm{R}_{n,k}(A') \rfloor$, for $A, A' \in \binom{\mathbb{Z}_n}{k}$. Put $A = \{a_1, \ldots, a_k\}$ and $A' = \{a'_1, \ldots, a'_k\}$, and we find that $\langle A, A' \rangle_R$ equals

$$\prod_{i=1}^{k} |[a_i, a_{i+1} - 1]_n| \prod_{j=1}^{k} |[a'_j, a'_{j+1} - 1]_n|$$

$$\Big(\sum_{i=1}^{k} \sum_{j=1}^{k} \frac{|[a_i, a_{i+1} - 1]_n \cap [a'_j, a'_{j+1} - 1]_n|}{|[a_i, a_{i+1} - 1]_n| \cdot |[a'_j, a'_{j+1} - 1]_n|}\Big).$$

The famous proof of Katona on the EKR theorem makes use of the technique of averaging over all circular arrangements. Is the inner product introduced above for any specific cyclic arrangement useful for getting anything in extremal set theory?

Stirling numbers

Stirling numbers of the first kind $s_{n,k}$:
$(-1)^{n-k}s(n,k)$ counts the number of permutations of n elements with k disjoint cycles.

Stirling numbers of the second kind $S(n,k)$: it counts the number of ways to partition a set of n elements into k nonempty subsets.

Stirling numbers of the third kind $L(n,k)$ (Lah number): it counts the number of partitions of n elements into k non-empty linearly ordered sets (chains).

Change of basis formula

falling factorial $(x)_n = x(x-1)\cdots(x-n+1)$;
rising factorial $x^{(n)} = x(x+1)\cdots(x+n-1)$.

$\{x^n\}$, $\{(x)_n\}$ and $\{x^{(n)}\}$ are three bases of the linear space of all polynomials in the variable x. The three bases are connected by Stirling numbers.

$x^n = \sum_{k=0}^{n} S(n,k)(x)_k, \quad (x)_n = \sum_{k=0}^{n} s(n,k)x^k$.

$x^n = \sum_{k=0}^{n}(-1)^{n-k}S(n,k)x^{(k)}, \quad x^{(n)} = \sum_{k=0}^{n}(-1)^{n-k}s(n,k)x^k$.

$(x)_n = \sum_{k=0}^{n}(-1)^{n-k}L(n,k)x^{(k)}, \quad x^{(n)} = \sum_{k=0}^{n} L(n,k)(x)_k$.

Can we get anything parallel to the analytic/combinatorial theory of Stirling numbers in the setting of our multibody dynamics problem on discrete circles?

The study of $\mathrm{R}_{n,2}$ is related to trees, consecutive-ones property, circular split systems, Gromov product (Farris transform), Kalmanson matrices and Robinsonian matrices. In this ongoing research, we intend to see what happens for general k and if that helps to do cluster analysis and to develop diversity decomposition theory (compared with split decomposition theory).

A six year old seminar at SJTU

We maintain an SJTU combinatorics seminar website since 2012:

Combinatorics Seminar at SJTU

A better report for this ongoing research should be given by Chengyang Qian there and also in G2R2.

THANK YOU! 谢谢!