From single-variable graph theory to multivariable graph theory:

Expansion properties

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Looking at the old facts in a new way

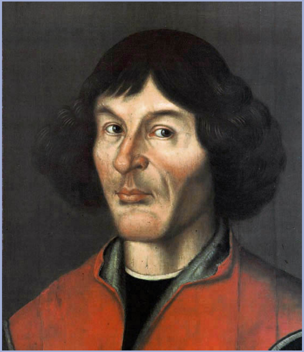

Figure: Nicolaus Copernicus, 19 February 1473 – 24 May 1543

https://en.wikipedia.org/wiki/Nicolaus_Copernicus

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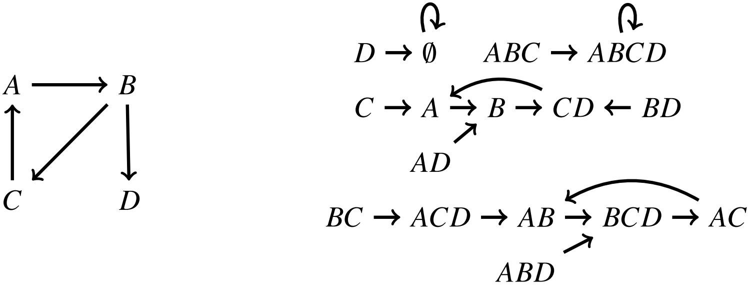
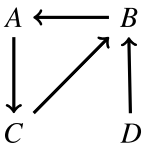


Figure: A digraph f on $\{A, B, C, D\}$ and its phase space $\mathcal{PS}_f$.

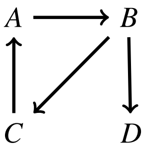
The phase space consists of those limit cycles together with the transient parts (in-trees attached to the limit cycles).

Reversal

To reflect a possible time reversal, we define the reversal of a digraph f on K to be the digraph $\overleftarrow{f}$ on K such that $i \in f(j)$ if and only if $j \in \overleftarrow{f}(i)$ for all $i, j \in K$.



$\overleftarrow{f}$



f

The correspondence between limit cycles in $\mathcal{PS}_f$ and $\mathcal{PS}_{\overleftarrow{f}}$

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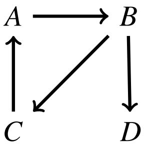
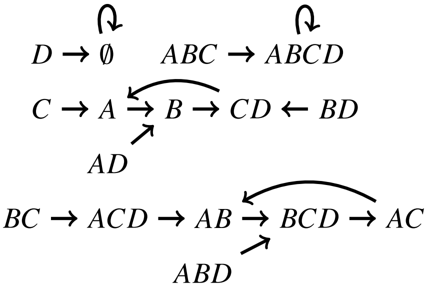


$\mathcal{PS}_f$



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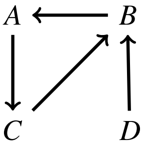
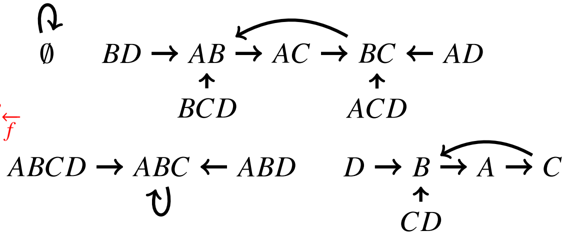


$\mathcal{PS}_{\overleftarrow{f}}$



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Let K be a set and t a positive integer. A digraph f on K in t variables is a map from K^t to 2^K.

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$$\mathrm{M}_f(A_1, \ldots, A_t) = (A_2, \ldots, A_{t-1}, \cup_{k_1 \in A_1, \ldots, k_t \in A_t} f(k_1, \ldots, k_t)).$$

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The last entry of M_f corresponds to the $\underbrace{K \times \cdots \times K}_{t+1}$ Boolean array whose $(k_1, \ldots, k_t, k_{t+1})$th entry is 1 if and only if $k_{t+1} \in f(k_1, \ldots, k_t)$.

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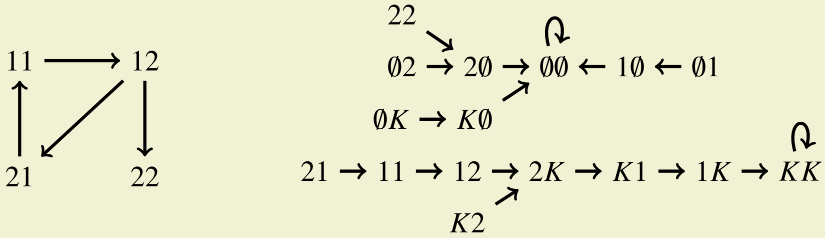

Figure: The De Bruijn form Γ_f and the phase space $\mathcal{PS}_f$ of a 2-variable digraph f on $K = [2] = \{1, 2\}$.

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One can also think of Γ_f as the particle version of f and $\mathcal{PS}_f$ as the wave version of f. Both versions encode full information about f in some way.

http://fossbytes.com/wp-content/uploads/2015/03/light-wave-particle-duality-.jpg

Primitive digraph

For $A = (A_1, \ldots, A_t) \in (2^K)^t$ and $B = (B_1, \ldots, B_t) \in (2^K)^t$, we write $A \preceq B$ provided $A_i \subseteq B_i$ for $i = 1, \ldots, t$.

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Let f be a t-variable digraph on K. It is clear that every vacant element of $(2^K)^t$ will reach $\emptyset^t$ in $\mathcal{PS}_f$ eventually.

We call f primitive provided every element from K^t will reach K^t in $\mathcal{PS}_f$, equivalently, every non-vacant element will go to the largest element K^t in $\mathcal{PS}_f$ and every vacant element will go to the minimum one $\emptyset^t$ in $\mathcal{PS}_f$.

A primitive 3-variable digraph

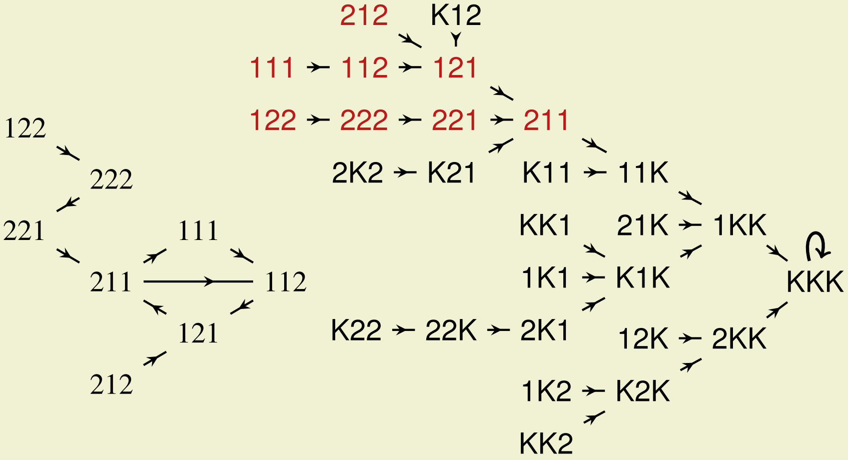

Figure: The De Bruijn form and part of the phase space induced by non-vacant vertices of a 3-variable digraph on $K = [2]$. This digraph is not primitive in the sense of Chang-Pearson-Zhang (SIMAX, 2011).

Hitting index

Let f be a t-variable digraph on K. For $a, b \in K^t$, we define $\mathcal{HI}_f(a,b)$ to be the set

$$\{n > 0 : \ b \leq \mathrm{M}_f^n(a) \in (2^K)^t\} = \{n > 0 : \ b \in \mathrm{M}_f^n(a) \in (2^K)^t\}$$

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For any $a, b \in K^t$ with $a \neq b$, the distance from a to b in f is defined to be $\mathrm{Dist}_f(a,b) = \min \mathcal{HI}_f(a,b)$.

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and call it the set of hitting indices of f from a to b.

For any $a, b \in K^t$ with $a \neq b$, the distance from a to b in f is defined to be $\mathrm{Dist}_f(a,b) = \min \mathcal{HI}_f(a,b)$. The diameter of f, denoted by $\mathrm{Dia}(f)$, is $\max_{a \neq b} \mathrm{Dist}_f(a,b)$.

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Understanding several from one?

Sometimes, this can be misleading.

A recent conjecture (Yuan-He-You, LAA, 2015) on characterizing primitive tensors in terms of irreducibility and local primitivity does not hold.

Large diameter

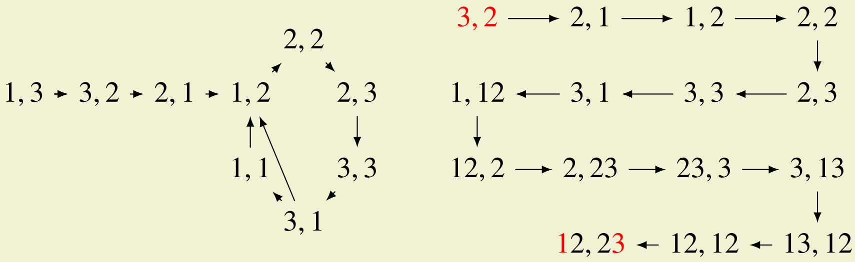


Figure: The De Bruijn form Γ_f of a 2-variable digraph f and part of the phase space $\mathcal{PS}_f$.

It holds

$$\mathrm{Dia}(f) = \mathrm{Dist}_f(32, 13) = 14 > 8 = 3^2 - 1 = |K|^t - 1 = |K^t| - 1.$$

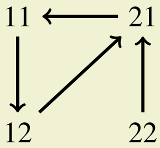
Let f be a digraph on K in t variables. The reversal of f, denoted by $\overleftarrow{f}$, is the digraph on K in t variables such that

$$k_1 \cdots k_t \to k_2 \dots k_{t+1}$$

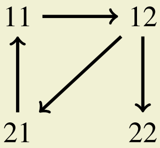
is an arc in Γ_f if and only if

$$k_t \dots k_1 \leftarrow k_{t+1} \cdots k_2$$

is an arc in $\Gamma_{\overleftarrow{f}}$.



$\Gamma_{\overleftarrow{f}}$



Γ_f

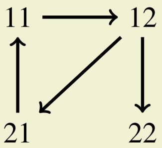
Time in the mirror

Figure: https://en.wikipedia.org/wiki/Mirror

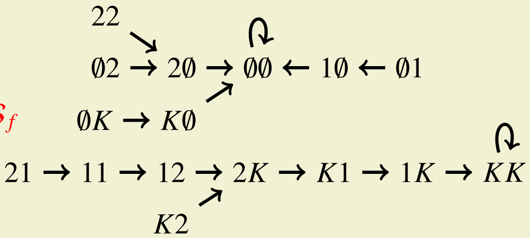
Though the relationship between $\Gamma_{\overleftarrow{f}}$ and Γ_f is easy to understand, it seems not so easy to see the relationship between the dynamical behaviour of $\mathrm{M}_{\overleftarrow{f}}$ and M_f, indicating the magic of the arrow of time.

A 2-variable digraph and its reversal: Different numbers of limit cycles

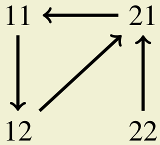
Γ_f



$\mathcal{PS}_f$



$\Gamma_{\overleftarrow{f}}$



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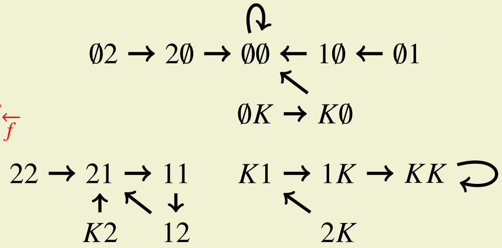

Magic mirror

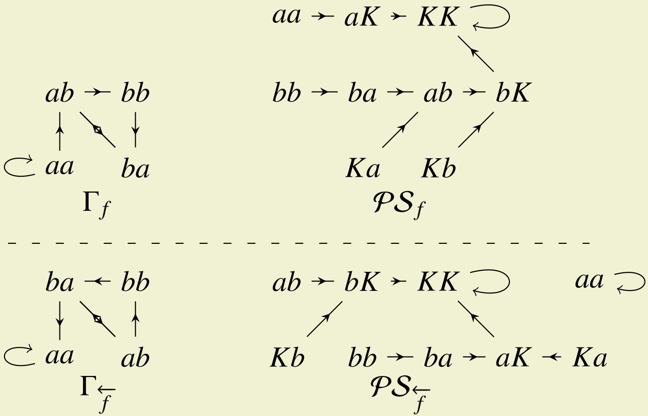

Figure: The digraph f on $K = \{a, b\}$ is primitive while its reversal $\overleftarrow{f}$ is not strongly connected. We do not include those vacant vertices when displaying the phase spaces.

A symmetric 2-variable digraph

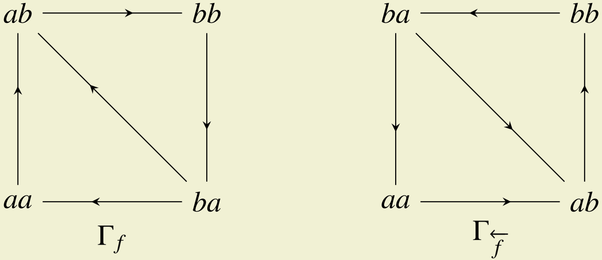

Figure: $f = \overleftarrow{f}$.

We call f symmetric provided f coincides with $\overleftarrow{f}$.

Let f be a strongly connected t-variable digraph on K.

Lemma

For all $a, b \in K^t$,

$$\gcd(\mathcal{HI}_f(a, a)) = \gcd(\mathcal{HI}_f(b, b)).$$

Periodic classes

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We define the period of f to be $\mathrm{per}(f) = \gcd(\mathcal{HI}_f(a,a)), a \in K^t$.

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We define the period of f to be $\operatorname{per}(f) = \gcd(\mathcal{HI}_f(a,a)), a \in K^t$.

We say $a, b \in K^t$ are equivalent for f provided $\gcd(\mathcal{HI}_f(a,b))$ is a multiple of $\operatorname{per}(f)$. Each such equivalent class for f is called a periodic class of f.

Periodic classes, Cont'd

Let f be a strongly connected t-variable digraph on K with $\mathrm{per}(f) = p$. Then the periodic classes of f can be enumerated as $C_1, \ldots, C_p$ such that the following hold:

(i) $(C_1, \ldots, C_p)$ is a longest cycle in $\mathcal{PS}_f$;

(ii) $\gcd(\mathcal{HI}(a,b)) \equiv j - i \pmod{p}$ for $a \in C_i$ and $b \in C_j$;

(iii) there are p nonempty subsets $A_1, \ldots, A_p$ of K, which are not necessarily distinct, such that $C_i = A_i \times A_{i+1} \times \cdots \times A_{i+t-1} \subseteq K^t, i \in [p]$, where the subscripts are calculated modulo p.

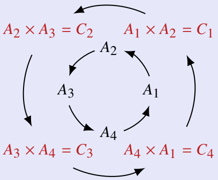

Figure: Periodic classes of a 2-variable strongly connected digraph.

Set of periods

Let $\mathcal{P}_{t,k}$ be the set of positive integers which can be the period of a t-variable strongly connected digraph on $[k]$ and let $\mathcal{P}_t = \cup_{k>0}\mathcal{P}_{t,k}$.

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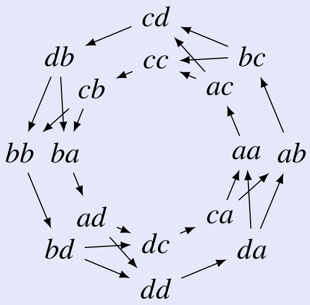
It is known that $\mathcal{P}_t \supseteq \{1, 2^t, 3^t, \ldots\}$. It is also known that $\mathcal{P}_{2,4} = \{1, 4, 8, 9, 10, 11, 12, 14, 16\}$.



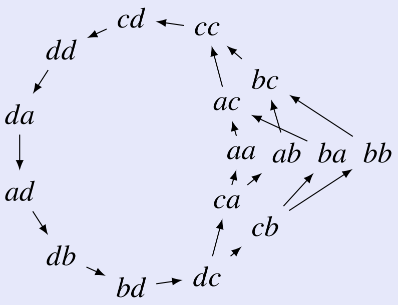


Figure: The De Bruijn forms of 2-variable digraphs with peroids 8 and 11.

Set of periods, Cont'd

Theorem

For any positive integer t, the set $\mathbb{N} \setminus \mathcal{P}_t$ is finite. In particular, $\mathbb{N} \setminus \mathcal{P}_1 = \emptyset \subsetneq \mathbb{N} \setminus \mathcal{P}_2 = \{2, 3, 5, 6, 7\} \subsetneq \mathbb{N} \setminus \mathcal{P}_3$.

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Question

- *Is it true that* $\mathcal{P}_1 \supsetneq \mathcal{P}_2 \supsetneq \mathcal{P}_3 \supsetneq \mathcal{P}_4 \supsetneq \cdots$*?*
- *Is it true that* $\cap_{i>0} \mathcal{P}_i = \{1\}$*?*

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For a primitive t-variable digraph f on K, we define

$$\mathrm{g}(f) = \min\{i : \mathrm{M}_f^i(A) = K^t \text{ for all } A \in K^t\}$$

and call it the primitive exponent of f.

Lemma

Let f be a strongly connected t-variable digraph on K. There exists a positive integer N_f such that for any $a, b \in K^t$ and any $p \in \mathcal{HI}(a, b)$, we can find a sequence of integers $0 = p_0 < p_1 < p_2 < \cdots < p_s = p$ such that $p_i - p_{i-1} \leq N_f$ for all i, $\{p_i - p_{i-1} : \ 1 \leq i \leq s-1\} \subseteq \mathcal{HI}(a, a)$ and $p_s - p_{s-1} \in \mathcal{HI}(a, b)$.

Lemma

Let f be a strongly connected t-variable digraph on K. There exists a positive integer N_f such that for any $a, b \in K^t$ and any $p \in \mathcal{HI}(a, b)$, we can find a sequence of integers $0 = p_0 < p_1 < p_2 < \cdots < p_s = p$ such that $p_i - p_{i-1} \leq N_f$ for all i, $\{p_i - p_{i-1} : \ 1 \leq i \leq s-1\} \subseteq \mathcal{HI}(a, a)$ and $p_s - p_{s-1} \in \mathcal{HI}(a, b)$.

We reserve N_f for the smallest possible number to satisfy the above lemma and call it the split number of f.

Corollary

If f is primitive multivariable digraph, then

$$\mathrm{g}(f) \leq \mathrm{Dia}(f) + N_f(N_f - 1) \leq \mathrm{Dia}(f)(\mathrm{Dia}(f) + 2).$$

Let $\mathscr{D}_{t,k}$ be the set of primitive t-variable digraphs on $[k]$ and let

$$\gamma(t,k) := \max_{f \in \mathscr{D}_{t,k}} \mathrm{g}(f).$$

Proposition

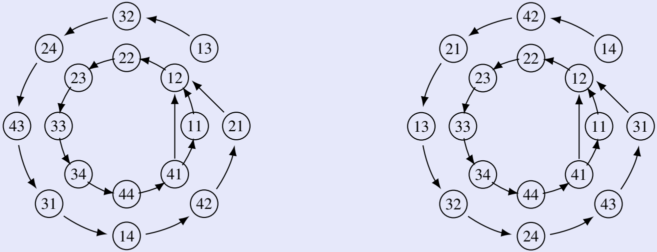
- *Wielandt's bound (1959):* $\gamma(1,k) = (k-1)^2 + 1$*;*
- $\gamma(2,1) = 1, \gamma(2,2) = 7, \gamma(2,3) = 23 = (2 \times 3 - 1)^2 - 2$*;*
- $\gamma(2,k) \geq (2k-1)^2 + 1$ *when* $k \geq 4$*;*
- $k^t \leq \gamma(t,k)$ *for any positive integer* t*.*

Let r_k be the minimum number of multiplications to multiply two k by k matrices (rank of the k by k matrix multiplication tensor). By coincidence (?), $r_1 = 1, r_2 = 7, r_3 \leq 23$. Can we expect $r_k \leq \gamma(2,k)$?

Wielandt-like constructions

Figure: $\gamma(2,2) = \mathrm{g}(\alpha) = 7, \gamma(2,3) = \mathrm{g}(\beta) = 23.$

Wielandt-like constructions, Cont'd



Γ_{f_4} $\Gamma_{f_4'}$

Figure: $\gamma(2,4) \geq \mathrm{g}(f_4) = \mathrm{g}(f_4') = 50.$

A general construction

- Pick $k \geq 2$.
- Let $\widehat{f_k}$ be the 2-variable digraph on $[k]$ such that $\Gamma_{\widehat{f_k}}$ has arc set $\{(i, j) \to (j, i+1) \pmod{k}\}$.

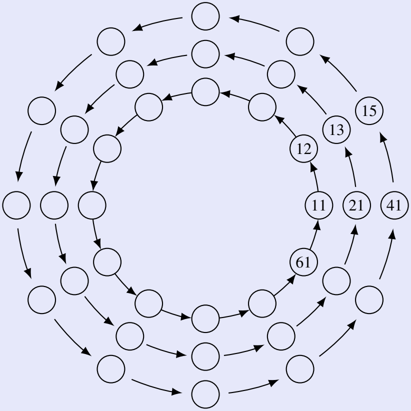


Figure: Γ_{f_6} and $\Gamma_{\widehat{f_6}}$.

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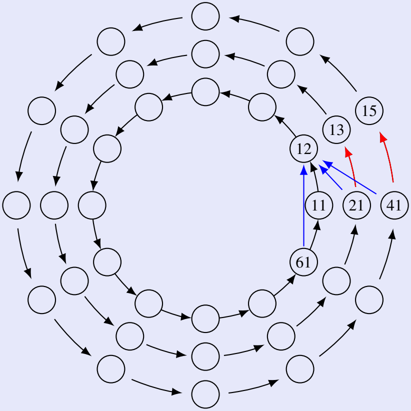
- Let $\widehat{f_k}$ be the 2-variable digraph on $[k]$ such that $\Gamma_{\widehat{f_k}}$ has arc set $\{(i,j) \to (j, i+1) \pmod{k}\}$.
- Let f_k be the digraph such that $\Gamma_{f_k} = \Gamma_{\widehat{f}} - \lceil \frac{k-2}{2} \rceil$ red edges $+ \lceil \frac{k}{2} \rceil$ blue edges.



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- Let f_k be the digraph such that $\Gamma_{f_k} = \Gamma_{\widehat{f}} - \lceil \frac{k-2}{2} \rceil$ red edges $+\lceil \frac{k}{2} \rceil$ blue edges.
- $g(f_k) = 4k^2 - 4k + 2 = (2k-1)^2 + 1$.

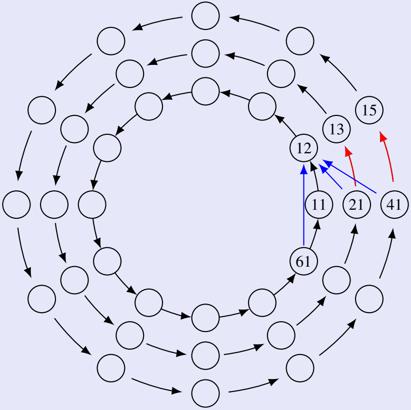

Figure: Γ_{f_6} and $\Gamma_{\widehat{f_6}}$.

Expansion property

Let f be a strongly connected t-variable digraph on K. For $A = (A_1, \dots, A_t) \in (2^K)^t$, the size of A is defined to be $\|A\| = \sum_{i=1}^{t} |A_i|$.

Lemma

Assume that S is a strongly connected component of $\Gamma(f)$ which contains at least one arc. Let Gir_S and Dia_S be the girth and diameter of the digraph induced by component S, respectively. For $a \in S$, if $\mathrm{M}_f^h(a)$ is transient for some $h \in \mathbb{N}$, then $\left\|\mathrm{M}_f^h(a)\right\| \geq \frac{h-\mathrm{Dia}_S}{\mathrm{Gir}_S} + t$.

Theorem

Take $a \in K^t$ and $h \in \mathbb{N}$. If $\mathrm{M}_f^h(a)$ is transient, then

$$\left\|\mathrm{M}_f^h(a)\right\| \geq \frac{h}{k^t - 1} + t - 1.$$

A tensor generalization of Coxson-Larson-Schneider

Let f be a single-variable digraph on K. If a singleton set B can be reached by A in $\mathcal{PS}_f$, then, according to Coxson-Larson-Schneider (LAA, 1987), $B = \mathrm{M}_f^h(A)$ for some $h \leq |K| - 1$.

Theorem

Let f be a strongly connected t-variable digraph on K. Take two subsets A and B of K such that A can reach B in $\mathcal{PS}_f$. If B is transient for M_f, then there exists $h \in \mathbb{N}$ such that $B = \mathrm{M}_f^h(A)$ and

$$h \leq (\|B\| - t + 1)(|K|^t - 1).$$

A generalization of Wielandt's bound

Theorem

For any positive integers t and k,

$$k^t \leq \gamma(t,k) \leq t(k-1)(k^t-1)+1.$$

When $(t,k)=(2,2)$ and when $t=1$, the second inequality holds as an equality.

- From Moore graphs to Moore multivariable symmetric digraphs?
- Multivariable Cayley digraphs?
- Multivariable posets (acyclic transitive multivariable digraphs)?
- Weighted multivariable digraphs (replacing $K^t \mapsto 2^K$ by $K^t \mapsto \mathbb{R}^K$ or other weighting methods) and hence spectral multivariable graph theory?
- Guessing number game on multivariable digraphs and network coding?
- $\cdots$

A t-variable digraph f on K is said to be a tournament provided the arc set of Γ_f contains exactly one of the following two arcs,

$$k_1 k_2 \cdots k_t \to k_2 k_3 \cdots k_{t+1}$$

and

$$k_{t+1} k_t \cdots k_2 \to k_t k_{t-1} \cdots k_1$$

for any sequence $k_1 \cdots k_{t+1} \in K^{t+1}$. Note that, for a tournament f, Γ_f will not contain any arc of the form

$$k_1 k_2 \cdots k_t \to k_2 k_3 \cdots k_{t+1}$$

where $k_i = k_{t+2-i}$ for $i \in [t+1]$, as in this case

$$k_1 k_2 \cdots k_t \to k_2 k_3 \cdots k_{t+1}$$

and

$$k_{t+1} k_t \cdots k_2 \to k_t k_{t-1} \cdots k_1$$

become the same arc.

THANK YOU!