Some combinatorial properties of the fundamental polytope

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Introduction

Distance function and metric

Let X be a set and d be a map from $X \times X$ to $\mathbb{R}$.

We call d a distance function on X provided

- $\mathrm{d}(x,x) = 0$ and
- $\mathrm{d}(x,y) + \mathrm{d}(y,z) \geq \mathrm{d}(x,z) \geq 0$

hold for all $x, y, z \in X$.

We call d a metric on X provided

- $\mathrm{d}(x,x) = 0$ and
- $\mathrm{d}(x,y) + \mathrm{d}(z,y) \geq \mathrm{d}(x,z)$

hold for all $x, y, z \in X$.

Note that a metric is just a symmetric distance function. We call (X, d) a metric space if d is a metric and call (X, d) a directed metric space if d is a distance function.

KR-polytope and L-polytope of a distance function

For each $a \in X$, let δ_a be the function on X which takes value 1 at a and value 0 elsewhere. For any $a, b \in X$, let $L_{a,b}$ denote the closure of $\{\frac{\delta_a - \delta_b}{t} : t > \mathrm{d}(a,b)\}$ in $\mathbb{R}^X$.

Let $V_0(X) = \{f \in \mathbb{R}^X : \sum_{x \in X} f(x) = 0\}$ (Hyperplane).

KR-polytope: $P_{\mathrm{d}} = \mathrm{conv}(\cup_{a,b \in X} L_{a,b}) \subseteq V_0(X)$.

Its polar:

$$\begin{aligned} P_{\mathrm{d}}^{\Delta} &= \{f \in \mathbb{R}^X : \langle f, g \rangle \leq 1, \forall g \in P_{\mathrm{d}}\} \\ &= \{f \in \mathbb{R}^X : f(a) - f(b) \leq \mathrm{d}(a,b), \forall a, b \in X\} \end{aligned}$$

L-polytope: $Q_{\mathrm{d}} = P_{\mathrm{d}}^{\Delta} \cap V_0(X)$.

$|X| = 4$, $\mathrm{d}(x, y) = 1$ for all $x \neq y \in X$.

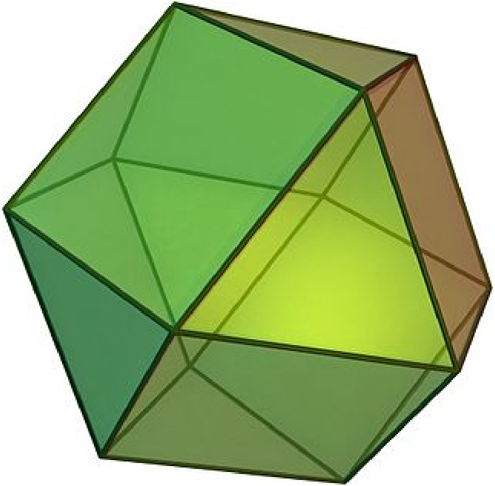

Figure: KR-polytope P_{d}. f-vector $= (f_0, f_1, f_2) = (12, 24, 14)$.

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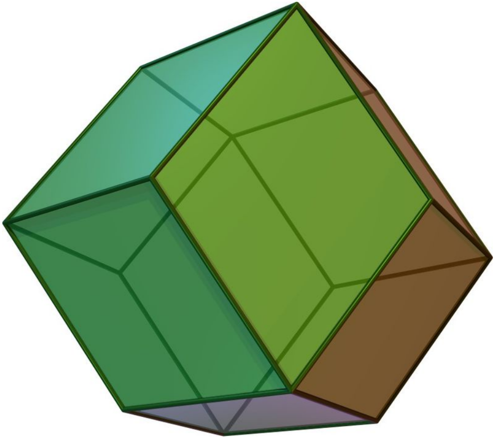


Figure: L-polytope Q_{d}. f-vector $= (f_0, f_1, f_2) = (14, 24, 12)$.

$X = \{a, b, c, d\}$, $\mathrm{d}(a, b) = \mathrm{d}(c, d) = 1$,
$\mathrm{d}(a, c) = \mathrm{d}(a, d) = \mathrm{d}(b, c) = \mathrm{d}(b, d) = 2$.

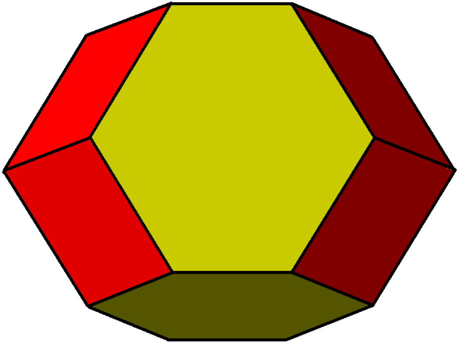


Figure: L-polytope Q_{d}.

$\mathrm{d}(a,b) = \mathrm{d}(c,d) = 2$, $\mathrm{d}(a,c) = \mathrm{d}(b,d) = 3$, $\mathrm{d}(a,d) = \mathrm{d}(b,c) = 4$.

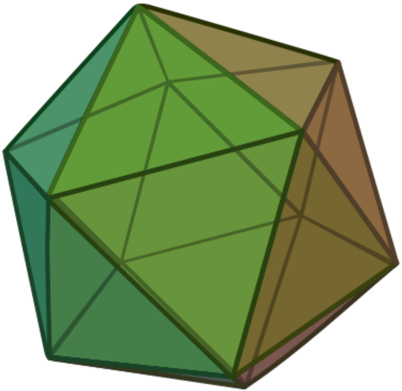


Figure: The KR-polytope P_{d} and the icosahedron have the same face lattice.

Asymmetric norm

Asymmetric Lipschitz norm on $V_0(X)$: Given $f \in V_0(X)$,

$$\| f \|_L = \min\{k : f(a) - f(b) \leq k \,\mathrm{d}(a,b), \forall a, b \in X\}.$$

The L-polytope Q_{d} is the unit ball of the Lipschitz norm on $V_0(X)$ associated with the distance function d.

If (X, d) is a metric space, then the KR-polytope P_{d} is the unit ball of the (asymmetric) Kantorovic-Rubinstein norm on $V_0(X)$ associated with the metric d.

Vershik calls P_{d} the fundamental polytope of the metric d and he suggests to study the convex geometry of the fundamental polytope.

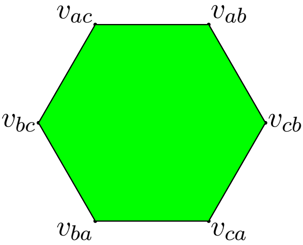
Problems of Vershik

- Characterize the face lattice of the KR-polytope, say calculating its f-vector.
- Provide sufficient conditions for two metrics to be equivalent, meaning that the face lattices of their KR-polytopes are isomorphic.
- Classify all the n-point metrics up to equivalence.

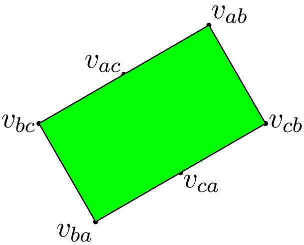
General metric

3-point metric (KR-polytope P_{d})

$\mathrm{d}(a,b) = \mathrm{d}(b,c) = \mathrm{d}(a,c) = 1.$



$\mathrm{d}(a,b) = \mathrm{d}(b,c) = 1, \mathrm{d}(a,c) = 2.$



Theorem (Melleray, Petrovy, Vershik)

Let X be an n-element set and d a symmetric function from $X \times X$ to $\mathbb{R}_{\geq 0}$ with $\mathrm{d}^{-1}(0) = \{(x,x) : x \in X\}$. Then, (X,d) is a metric space if and only if v_{ab} ($= \frac{\delta_a - \delta_b}{\mathrm{d}(a,b)}$) lies in the $(n-2)$-skeleton of P_d for every $\{a,b\} \in \binom{X}{2}$.

Theorem

Let X be an n-element set and d a function from $X \times X$ to $\mathbb{R}_{\geq 0}$ with $\mathrm{d}^{-1}(0) = \{(x,x) : x \in X\}$. Then, (X,d) is a directed metric space if and only if v_{ab} lies in the $(n-3)$-skeleton of P_d for every $\{a,b\} \in \binom{X}{2}$ or when (X,d) can be isometrically embedded into the real line $\mathbb{R}$.

Alternating inequality

Let (X, d) be a metric. For a sequence θ of distinct elements $x_1, x_2, \ldots x_k$ in X, the alternating inequality for θ tells us the sign of

$$\mathrm{d}(x_1, x_k) - \sum_{i=1}^{k-1} (-1)^i \, \mathrm{d}(x_i, x_{i+1})$$

and

$$\mathrm{d}(x_1, x_k) - \sum_{i=1}^{k-1} (-1)^{i-1} \, \mathrm{d}(x_i, x_{i+1}).$$

Example (Not all alternating relations are possible)

For sequence a, b, c in X,

$$\begin{aligned}
\mathrm{d}(a,c) + \mathrm{d}(b,c) \geq \mathrm{d}(a,b) \quad &\Rightarrow \quad \mathrm{d}(a,c) \geq \mathrm{d}(a,b) - \mathrm{d}(b,c), \\
\mathrm{d}(a,c) + \mathrm{d}(a,b) \geq \mathrm{d}(b,c) \quad &\Rightarrow \quad \mathrm{d}(a,c) \geq -\mathrm{d}(a,b) + \mathrm{d}(b,c).
\end{aligned}$$

A sufficient condition

Theorem

If two metrics d_1 *and* d_2 *on* $|X|$ *are not equivalent, then there exists a sequence* θ *of length at most* $|X|$ *such that* d_1 *and* d_2 *have different alternating relations for* θ*.*

Tight span

Given a finite metric space (X, d), we construct the unbounded polyhedron

$$P(X, \mathrm{d}) = \{f \in \mathbb{R}^X : f(a) + f(b) \geq \mathrm{d}(a, b),\ \forall a, b \in X\}.$$

Tight span is the union of bounded faces of $P(X, \mathrm{d})$:

$$\mathbf{T}(X, \mathrm{d}) = \{f \in \mathbb{R}^X : f(a) = \sup_{b \in X}\{\mathrm{d}(a, b) - f(b)\},\ \forall a \in X\}.$$

Isbell (1964): $\mathbf{T}(X, \mathrm{d})$ is a smallest $injective$ metric space containing an $isometric\ copy$ of X.

Dress (1984): When (X, d) is a tree metric, $\mathbf{T}(X, \mathrm{d})$ is a tree which encodes the metric (X, d).

Projecting tight span into L-polytope

$$\begin{array}{ccc} X & \xrightarrow{x \mapsto k_x} & \mathbf{T}(X, \mathrm{d}) \\ & & \downarrow \pi \\ & & Q_{\mathrm{d}} \end{array}$$

$k_x : y \mapsto \mathrm{d}(x, y)$

$\pi : f \mapsto f - \frac{\sum_{x \in X} f(x)}{n} 1 \in V_0(X).$

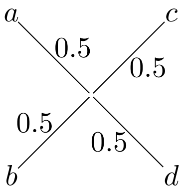
Theorem

$\pi(\mathbf{T}(X, \mathrm{d})) \subseteq Q_{\mathrm{d}}$, and $\pi(k_x)$ is vertex of Q_{d} if k_x is a dimension zero face of $\mathbf{T}(X, d)$ for each $x \in X$.

Tree metric

Tree metric

$|X| = 4$, $\mathrm{d}(a, b) = 1$ for all $a \neq b \in X$.



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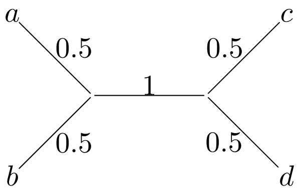
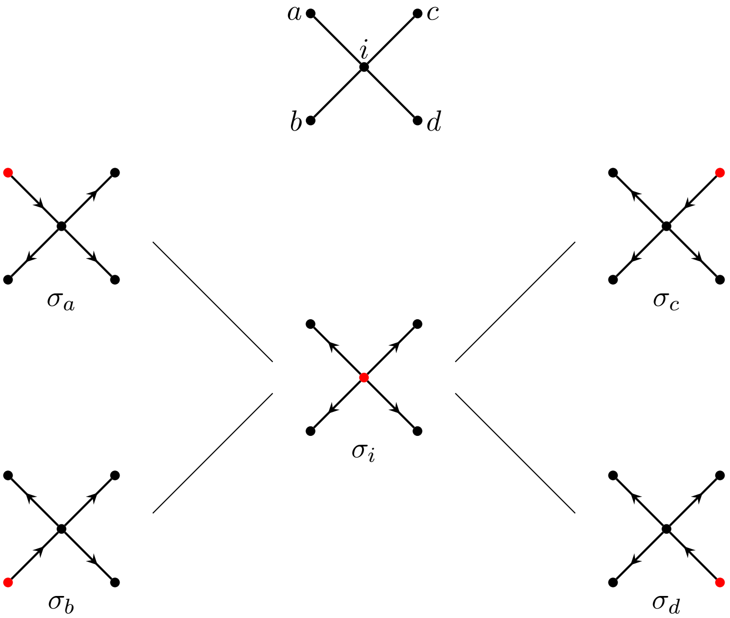
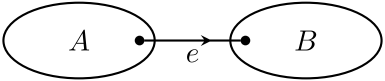


A tree on n vertices and its n single-source orientations



From an oriented X-tree to a point in $V_0(X)$



Let T be a tree with leaf set X where each edge e is assigned a weight $\omega(e)$. Fix an orientation σ of T. For each edge e of T, define a vector $f_e \in V_0(X)$ such that

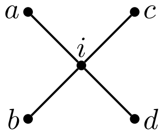
$$f_e(v) = \begin{cases} \frac{\omega(e)|B|}{|X|}, & \text{if } v \in A, \\ -\frac{\omega(e)|A|}{|X|}, & \text{if } v \in B, \end{cases}$$

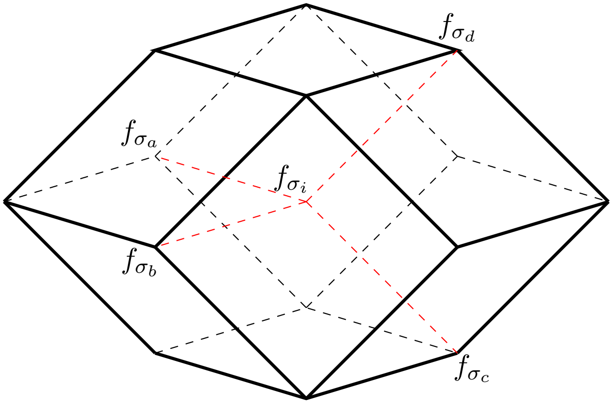
where A is the set of leaves at the head side of e and B is the set of leaves at the tail side of e.

We associate with the weighted oriented tree (T, σ, ω) the vector

$$f_{T,\sigma,\omega} = \sum_{e \in E(T)} f_e \in V_0(X).$$

Embedding a tree into its L-polytope





Theorem

Let T be a weighted tree and let (X, d_T) be the induced tree metric. For all the 1-source orientations of T we construct a graph $\widetilde{T} = (V, E)$ such that

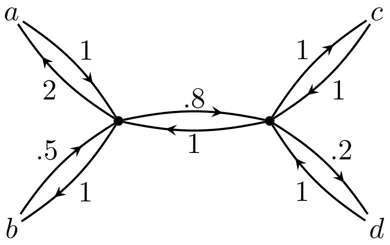
- $V = \{f_{T,\sigma} : \sigma$ *is a 1-source orientation of T*$\}$ *and*
- $E = \{(f_{T,\sigma_1}, f_{T,\sigma_2}) : \sigma_1$ *and* σ_2 *differ at exactly one edge*$\}$.

Then

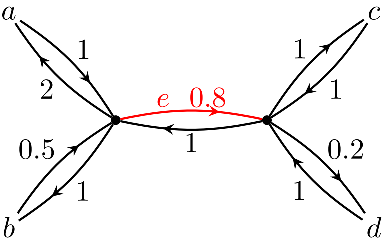
$$\widetilde{T} = \pi(\mathbf{T}(X, \mathrm{d}_T)).$$

Directed tree metric

We replace every edge of a tree by two arcs in reverse directions and assign positive weights to them which are not necessarily equal. Directed metrics induced by these kinds of directed trees are called directed tree metrics.



Minkowski decomposition



$$\mathrm{d}_e(x,y) = \begin{cases} 0.8, & \text{if } e \in x \to y; \\ 0, & \text{otherwise.} \end{cases}$$

Metric decomposition: $\mathrm{d}_T = \sum_{e \in E(T)} \mathrm{d}_e$.

Theorem

If (X, d_T) is a directed tree metric space, then

$$Q_{\mathrm{d}_T} = \sum_{e \in E(T)} Q_{\mathrm{d}_e},$$

where $E(T)$ refers to the set of all those arcs of T.

Directed tree metric $\Leftrightarrow$ L-polytope is zonotope

Theorem

Let (X, d) be a directed metric space. Then Q_{d} is a zonotope if and only if d is a directed tree metric.

f-vector of L-polytope

Let d_T be a directed tree metric displayed by the leaves of weighted directed tree T.

- $f_0(Q_{\mathrm{d}_T}) = 2\prod_v (2^{deg(v)} - 1)$, where v runs through all interior vertices of T.

Theorem

If T is a weighted directed binary tree, then the f-vector of Q_{d_T} is determined by

$$f_i = \sum_{k=1}^{\min\{i+1, n-i-1\}} 2^k 3^{n-i-k-1} \binom{n}{n+k-i-1} \binom{n-i-2}{k-1},$$

where $n = |X|$ and i ranges from 0 to $n-2$.

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where $n = |X|$ and i ranges from 0 to $n-2$.

Theorem

Let T and T' be two weighted directed trees. Then d_T and $\mathrm{d}_{T'}$ are equivalent if and only if T and T' are isomorphic as (unweighted) graphs.

Corollary

A weighted tree T is determined by the 1-skeleton of Q_{d_T}.

1-skeleton graph of Q_{d_T}

Proper orientation: An orientation of edges of T such that only leaves could be source/sink vertices.

Theorem

The vertex/edge sets of the 1*-skeleton graph of* Q_{d_T} *are as follows:*

- $V = \{f_{T,\sigma} : \sigma$ *is a proper orientations of* $T\}$,
- $E = \{(f_{T,\sigma_1}, f_{T,\sigma_2}) : \sigma_1$ *and* σ_2 *differ at exactly one edge*$\}$.

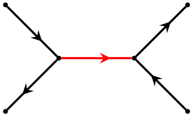


Figure: Two adjacent proper orientations.

The orientation graph

For any graph G, an orientation of its edge set is proper if all the source/sink vertices have degree at most 1.

Claim

Every graph admits a proper edge orientation.

We construct the orientation graph $O(G) = (V, E)$ by setting

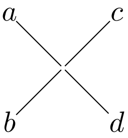
- $V = \{$proper orientations σ of $G\}$,
- $E = \{(\sigma_1, \sigma_2) : \sigma_1$ and σ_2 differs at exactly one edge$\}$.

Our questions on $O(G)$:

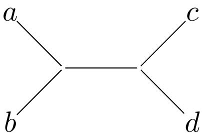
- Size?
- If the graph G has no degree-two vertices, is $O(G)$ connected? What about its vertex connectivity?
- Minimal degree?
- When can we reconstruct G from $O(G)$?

Four-point metric space

4-point metrics $(\mathrm{d}(x,y)+\mathrm{d}(y,z)>\mathrm{d}(x,z))$



$ad+bc=ac+bd=ab+cd$



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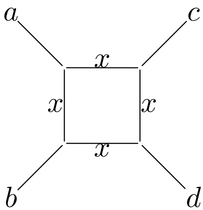

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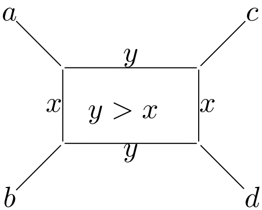


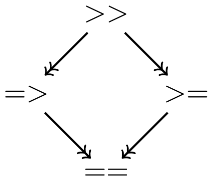
$ad+bc>ac+bd>ab+cd$

(ab is the shorthand of $\mathrm{d}(a,b)$.)

For sequence a, b, c, d in X,

$$\mathrm{d}(a,d)+\mathrm{d}(b,c) > \mathrm{d}(a,b)+\mathrm{d}(c,d) \Rightarrow \mathrm{d}(a,d) > \mathrm{d}(a,b)-\mathrm{d}(b,c)+\mathrm{d}(c,d).$$

Partial order on the fours metric classes:



Observation: $\mathrm{d}_1 \twoheadrightarrow \mathrm{d}_2$ if and only if P_{d_1} is a refinement of P_{d_2}. (Refinement: For every facet F_1 of P_{d_1}, there exists a facet F_2 of P_{d_2} such that $\mathrm{Vert}(F_1) \subseteq \mathrm{Vert}(F_2)$.)

Can this phenomenon be generalized to metric spaces of bigger size?

Theorem

There are in total 13 *nonequivalent proper metrics on a set of size four.*

Further research

- Relations with directed tight span (Hirai, Koichi)
- Relation with enriched category theory approach to metric spaces (Leinster, Willerton)
- Split decomposition of metrics w.r.t L-polytopes vs. that w.r.t tight span (Bandelt, Dress)
- Characterize the polytopes of some nice metric spaces (Hamming scheme, Johnson scheme)
- Various extremal problems about f-vectors

Thank You!