GRAPH DYNAMICAL SYSTEMS

Some combinatorial problems related to Markov chains

AG15, NIMS, August 3, 2015

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TOPOLOGICAL MARKOV CHAINS

Let K be a k-element set. The set of probability distributions on K is the $(k-1)$-dimensional probability simplex

$$\Delta^K = \{x \in \mathbb{R}^K = \mathbb{R}_{k\times 1}, x \geq 0, \sum_{i\in K} x_i = 1\}.$$

A **Markov chain of order t on K**, also known as a Markov chain with memory t or a t-step Markov shift on K, is a sequence of points $x(0), x(1), \ldots$ in Δ^K where each point $x(1+t+h)$ is linearly determined by its earlier t points $x(1+h), \ldots, x(t+h)$, written as $x(1+t+h) = Mx(1+h)\cdots x(t+h)$, where M is a t-linear map (hypermatrix/tensor of order $t+1$) and

$$(Mx(1+h)\cdots x(t+h))_i = \sum_{i_1,\ldots,i_t\in K} M_{ii_t\ldots i_1} x(1+h)_{i_1} \cdots x(t+h)_{i_t}.$$

FROM PROBABILISTIC TO TOPOLOGICAL

- A Markov chain of order t on K is determined by the initial t points from Δ^K as well as the t-linear map M which sends $\underbrace{\Delta^K \times \cdots \times \Delta^K}_{t}$ into Δ^K. If M varies in time, we have an inhomogeneous chain; If M is constant, it is a homogeneous chain.
- We replace Δ^K by 2^K (mapping a probability vector to its support) and choose coefficients from the Boolean semiring instead of from nonnegative reals, thus arriving at a nonparametric version of a Markov chain, called a **topological Markov chain**. The probability transition tensor now becomes a **Boolean tensor**. If the chain is homogeneous, the topological chain is known as a **t-step subshift of finite type** in symbolic dynamics [2, 3].

ONE-STEP TOPOLOGICAL MARKOV CHAIN

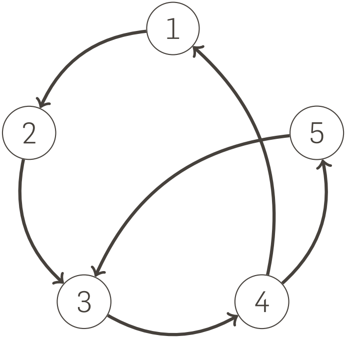


Abbildung: A digraph f representing a nonparametric 1-step Markov chain.

ONE-STEP TOPOLOGICAL MARKOV CHAIN

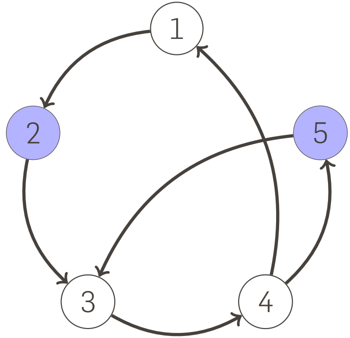


Abbildung: A digraph f and its evolving vertex subset

$\{2, 5\}$

ONE-STEP TOPOLOGICAL MARKOV CHAIN

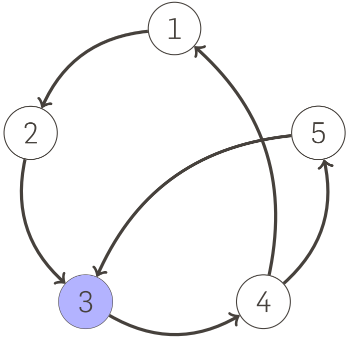


Abbildung: A digraph f and its evolving vertex subset

$\{2,5\} \to \{3\}$

ONE-STEP TOPOLOGICAL MARKOV CHAIN

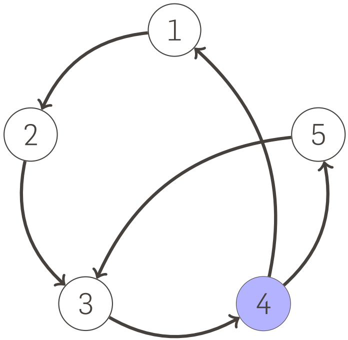


Abbildung: A digraph f and its evolving vertex subset

$\{2,5\} \to \{3\} \to \{4\}$

ONE-STEP TOPOLOGICAL MARKOV CHAIN

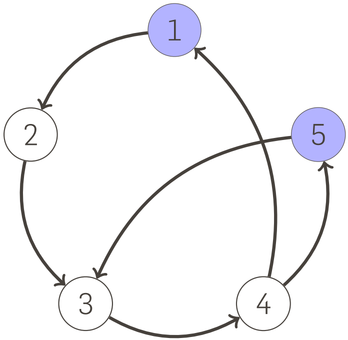


Abbildung: A digraph f and its evolving vertex subset

$\{2,5\} \to \{3\} \to \{4\} \to \{1,5\}$

ONE-STEP TOPOLOGICAL MARKOV CHAIN

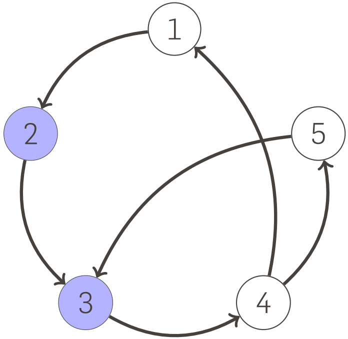


Abbildung: A digraph f and its evolving vertex subset

$\{2,5\} \to \{3\} \to \{4\} \to \{1,5\} \to \{2,3\}$

ONE-STEP TOPOLOGICAL MARKOV CHAIN

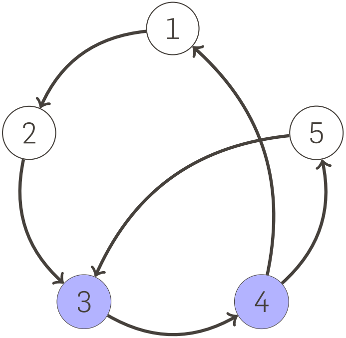


Abbildung: A digraph f and its evolving vertex subset

$$\{2,5\} \to \{3\} \to \{4\} \to \{1,5\} \to \{2,3\} \to \{3,4\}$$

ONE-STEP TOPOLOGICAL MARKOV CHAIN

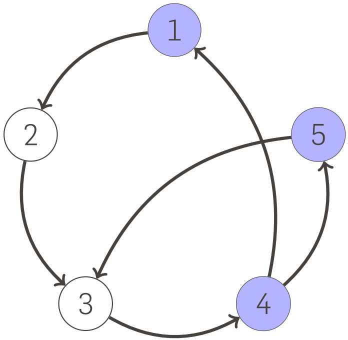


Abbildung: A digraph f and its evolving vertex subset

$$\{2,5\} \to \{3\} \to \{4\} \to \{1,5\} \to \{2,3\} \to \{3,4\} \to \{1,4,5\}$$

ONE-STEP TOPOLOGICAL MARKOV CHAIN

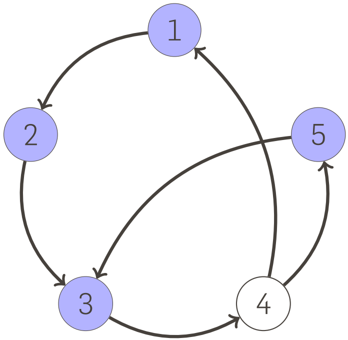


Abbildung: A digraph f and its evolving vertex subset

$$\{2,5\} \to \{3\} \to \{4\} \to \{1,5\} \to \{2,3\} \to \{3,4\} \to \{1,4,5\} \to \{1,2,3,5\}$$

PHASE SPACE $\mathcal{PS}_F$ OF THE DIGRAPH F

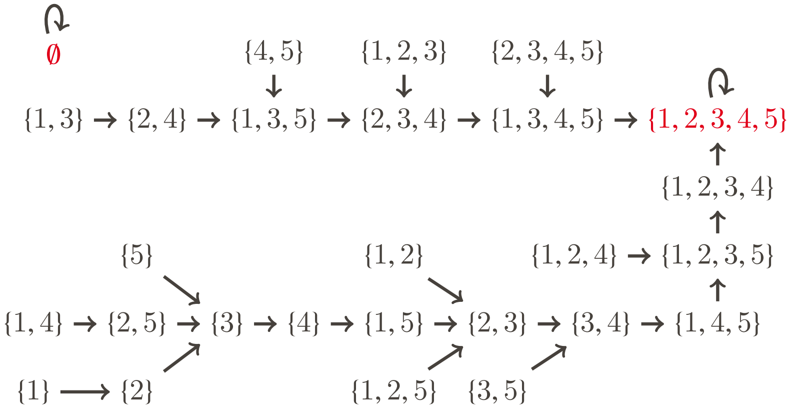


Abbildung: Besides ∅, all vertices will eventually visit $V(f)$.

PHASE SPACE $\mathcal{PS}_F$ OF THE DIGRAPH F

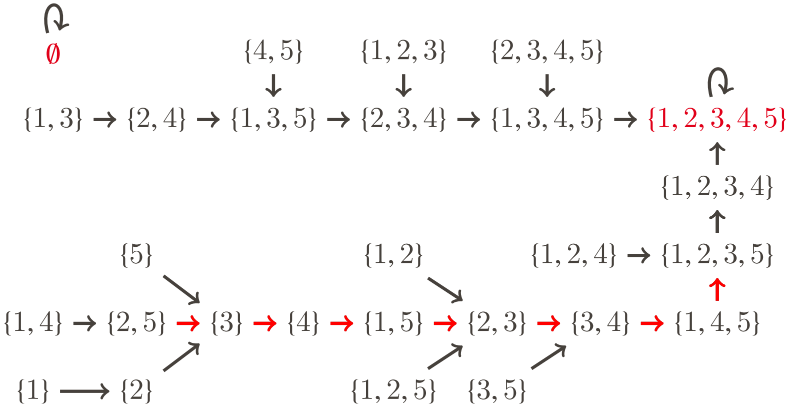


Abbildung: In seven steps $\{2, 5\}$ moves to a set which properly contains itself.

The digraph f stands for the local connection mechanism and its phase space $\mathcal{PS}_f$ displays the global evolving picture.

The theory of dynamical systems aims to relate a system's global behaviour to its local behaviour and the forces that shape it.

PRIMITIVITY: FROM SINGLETON SET TO WHOLE VERTEX SET

A Boolean linear map f on K is called **primitive** if every vertex from $2^K \setminus \{\emptyset\}$ will reach K in $\mathcal{PS}_f$. If f is primitive, the length of a longest path in $\mathcal{PS}_f$ is called the **primitive exponent** of f and is denoted $g(f)$.

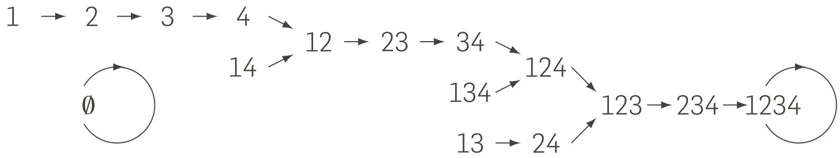
Every digraph f is identified with a corresponding Boolean linear map.

WIELANDT-TYPE MATRICES

Take a positive integer $k \geq 2$ and choose $i \in \{1, \ldots, k-1\}$ satisfying $\gcd(i, k) = 1$. A **Wielandt-type matrix/digraph** $W_{k;i}$ is the matrix/digraph with vertex set $\mathbb{Z}/k\mathbb{Z}$ and arc set $\{i \to i+1 : \ i = 1, \ldots, k\} \cup \{k \to 1+i\}$.

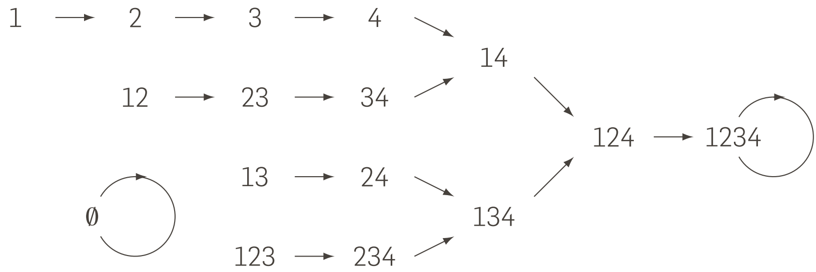
$$W_{4;1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, W_{4;3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, W_{5;4} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$\mathcal{PS}_{W_{4;1}}$



$g(W_{4;1}) = 10 = (4-1)^2 + 1.$

$\mathcal{PS}_{W_{4;3}}$



In $6 = 2 \times 4 - 2$ steps every vertex other than $\emptyset$ reaches 1234.

Helmut Wielandt (1959)

Let f be a primitive digraph with k vertices. Then the primitive exponent of f is at most $(k-1)^2+1$ and the bound is attained if and only if the digraph f is isomorphic to $W_{k;1}$.

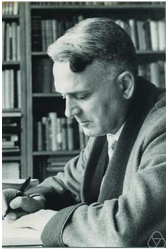


Abbildung: Helmut Wielandt, 19 December 1910 – 14 February 2001

MORE GENERAL REACHABILITY RESULTS

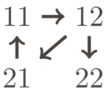
W., Zeying Xu, Yinfeng Zhu

Let k be a positive integer and let f be a digraph on $[k]$. Let $Y = f^h(X)$ where $X \subseteq [k]$ and $h \in \mathbb{N}$.

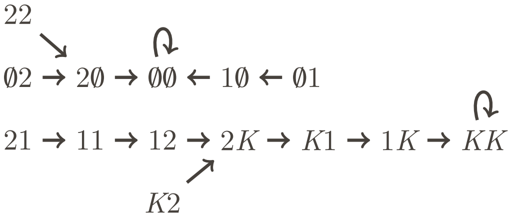
(A) If Y falls into a strongly connected component of f, then there exists $h' \in \mathbb{N}$ such that $h' \leq |Y| L_f$ and $Y = f^{h'}(X)$, where L_f is the length of a longest path in f.

(B) If f itself is irreducible, then there exists $h' \in \mathbb{N}$ such that $h' \leq |Y| \mathrm{D}_f$ and $Y = f^{h'}(X)$, where D_f is the diameter of f.

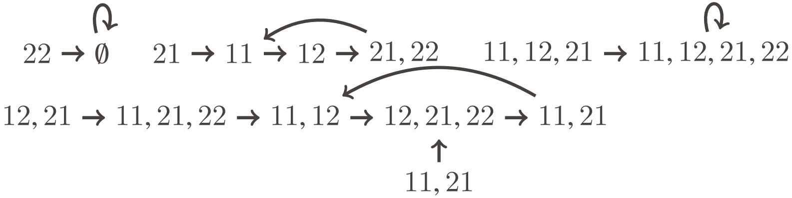
WAVE-PARTICLE DUALITY FOR HIGHER-ORDER MARKOV CHAIN



A 2-step MC on $K = [2]$

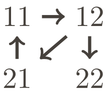


Phase space of the 2-step MC (wave view)

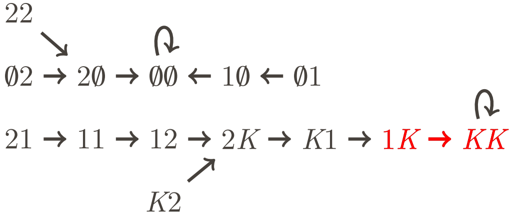


Phase space of the encoded 1-step MC on 2^K (particle view)

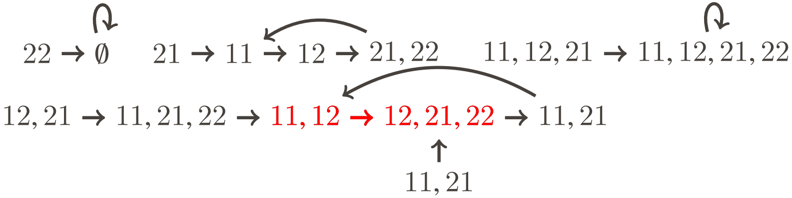
WAVE-PARTICLE DUALITY FOR HIGHER-ORDER MARKOV CHAIN



A 2-step MC on $K = [2]$



Phase space of the 2-step MC (wave view)



Phase space of the encoded 1-step MC on 2^K (particle view)

PRIMITIVE TENSOR (DIRECTED HYPERGRAPH)

A Boolean t-linear map f on K is called **primitive** if every vertex from $(2^K \setminus \{\emptyset\})^t$ will reach $(K, \ldots, K)$ in $\mathcal{PS}_f$. If f is primitive, the length of a longest path in $\mathcal{PS}_f$ is called the **primitive exponent** of f.

Every spanning subdigraph f of the $(t-1)$th line digraph of a complete digraph (a De Bruijn digraph) is identified with a Boolean t-linear map.

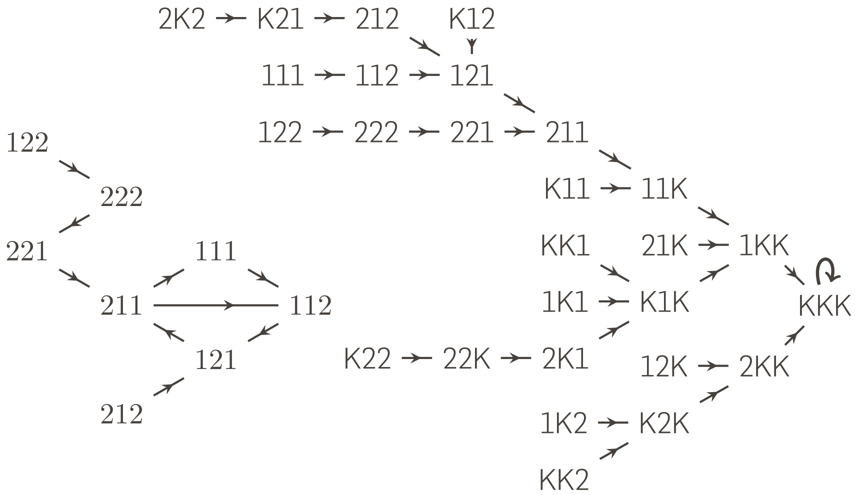


Abbildung: A primitive 3-linear map and part of its phase space.

Conjecture (W., Zeying Xu, Yinfeng Zhu)

Take an integer $k \geq 3$. The maximum primitive exponent for which a primitive Boolean 2-linear map on $[k]$ can achieve is $5k^2 - 8k + 2$.

The Perron-Frobenius Theory for nonnegative matrices basically give all information of a 1-step Markov chain.

For a higher-order Markov chain, to establish corresponding higher-order Perron-Frobenius Theory has been a hot topic in the study of tensors.

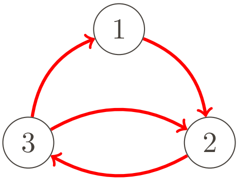
The process of going from linear to multilinear/polynomial seems to suggest many nontrivial problems.

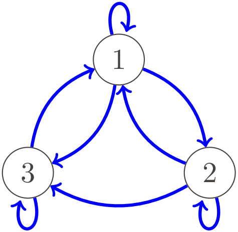
PRIMITIVE INHOMOGENEOUS CHAIN

The Boolean linear dynamical system $(2^K, \mathcal{F})$ is **primitive** provided every walk of length $2^k - 2$ in $\mathcal{PS}_\mathcal{F}$ starting from a vertex other than $\emptyset$ will reach K. If $\mathcal{F}$ is primitive, the length of a longest path in $\mathcal{PS}_\mathcal{F}$ is called the **primitive exponent** of $\mathcal{F}$ and is denoted by $g(\mathcal{F})$.

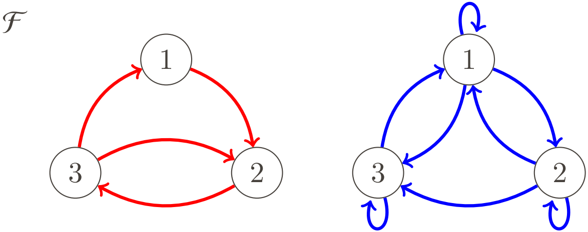
PRIMITIVE DIGRAPH SET

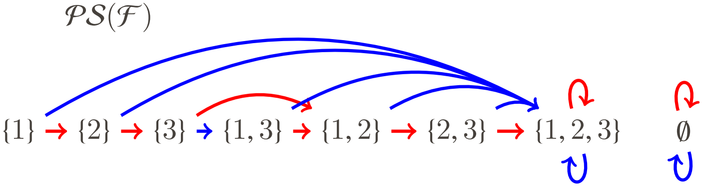
$\mathcal{F}$





PRIMITIVE DIGRAPH SET





In $2^3 - 2 = 6$ steps every nonempty set is arriving at $\{1, 2, 3\}$.

A PROBLEM OF PROTASOV

Let $g_{k,t}$ be the maximum possible value of $g(\mathcal{A})$ where $\mathcal{A}$ is a primitive set of Boolean linear maps on $[k]$ of size t. Protasov suggested to estimate $g_{k,t}$.

V. Yu. Protasov, *Semigroups of non-negative matrices*, Communications of the Moscow Mathematical Society 65 (2010) 1186–1188.

Besides Wielandt's bound of $g_{k,1} = (k-1)^2 + 1$ and the trivial bound of $g_{k,t} \le 2^k - 2$, very little is known about $g_{k,t}$.

$$\begin{cases} g_{2,2} = 2 = 1 \times 2 = 2^2 - 2 \\ g_{3,2} = 6 = 2 \times 3 = 2^3 - 2 \\ g_{4,2} = 12 = 3 \times 4 = 2^4 - 2^2 \\ g_{5,2} \ge 23 \quad (24 = 2^5 - 2^3 < 30 = 2^5 - 2) \\ g_{6,2} \ge 39 \quad (48 = 2^6 - 2^4) \end{cases}$$

GENERAL REACHABILITY

Conjecture (W., Zeying Xu, Yinfeng Zhu)

Let $\mathcal{F}$ be a primitive set of Boolean linear maps on $[k]$. For any $A, B \in 2^{[k]}$, it holds $\mathrm{dist}_{\mathcal{PS}_{\mathcal{F}}}(A, B) \leq |B|k^{|\mathcal{F}|}$ as long as A can reach B in $\mathcal{PS}_{\mathcal{F}}$.

Cohen-Sellers suggested to estimate the parameter γ_k, where $\gamma_k = \min\{t : \; g_{k,t} = 2^k - 2\}$.

Cohen-Sellers, 1982

For every $k \geq 2$, it holds $\gamma_k \leq 2^k - 2$.

Joel E. Cohen, Peter H. Sellers, *Sets of nonnegative matrices with positive inhomogenous products*, Linear Algebra and its Applications 47 (1982) 185–192.

W., Yinfeng Zhu

For every $k \geq 2$, it holds $\gamma_k \leq k$.

Given a set $\mathcal{A}$ of Boolean linear maps on $[k]$, can we find a set $\mathcal{F}$ of Boolean linear maps on $[k]$ such that $g(\mathcal{F} \cup \mathcal{A}) = 2^k - 2$?

We let $\gamma(\mathcal{A})$ be the minimum size of such $\mathcal{F}$ if it exists and let $\gamma(\mathcal{A}) = \infty$ otherwise.

W., Yinfeng Zhu

Take $k \geq 2$. For a set $\mathcal{A}$ of Boolean linear maps on $[k]$, it holds $\gamma(\mathcal{A}) \leq 2^k - 2$ if and only if the only possible cycles in $\mathcal{PS}_{\mathcal{A}}$ are loops at $\emptyset$ and $[k]$.

Xinmao Wang, W., Ziqing Xiang

It holds $\gamma(W_{k,k-1}) \leq \binom{k-2}{\lfloor (k-2)/2 \rfloor}$ for all integers $k \geq 2$.

GRAPH-INDEXED MARKOV CHAINS

GRAPH-INDEXED MARKOV CHAIN

If we think of time not as infinite path (integers) but as a general graph, we go from usual Markov chains to graph-indexed Markov chains.

Take a connected graph G. A function f from $V(G)$ to integers is **Lipschitz** if

$$\min f = 0$$

and

$$0 \leq |f(u) - f(v)| \leq \mathrm{dist}_G(u, v)$$

for all $u, v \in V(G)$. The **range** of such a map f is $\max f$. The average range of all Lipschitz functions on G is called the **average range of G**.

GRAPH-INDEXED MARKOV CHAIN: CONTD.

Conjecture (Loebl-Nešetřil-Reed, 2003)

The maximum average range among all connected graphs with n vertices is achieved by the n-vertex path.

W., Zeying Xu, Yinfeng Zhu

Among all trees with the same number of vertices, the path is the unique tree whose average range takes the maximum value.

THANKS!

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