COMBINATORICS OF THE LIT-ONLY σ-GAME

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International conference "Geometry, Topology, and Applications" dedicated to the upcoming 70th birthday of Nikolay Dolbilin

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Blackboard of Nikolay: Tilings, quasicrystals, polyhedra, Ising model, mathematics education ...

Figure 1 : Nikolay Petrovich Dolbilin, June 18, 2013.

`http://www.mathnet.ru/php/presentation.phtml?option_lang=eng&presentid=7018`

The talk

The talk is based on the working paper

W., Ziqing Xiang, Reachability of the phase space of the lit-only σ-game on a graph.

The work on this game has short history and till now only a few other researchers (John Goldwasser, Hau-wen Huang, William Klostermeyer, Xinmao Wang and Chih-wen Weng) write papers directly related to it.

We will report to you that this short history already points towards the possibility that the game possesses nice underlying mathematical structures; We will also discuss some other combinatorics topics which have surprising connections to this study.

Our blackboard

Figure 2 : Ziqing Xiang, September 19, 2013.

Lit-only σ-game

Pick a digraph D. For every $v \in V(D)$, construct a map $\mathcal{T}_v \in \mathrm{End}(\mathbb{F}_2^{V(D)}) : x \to \mathcal{T}_v(x)$ by setting $\mathcal{T}_v(x) = A_v x$ where

$$A_v = I + \chi_{N_D(v)}\chi_v^\top = I + A(D)^\top E_{vv},$$

namely

$$\mathcal{T}_v(x)(w) = \begin{cases} x(w), & vw \notin A(D), \\ x(w) + x(v), & vw \in A(D). \end{cases}$$

The map $\mathcal{T}_v$ is a transvection when v is not a loop, and is a projection when v is a loop.

The phase space of the lit-only σ-game on a digraph D is the digraph $\mathbb{PS}^*(D)$ with:

- $V(\mathbb{PS}^*(D)) = \mathbb{F}_2^{V(D)}$;
- $A(\mathbb{PS}^*(D)) = \{(x, \mathcal{T}_v(x)) : x \in \mathbb{F}_2^{V(D)}, v \in V(D), x \neq \mathcal{T}_v(x)\}$.

Phase space: An example

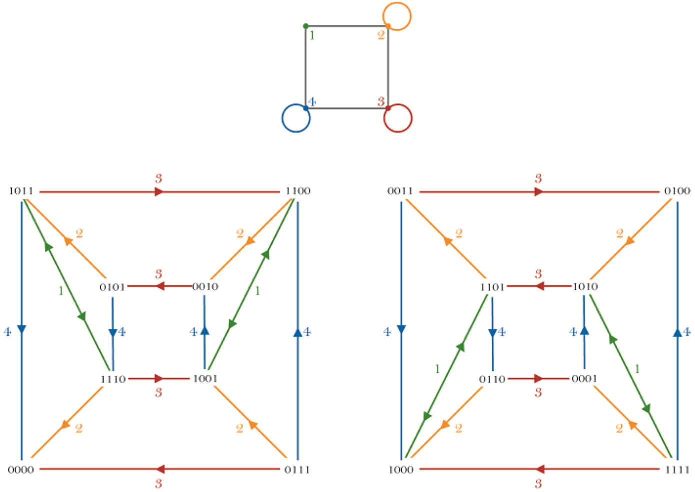

Figure 3 : A graph and the corresponding phase space.

Lit-only group

If the digraph D is loopless, $\{\mathcal{T}_v : v \in V(D)\}$ is a set of transvections and so we can consider the multiplicative group generated by it, which we call the lit-only group of G and denote by $\mathrm{LOG}(D)$.

The problem

The problem: Is it possible to classify the reachability types of the phase spaces of the lit-only σ-game?

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This can be achieved in the graph case; We indeed obtain a classification of all graphs according to the dynamical behavior of the lit-only σ-game. We even classify the lit-only group for the loopless graphs and find that the lit-only group basically determines the phase space, and vice versa.

Computer experiments suggest that a good classification still exists in the general digraph case.

Notation

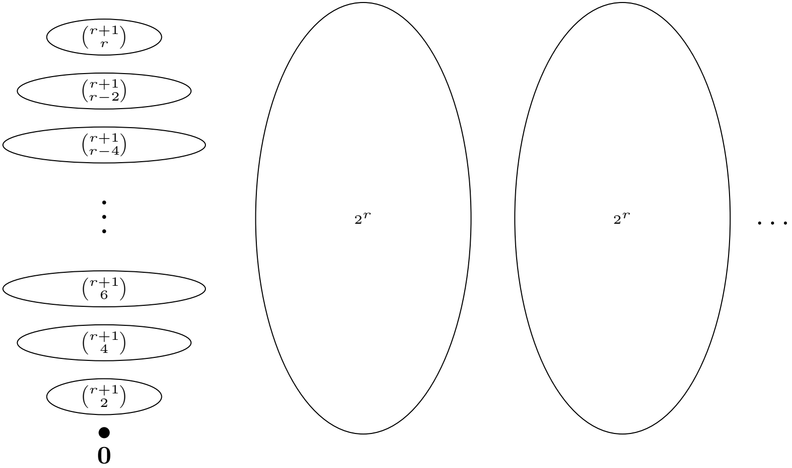
Let D be a strongly connected digraph, let $V = \mathbb{F}_2^{V(D)}$, let $L(D)$ be the set of loops in D, let W be the row space of $\mathbb{A}(D)$ (binary code of D) and let $r = \dim W$.

When the digraph D is symmetric, we write it as G.

For the diagrams in this section, each column stands for a coset of W in V, each circle stands for a strongly connected component with its size indicated inside.

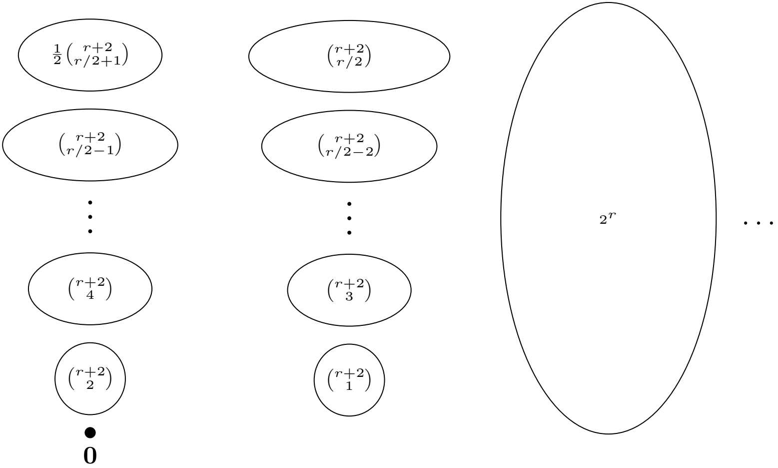
Some undefined concepts will be explained in the last part of this talk.

Connected loopless graph I: line graph $G = \mathscr{L}(H)$, $|V(H)| \equiv 1 \pmod 2$



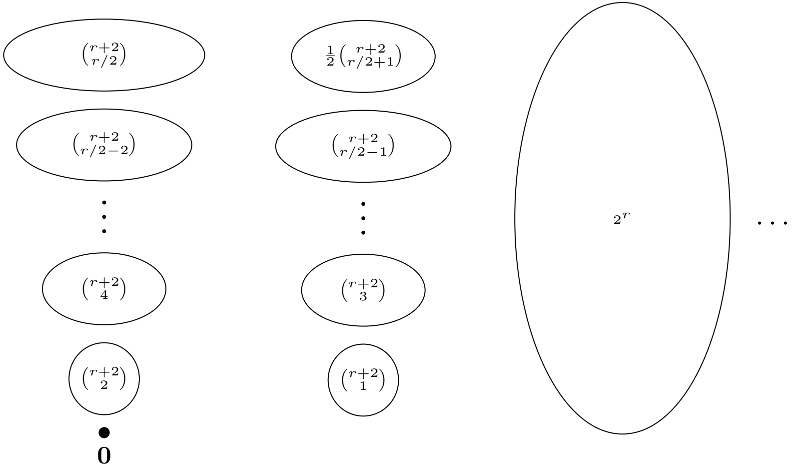
$$\mathrm{LOG}(G) \cong \mathrm{Sym}_{\dim W+1} \ltimes W^{\dim V - \dim W}, \dim W = |V(H)| - 1$$

Connected loopless graph II: line graph $G = \mathscr{L}(H)$, $|V(H)| \equiv 0 \pmod 4$



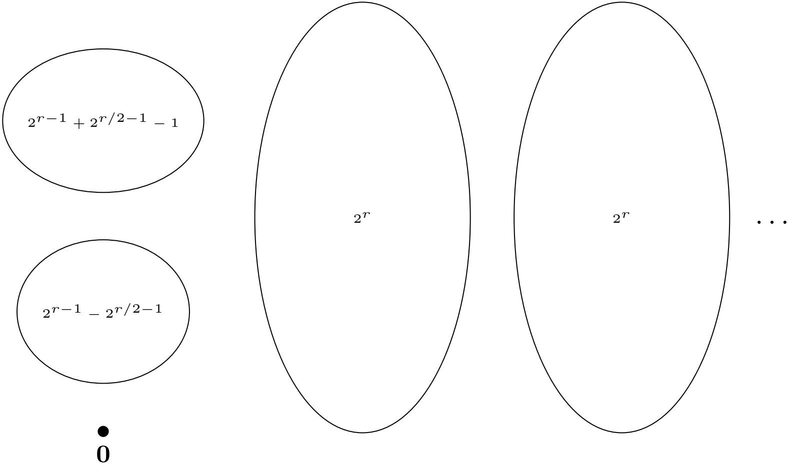
$$\mathrm{LOG}(G) \cong \mathrm{Sym}_{\dim W+2} \ltimes W^{\dim V - \dim W - 1}, \dim W = |V(H)| - 2$$

Connected loopless graph III: line graph $G = \mathscr{L}(H)$, $|V(H)| \equiv 2 \pmod 4$



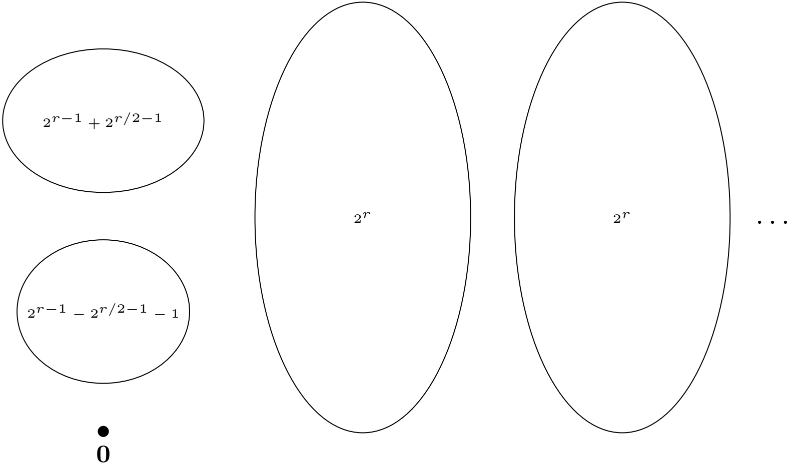
$$\mathrm{LOG}(G) \cong \mathrm{Sym}_{\dim W+2} \ltimes W^{\dim V - \dim W - 1}, \dim W = |V(H)| - 2$$

Connected loopless graph IV: marble non-line graph G, q_G has Arf invariant 0



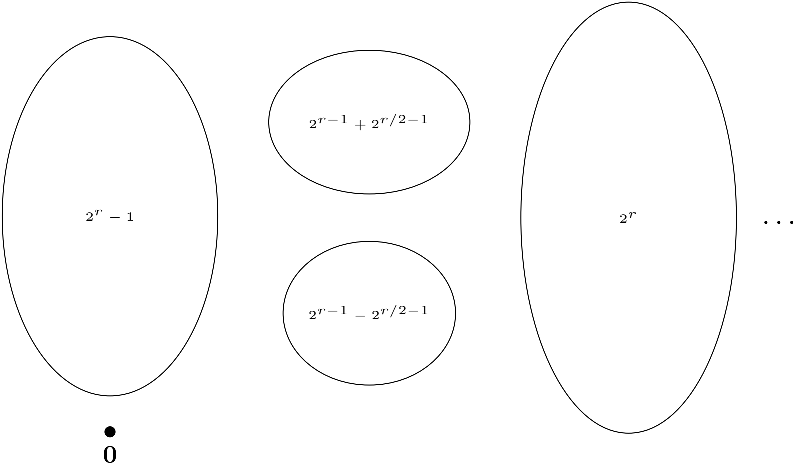
$$\mathrm{LOG}(G) \cong \mathrm{O}^+(W) \ltimes W^{\dim V - \dim W}$$

Connected loopless graph V: marble non-line graph G, q_G has Arf invariant 1



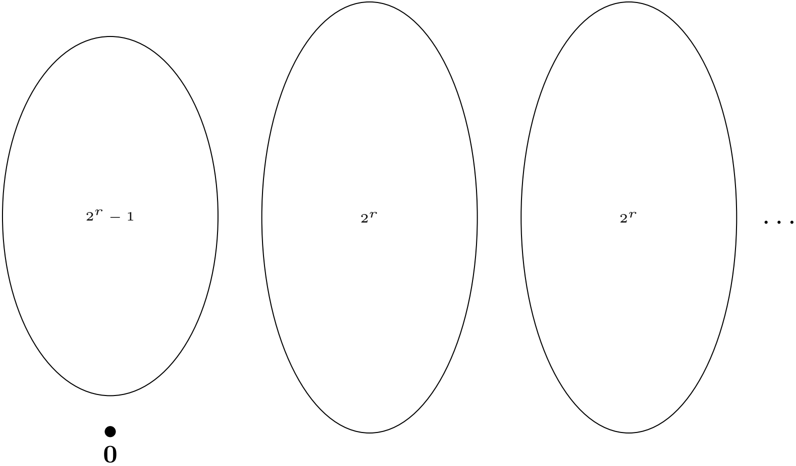
$$\mathrm{LOG}(G) \cong \mathrm{O}^-(W) \ltimes W^{\dim V - \dim W}$$

Connected loopless graph VI: non-marble non-line graph G



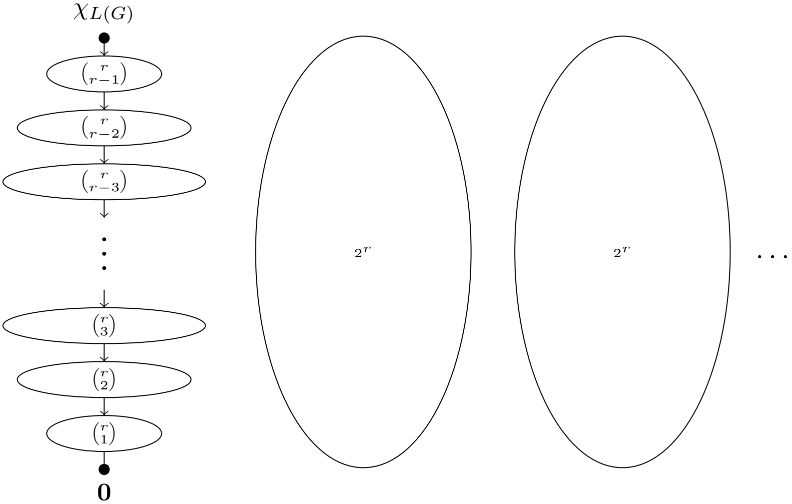
$$\mathrm{LOG}(G) \cong \mathrm{Sp}(W) \ltimes W^{\dim V - \dim W - 1}$$

Strongly connected loopless non-symmetric digraph D (conjecture)



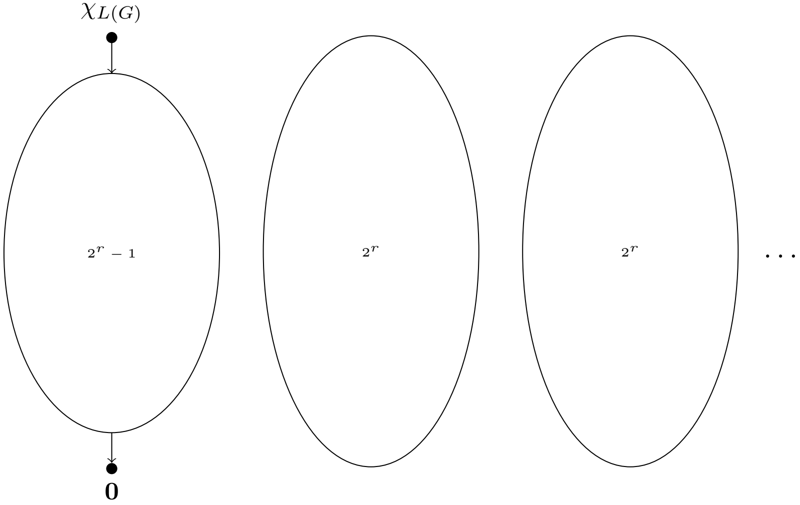
$$\mathrm{LOG}(D) \cong \mathrm{SL}(W) \ltimes W^{\dim V - \dim W}$$

Connected graph with loops I: line graph $G = \mathscr{L}(H)$

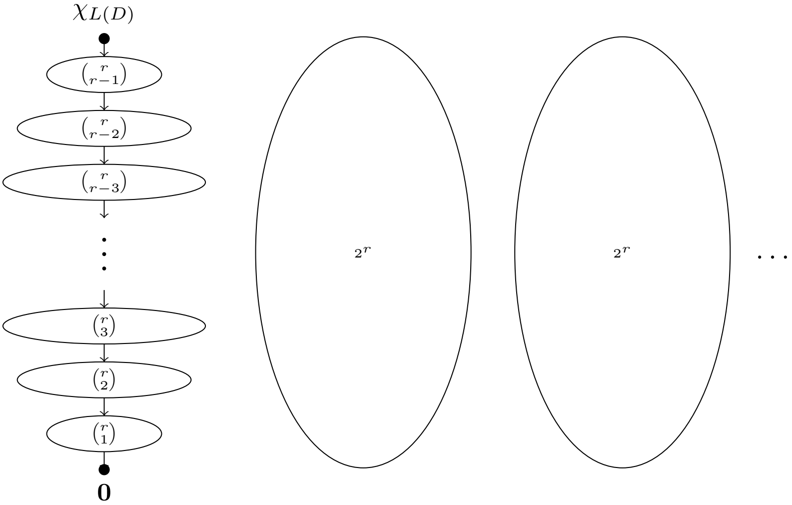


$$\dim W = |V(H)|$$

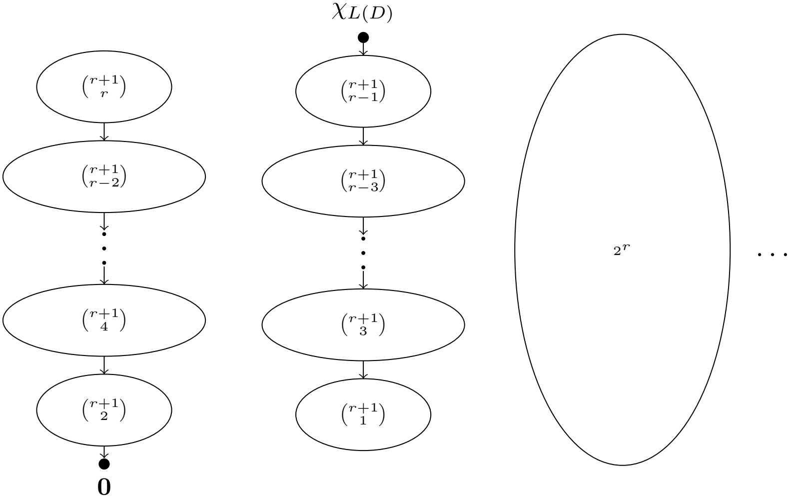
Connected graph with loops II: non-line graph G



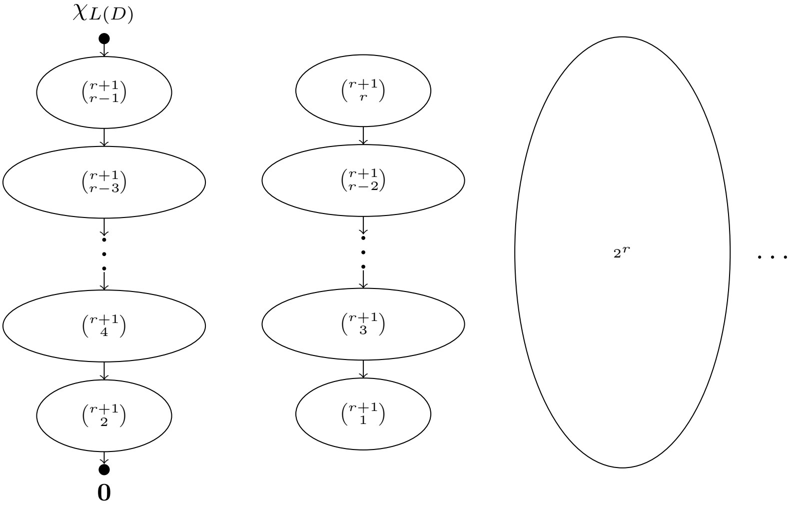
Strongly connected non-symmetric digraph with loops I (conjecture)



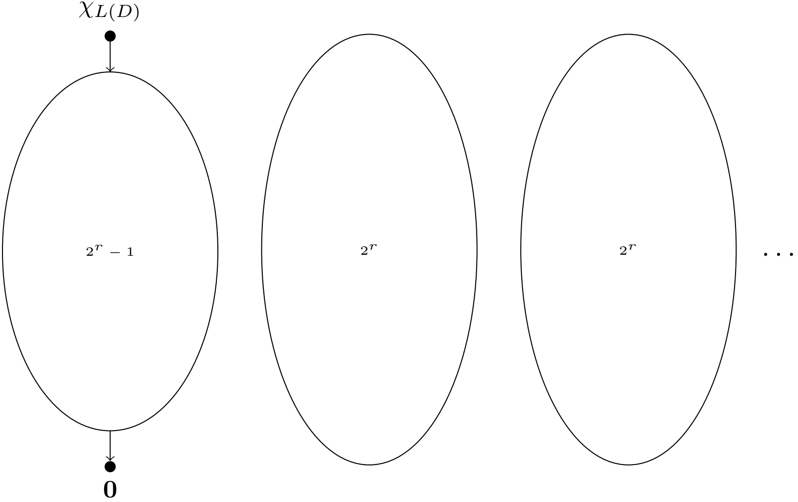
Strongly connected non-symmetric digraph with loops II (conjecture)



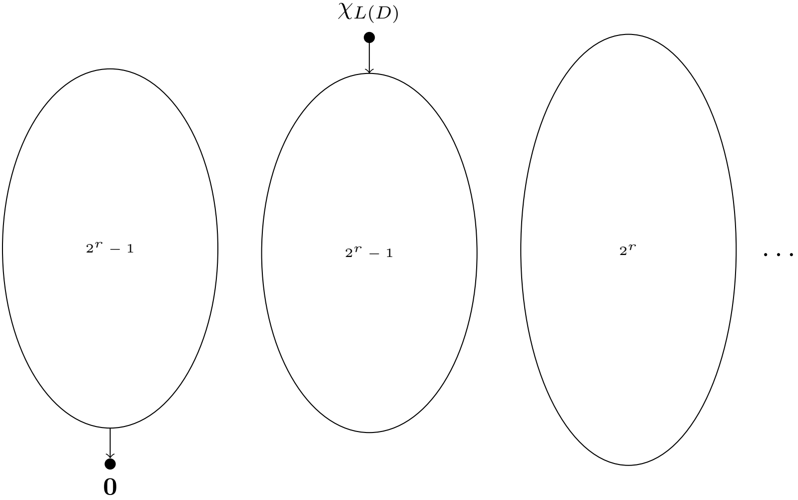
Strongly connected non-symmetric digraph with loops III (conjecture)



Strongly connected non-symmetric digraph with loops IV (conjecture)



Strongly connected non-symmetric digraph with loops V (conjecture)



Summary

In all, we conjecture that strongly connected loopless digraphs have $6 + 1 = 7$ combinatorial types while strongly connected digraphs with loops have $2 + 5 = 7$ combinatorial types.

The symmetric digraph case is already verified.

Mixed graphs

A mixed graph M consists of a pair of finite sets, its vertex set $V(M)$ and its face set $F(M)$, and its boundary maps $\partial_M^+, \partial_M^- : F(M) \to \binom{V(M)}{1} \cup \binom{V(M)}{2}$, which satisfies

$$|\partial_M^+(f)| = |\partial_M^-(f)|$$

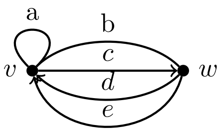
and

$$\partial_M^+(f) = \partial_M^-(f)$$

when $|\partial_M^+(f)| = |\partial_M^-(f)| = 2$.

An example

Roughly speaking, mixed graphs allow directed edges, undirected edges, parallel edges and loops.
We assume ∂^+ stands for head, and ∂^- stands for tail.



Multidigraph	$E(M) = L(M)$
Multigraph	$A(M) = L(M)$
Simple mixed graph	$(\partial_M^+, \partial_M^-)$ is injective
Digraph	Simple multidigraph
Graph	Simple multigraph

Vertices	$V(M)$	$\{v, w\}$
Faces	$F(M)$	$\{a, b, c, d, e\}$
Edges	$E(M)$	$\{a, b\}$
Arcs	$A(M)$	$\{a, c, d, e\}$
Loops	$L(M)$	$\{a\}$

Face	∂_M^+	∂_M^-
a	$\{v\}$	$\{v\}$
b	$\{v, w\}$	$\{v, w\}$
c	$\{w\}$	$\{v\}$
d	$\{v\}$	$\{w\}$
e	$\{v\}$	$\{w\}$

Figure 4 : A mixed graph M and its boundary maps.

Laplacian of mixed graphs

An orientation O of a mixed graph M is a multidigraph, which is defined on the same vertex set, equipped with a mapping $o: F(M) \to F(O)$ that satisfies for every $f \in F(M)$, $\partial_O^\circ(o(f)) \subseteq \partial_M^\circ(f)$ for sign $\circ \in \{+,-\}$ and $\bigcup_{\circ\in\{+,-\}} \partial_O^\circ(o(f)) = \bigcup_{\circ\in\{+,-\}} \partial_M^\circ(f)$.

Let M be a mixed graph and O be an orientation of M. Let

$$\mathbb{B}_O^\circ(M) = \operatorname{sgn}(\circ)(2\,\mathbb{B}^\circ(O) - \mathbb{B}^\circ(M))$$

where $\mathbb{B}^\circ$ is the incidence matrix of relation ∂° for a sign $\circ \in \{+,-\}$.

The Laplacian of the mixed graph M is

$$\Delta(M) = \mathbb{B}_O^+(M)^\top\,\mathbb{B}_O^-(M).$$

Delicious graphs

A delicious graph is a mixed graph that has a unique functional orientation, and a unique injective orientation. The sign of a delicious graph M, denoted by $\operatorname{sgn}(M)$, is

$$\prod_{C \text{ is a cycle of } M} (-1)^{|V(C)|-1}.$$

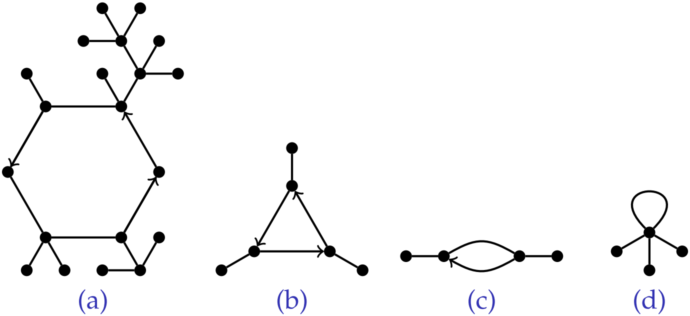


Figure 5 : Delicious graphs with cycles of lengths 6, 3, 2 and 1.

A generalization of the matrix-tree theorem

Theorem 1

For a mixed graph M,

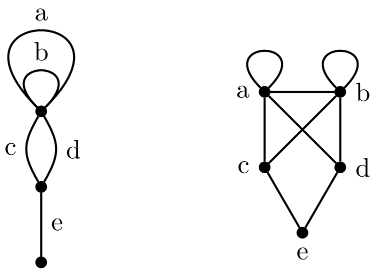
$$\det \Delta(M) = (-1)^{|V(M)|} \sum_{\textit{Spanning delicious sub-mixed-graph } D \textit{ of } M} \operatorname{sgn}(D)$$

where the determinant is calculated over $\mathbb{Z}$.

A corollary of the theorem is the principle minor matrix-tree theorem.

Line graphs

Given a multigraph G, its line graph, denoted by $\mathscr{L}(G)$, is the graph with vertex set $V(\mathscr{L}(G)) = E(G)$ and edge set $E(\mathscr{L}(G)) = \{\{e,f\} : |\partial_G(e) \cap \partial_G(f)| = 1, e,f \in E(G)\}$. A line graph is ordinary if it is the line graph of a graph.



(a) G (b) $\mathscr{L}(G)$

Figure 6 : A multigraph and its line graph.

Edge clique partition

Definition 2

Let G be a graph. An edge clique partition of G is a hypergraph $\mathcal{K}$ on the vertex set $V(G)$ and edge set $\mathcal{K} \subseteq 2^{V(G)} \setminus \{\emptyset\}$ such that the following conditions hold:

(i) For every $v \in V(G) \setminus L(G)$, it holds $E_{\mathcal{K}}(v) \in \binom{\mathcal{K}}{2}$ when $N_G(v) \neq \emptyset$ and $E_{\mathcal{K}}(v) \in \binom{\mathcal{K}}{1}$ when $N_G(v) = \emptyset$;

(ii) For every $v \in L(G)$, it holds $E_{\mathcal{K}}(v) \in \binom{\mathcal{K}}{1}$;

(iii) The set $E(G) \setminus L(G)$ is the disjoint union of $\binom{k}{2}$ where k runs through all elements of $\mathcal{K}$;

(iv) For each $k \in \mathcal{K}$, it happens $|k \cap L(G)| \leq 1$.

Graphs with two edge clique partitions

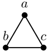
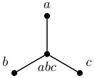
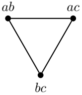
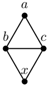
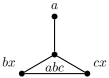
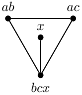
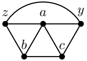
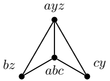
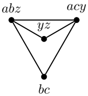
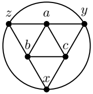
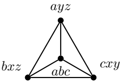
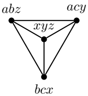
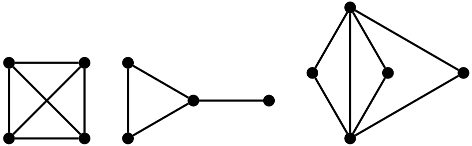
	Graph	Type A ECP	Type B ECP
$X = \emptyset$ $Y = \emptyset$ $Z = \emptyset$			
$X = \{x\}$ $Y = \emptyset$ $Z = \emptyset$			
$X = \emptyset$ $Y = \{y\}$ $Z = \{z\}$			
$X = \{x\}$ $Y = \{y\}$ $Z = \{z\}$			

Table 1 : Graphs and the dual hypergraphs of their two edge clique partitions (ECP).

Difference between K_3-free graphs and K_4-free graphs

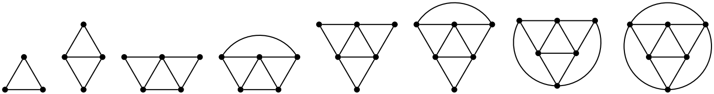
Theorem 3

{Fig. 7-free graphs} $=$ *{K_3-free graphs}* $\cup$ *Fig. 8.*



(a) K_4 (b) The paw (c) $\overline{K_3 \cup 2K_1}$

Figure 7 : Forbidden induced subgraphs for triangle-free graphs.



(a) K_3 (b) The diamond (c) The gem (d) The wheel W_4 (e) The sun S_3 (f) $\overline{E_6}$ (g) $\overline{P_4 \cup K_2}$ (h) $\mathscr{L}(K_4)$

Figure 8 : Eight graphs.

Ordinary line graph

Theorem 4 (Generalization of Krausz's theorem)

A graph is an ordinary line graph if and only it has an edge clique partition.

Theorem 5 (Generalization of Whitney's theorem)

Except the four graphs in the first column of Tab. 1, a connected ordinary line graph has a unique edge clique partition.

Characterization of loopless ordinary line graphs

Theorem 6 (Beineke's characterization of loopless ordinary line graphs)

A loopless graph is an ordinary line graph if and only if it does not contain any of the nine graphs in Fig. 9 as a vertex induced subgraph.

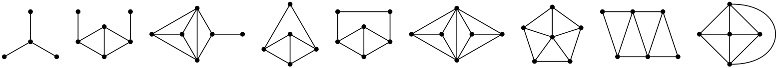


Figure 9 : Nine forbidden induced subgraphs for loopless ordinary line graphs.

Three loopless line graphs which are not ordinary

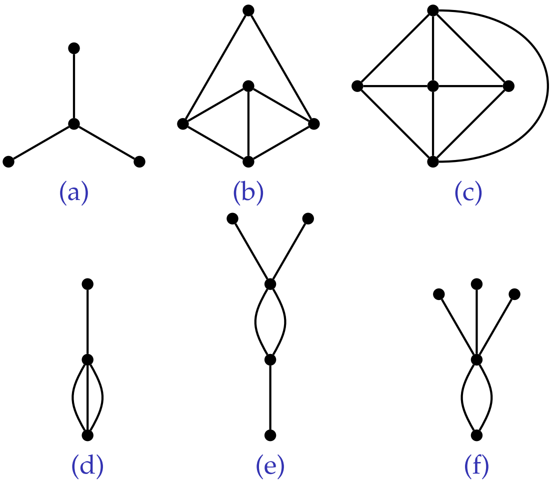


Figure 10 : Three line graphs from the list of Beineke and their root multigraphs.

Characterization of loopless line graphs

Theorem 7

For a loopless graph G, the following statements are equivalent.

- *The graph G is a line graph.*
- *The graph G does not contain any graph in a set of thirty-two forbidden graphs as an induced subgraph.*
- *Every connected 6-vertex induced subgraph of G is a line graph.*
- *Every connected nonsingular 6-vertex induced subgraph of G is one of the eleven line graphs of 7-vertex trees.*

The thirty-two forbidden graphs

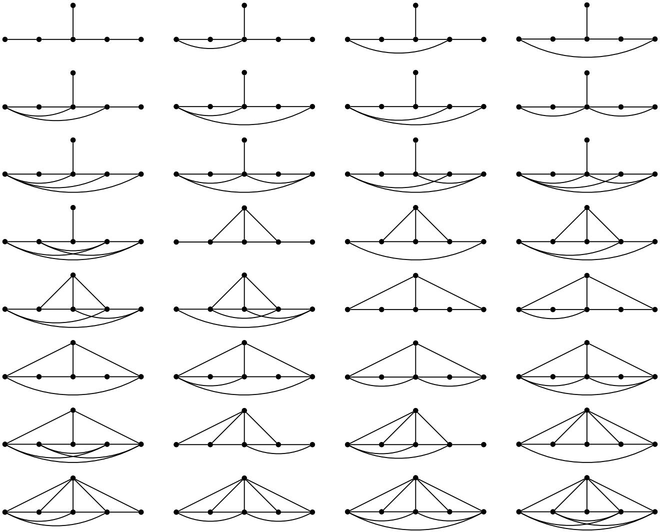


Figure 11 : The 32 forbidden subgraphs for loopless line graphs.

The eleven line graphs of 7-vertex trees

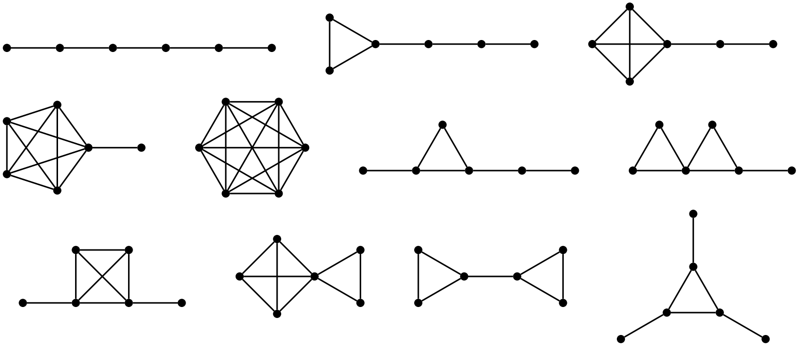


Figure 12 : Eleven 6-vertex line graphs of trees.

Two observations

- There are 43 connected nonsingular 6-vertex graphs. They consist of the 32 forbidden graphs (Fig. 11) and the 11 line graphs of 7-vertex trees (Fig. 12).
- All the 32 forbidden graphs are type V connected loopless graphs and have a lit-only group isomorphic to $W(E_6)$, the Weyl group of root system of type E_6 which has order $51840 = 2^7 3^4 5 = 72 \times 6!$. All those 11 line graphs are type I connected loopless graphs and have S_7 as their lit-only group. The size of S_7 is $7! = 5040$.

Image of the Coxeter groups

Given a loopless digraph D, let $M \in \mathbb{Z}^{V(D)\times V(D)}$ be the matrix defined by

$$M(u,v) = \begin{cases} 1, & u = v, \\ 2, & uv, vu \notin A(D), \\ 3, & uv, vu \in A(D), \\ 2, & uv \in A(D), vu \notin A(D), N_D(v) = \emptyset, \\ 4, & uv \in A(D), vu \notin A(D), N_D(v) \neq \emptyset, \\ 2, & uv \notin A(D), vu \in A(D), N_D(u) = \emptyset, \\ 4, & uv \notin A(D), vu \notin A(D), N_D(u) \neq \emptyset. \end{cases}$$

Theorem 8

There exists a group epimorphism from the Coxeter group corresponding to the Coxeter matrix M to the lit-only group $\mathrm{LOG}(D)$, which is generated by transvections.

Number of reduced words in the lit-only groups of the 32 forbidden induced subgraphs of line graphs

1, 6, 20, 50, 105, 195, 329, 514, 754, 1048, 1389, 1765, 2159, 2549, 2911, 3222, 3461, 3611, 3662, 3611, 3461, 3222, 2911, 2549, 2159, 1765, 1389, 1048, 754, 514, 329, 195, 105, 50, 20, 6, 1
1, 6, 21, 57, 133, 280, 547, 1011, 1734, 2705, 3852, 5068, 6213, 6883, 6711, 5750, 4413, 3088, 1919, 992, 372, 80, 4
1, 6, 21, 58, 134, 267, 475, 769, 1151, 1618, 2154, 2723, 3290, 3820, 4239, 4508, 4624, 4522, 4186, 3692, 3100, 2382, 1674, 1125, 685, 364, 170, 64, 16, 2
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1, 6, 22, 65, 164, 370, 749, 1343, 2148, 3128, 4211, 5195, 5860, 6145, 5951, 5261, 4305, 3238, 2111, 1081, 392, 88, 6
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Characterization of line graphs with loops

Theorem 9

Let G be a connected graph with at least one loop. Then G is a line graph if and only if G does not contain any graph in Fig. 13 as a vertex induced subgraph.

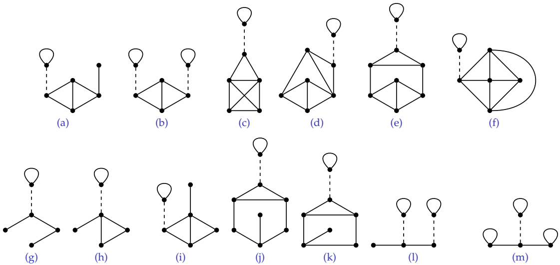


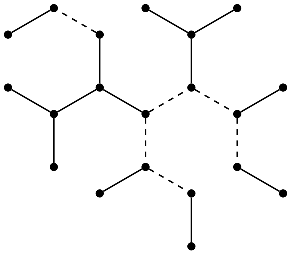
Figure 13 : Thirteen classes of forbidden subgraphs for line graphs with at least one loop. Dashed lines stand for paths of length zero or more.

Cherry forest and Euler characteristic

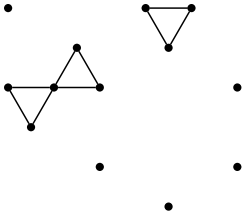
Let T be a tree with an even number of vertices. Removing those edges whose deletion cause an even component, we obtain from T its unique spanning forest $\widetilde{T}$ such that each vertex has odd degree in $\widetilde{T}$. For a graph G, the Euler characteristic of G over $\mathbb{F}_2$, denoted by $Q(G)$, is $|V(G)| - |E(G)| \pmod 2$.

Theorem 10

$Q(\mathscr{L}(\widetilde{T})) = |V(T)|/2 \pmod 2$.



(a) A tree T. The concrete edges are the edges in $\widetilde{T}$.



(b) The line graph $\mathscr{L}(\widetilde{T})$

Marble graphs

A loopless graph G is a marble graph if there exists a quadratic form q_G on the binary code of G that satisfies

$$q_G(\mathbb{A}(G)^\top x) = Q(G[\mathrm{Supp}(x)])$$

for every $x \in \mathbb{F}_2^{V(G)}$.

Theorem 11

Let G be the line graph of a connected loopless multigraph H. Then G is a marble graph if and only if $|V(H)| \not\equiv 2 \pmod 4$.

Critical subgraph

A critical subgraph of a graph G is a nonsingular vertex induced subgraph H with $\operatorname{rank} \mathbb{A}(G) = \operatorname{rank} \mathbb{A}(H)$.

Theorem 12

Critical subgraphs of a connected line graph correspond to "almost" spanning trees in the root multigraph.

Theorem 13

For each of the following graph classes $\mathcal{C}$ and every graph $G \in \mathcal{C}$, there exists graph $H \in \mathcal{C}$ such that H is a critical subgraph of G.

- *Connected graphs.*
- *Connected loopless line graphs.*
- *Connected line graphs with loops.*
- *Connected loopless non-line graphs.*
- *Connected non-line graphs with loops.*

Gaussian elimination

Let G be a graph and v be a loop vertex of G. Let $G \boxminus v$ be the graph with vertex set $V(G) \setminus \{v\}$ and edge set $E(G) \triangle \binom{N_G(v)}{1} \triangle \binom{N_G(v)}{2}$.

$$\begin{array}{cc} & \begin{array}{ccccc} v & & & & \end{array} \\ \begin{array}{c} v \\ \\ \\ \\ \\ \\ \end{array} & \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ 1 & & & & * \\ \vdots & & A & & * \\ 1 & & & & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \end{pmatrix} \end{array} \rightarrow \begin{array}{cc} & \begin{array}{ccccc} v & & & & \end{array} \\ \begin{array}{c} v \\ \\ \\ \\ \\ \\ \end{array} & \left(\begin{array}{c|cccc} 1 & 0 & \cdots & 0 & 0 \\ \cline{2-5} 0 & & & & * \\ \vdots & & A - J & & * \\ 0 & & & & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \end{array}\right) \end{array}$$

A double covering from $\mathbb{PS}^*(G)$ to $\mathbb{PS}^*(G \boxminus v)$: $wv \in E$

$$\phi_0(\alpha)|_{V(D\boxminus v)} = \alpha, \phi_0 + \phi_1 = \chi_{N_D(v)}, \phi_i(\alpha)(v) = i, \forall \alpha \in \mathbb{F}_2^{V(D\boxminus v)}, i \in \mathbb{F}_2 .$$

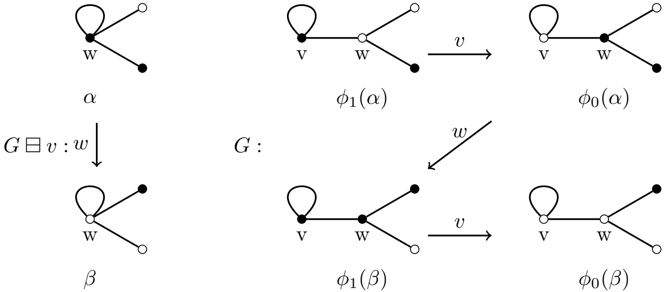


Figure 14 : For $w \in N_G(v) \setminus \{v\}$, a w-arc is lifted to two length two walks. A bullet stands for 1 while a circle stands for 0.

A double covering from $\mathbb{PS}^*(G)$ to $\mathbb{PS}^*(G \boxminus v)$: $wv \notin E$

$$\phi_0(\alpha)|_{V(D \boxminus v)} = \alpha, \phi_0 + \phi_1 = \chi_{N_D(v)}, \phi_i(\alpha)(v) = i, \forall \alpha \in \mathbb{F}_2^{V(D \boxminus v)}, i \in \mathbb{F}_2 .$$

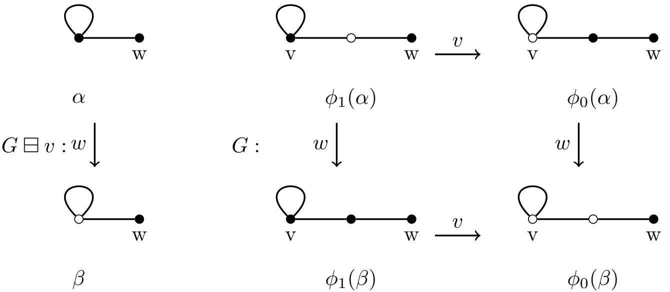


Figure 15 : For $w \in V(G) \setminus N_G(v)$, a w-arc is lifted to two w-arcs. A bullet stands for 1 while a circle stands for 0.

A generalization of Sutner's Theorem

Theorem 14 (Sutner's Theorem)

$\chi_{L(G)}$ *lies in the row space of* $\mathbb{A}(G)$.

Theorem 15 (A generalization of Sutner's Theorem)

$\chi_{L(G)} \xrightarrow{*}_G \mathbf{0}$.

Proof.

Induct on the number of vertices.
Pick a loop vertex $v \in L(G)$. By induction hypothesis,
$\chi_{L(G \boxminus v)} \xrightarrow{*}_{G \boxminus v} \mathbf{0}$.

$$\chi_{L(G)} = \phi_1(\chi_{L(G \boxminus v)}) \xrightarrow{v}_G \phi_0(\chi_{L(G \boxminus v)}) \xrightarrow{*}_G \phi_0(\mathbf{0}) = \mathbf{0}\,.$$

$\square$

The slides (and the paper) will be available at

http://math.sjtu.edu.cn/faculty/ykwu/home.php

Figure 16 : Nikolay Petrovich Dolbilin, June 18, 2013.

Уважаемый Николай Петрович!

Your mathematics blackboard gives us great pleasure and I wish that the presentation of this game might relax you for a bit while.