Spanning connectivity and Hamiltonian thickness of graphs and interval graphs

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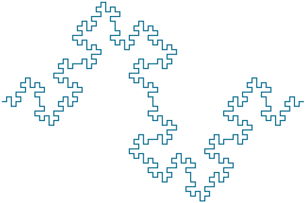

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ICOSIAN GAME: A VOYAGE AROUND THE WORLD

Figure : http://mathhmagic.blogspot.com/2011/05/icosian-game-tetris.html

In 1859, Sir William Rowan Hamilton used his Icosian calculus to show that one can always complete a cyclic tour through all vertices of the Icosian (dodecahedral graph), no matter what path of five vertices is chosen as a start.

SPANNING CONNECTIVITY

- Connectivity property: existence of a basic connection pattern (cycle subgraph); moreover, any path of length four can be packed into the cycle, ...
- Spanning property: the cyclic tour passes through all vertices.

We can consider other connection patterns and impose different robustness requirement for the existence of such a pattern and in this way we can define different families of graphs which can be said to be of high spanning connectivity.

Plan of the talk

1. Describe several desired properties for graphs with good "spanning connectivity".
2. Explain what is Hamiltonian thickness and how does it help us understand (sparse) graphs with good spanning connectivity.
3. Give precise statements of a few sample results.

BASIC CONNECTION PATTERN: ROOTED PATH SYSTEM

Let $(s_i, t_i) \in \binom{V(G)}{2}, 1 \le i \le k$, be k pairs of vertices. A k-path system rooted at $(s_i, t_i), 1 \le i \le k$, is a set of internally-vertex-disjoint s_i, t_i-paths P_i, $1 \le i \le k$, such that no vertex from $\cup_{i=1}^{k}\{s_i, t_i\}$ can appear as an inner vertex of P_i.

When all (s_i, t_i) coincide with (s, t), a path system defined above is called a k-rail between s and t.

When $s_i = s$ for all i and $U = \{t_1, \ldots, t_k\} \subseteq \binom{V(G)}{k}$, we call such a path system a $k-(x; U)$-fan, or simply a k-fan.

A cycle is a 2-rail. A path is a 1-fan and is also a 1-rail when it is of positive length.

ENDPOINT CONSTRAINTS

- Traceable graph: Hamiltonian path (spanning 1-fan)
- Hamiltonian connected: spanning 1-rail between every two different vertices
- Spanning k-rail
- Spanning k-rails between all possible given endpoints
- Hamiltonian graph: Hamiltonian cycle (spanning 2-rail between all given endpoints)
- Spanning k-fan
- Spanning rooted path system
- ...

PACKABILITY

A linear forest is a forest in which every component is a path. The size of a linear forest is the sum of its number of edges and its number of isolated vertices.

- Each linear forest of size k can be packed into a spanning path/cycle/k-rail...
- Given any linear forest F of size k and any set R of given endpoints (with some natural property [1]), there exists a spanning path/cycle/k-rail...rooted at R passing through F.

[1] You could not expect an s, t-path passing through a linear forest F in which s or t has degree 2.

ORDERING CONSTRAINTS

In addition to the requirement of passing through a linear forest F, we may hope that for any specified ordering of the components of F we can find a Hamiltonian cycle/path which traverses these components according to that ordering.

EXTENSION PROPERTY

- A graph on n vertices is cycle extendable if for every cycle C of less than n vertices there is another cycle containing all vertices of C plus a single new vertex.
- path extendable
- k-rail extendable
- k-fan extendable
- Extension of rooted path system

FAULT-TOLERANCE

- The subgraphs obtained from G by deleting at most/exactly k vertices all have a specified spanning connectivity property.
- The subgraphs obtained from G by deleting at most/exactly k edges all have a specified spanning connectivity property.
- For all possible t_1, t_2 satisfying $t_1 + t_2 \leq k$, the subgraphs obtained from G by deleting t_1 vertices and t_2 edges all have a specified spanning connectivity property.
- ...

FORWARD DEGREE SEQUENCE

Let G be a graph on n vertices and let $\pi_1, \ldots, \pi_n$ be an ordering of $V(G)$. The forward degree of π_i w.r.t G and π is $d_{G,\pi}(i) = |N_G(\pi_i) \cap \{\pi_{i+1}, \ldots, \pi_n\}|$ and the forward degree sequence of G w.r.t π is $(d_{G,\pi}(1), \ldots, d_{G,\pi}(n))$.

Given a positive integer $k \leq n - 1 = |V(G)| - 1$, we say that π is a k-thick ordering of G provided $d_{G,\pi}(i) \geq \min\{k,\ n - i\}$ holds for all $i \in [n]$. The existence of a k-thick ordering implies that the graph is k-connected [4, Lema 9.3].

The concept of forward degree sequence already appears in the study of the chip firing game (divisors on graphs) [3, Remark 1.10, Lemma 3.2, Theorem 3.3] and in the study of the offer rejection problem [7].

THICK HAMILTONIAN ORDERING

Let G be a graph and π an ordering of $V(G)$.

For any positive integer k, we call π a k-thick Hamiltonian ordering of G if π itself is a k-thick ordering of G and along π we read a Hamiltonian path of G, namely $\pi_i\pi_{i+1} \in E(G)$ holds for each $i \in [n-1]$.

For any nonnegative integer k, we say that π is a $-k$-thick Hamiltonian ordering provided along the ordering π the graph splits into at most $k+2$ paths, namely $|\{i \in [n-1] : \pi_i\pi_{i+1} \notin E(G)\}| \leq k+1$.

The Hamiltonian thickness with respect to G of the ordering π, denoted $\mathcal{H}_G(\pi)$, is the maximum integer k such that π is a k-thick Hamiltonian ordering.

HAMILTONIAN THICKNESS OF A GRAPH

For every $v \in V(G)$, the height of v in G, denoted by $\mathcal{H}_G(v)$, is defined to be $\max\{\mathcal{H}_G(\pi) : \pi$ is an ordering of $V(G)$ starting at $v\}$.

For every graph G, the height (Hamiltonian thickness) of G is $\mathcal{H}(G) = \max_{v \in V(G)} \mathcal{H}_G(v)$.

If a graph G has n vertices, then $2 - n \leq \mathcal{H}(G) \leq n - 2$ if it is not the complete graph and $\mathcal{H}(G) = n - 1$ otherwise.

For any integer k, denote by F^k the set of graphs possessing a k-thick Hamiltonian ordering, namely $F^k = \{G : \mathcal{H}(G) \geq k\}$.

OUR MAIN DISCOVERIES

1. A graph with height at least $O(k)$ has various spanning connectivity involving rooted k-path systems. This means that we can construct sparse graphs with good spanning connectivity.
2. For interval graphs, almost all those spanning connectivity properties are equivalent and are characterized by the height parameter. The concept of Hamiltonian thickness is also useful in algorithm design.

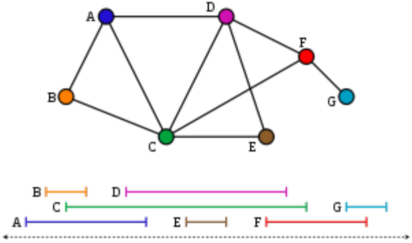


Figure : http://en.wikipedia.org/wiki/Interval_graph

IRRELEVANCE OF THE ROOT

Theorem 1
If G is an interval graph, then $|\mathcal{H}_G(v) - \mathcal{H}_G(w)| \leq 1$ *for all* $v, w \in V(G)$.

Conjecture 2
For every graph G and every $v, w \in V(G)$, *it holds* $|\mathcal{H}_G(v) - \mathcal{H}_G(w)| \leq 1$.

Most of our results are proved by induction. If Conjecture 2 is true, we should be able to get more results for general graphs and some existing results can be proved more easily.

SCATTERING NUMBER

For any graph H, let $c(H)$ denote the number of connected components of H. If G is a graph which is not complete, its *scattering number* [10] is $sc(G) = \max\{c(G-S) - |S| : S \subseteq V,\ c(G-S) > 1\}$. If G is the complete graph K_n, we make the convention that $sc(G) = 3 - n$.

Theorem 3

For every interval graph G, it holds $sc(G) + \mathcal{H}(G) = 2$.

If G is the Petersen graph, we have $sc(G) = -1$ and $\mathcal{H}(G) = 1$.

Theorem 3 allows us to derive almost all results in [5] from our work on Hamiltonian thickness.

Theorem 4 (Vertex-fault-tolerance, passing through edges)

Let $k \geq 2$ be an integer. If $G \in \mathcal{F}^k$ holds, then for all nonnegative integers t_1 and t_2 such that $t_1 + t_2 \leq k - 2$, for any set $S \in \binom{V(G)}{t_1}$ and any set $T \in \binom{E(G-S)}{t_2}$ that induces a linear forest, the graph $G - S$ contains a Hamiltonian cycle which passes through T.

Theorem 5 (Ordering constraint)

Let k be a positive integer. If $G \in \mathcal{F}^{2k}$, then for every linear forest F in G of size at most k, there is a Hamiltonian path of G which encounters the components of F in any specified order. If $G \in \mathcal{F}^{\lfloor \frac{3k}{2} \rfloor + 2}$ is an interval graph, then for every sequence $v_1, \ldots, v_k$ of distinct vertices of G there is a Hamiltonian cycle which encounters $v_1, \ldots, v_k$ in this order.

Theorem 6 (Routed path system)

Let k be a positive integer. If $G \in \mathcal{F}^{2k+1}$, then for any k (not necessary distinct) elements from $\binom{V(G)}{2}$, there is a spanning k-path system rooted at those k elements.

CHVÁTAL-ERDÖS TYPE RESULTS:

LOWER BOUND OF THE HAMILTONIAN THICKNESS

The Chvátal-Erdös Theorem [8] states that a 2-connected graph G is Hamiltonian if its independence number $\alpha(G)$ is bounded from above by its vertex connectivity $\kappa(G)$.

Theorem 7

Let G be a connected interval graph. Then, $\mathcal{H}(G) \geq \kappa(G) - \alpha(G) + 2$ *if G is not a complete graph and* $\mathcal{H}(G) = \kappa(G) - \alpha(G) + 1$ *otherwise.*

Theorem 8

Let G be a connected interval graph with at least 3 vertices and k be any nonnegative integer. If $\kappa(G(N_G[v])) - \alpha(G(N_G[v])) \geq k$ *holds for each* $v \in V(G)$*, then* $\mathcal{H}(G) \geq k + 2$ *unless G is a complete graph in which case we have* $\mathcal{H}(G) \geq k + 1$.

EQUIVALENT CHARACTERIZATIONS

Theorem 9

Let G be an interval graph and let $k \geq 3$ be an integer. The following statements are all equivalent.

- $G \in F^k$.
- *There exists a spanning k-rail of G between every two different vertices.*
- *The deletion from G of any vertex subset of size at most $k-1$ results in a traceable graph.*
- $G - F$ *is Hamiltonian for every set* $F \subseteq V(G) \cup E(G)$ *with* $|F| \leq k-2$.
- *The deletion from G of any vertex subset of size at most $k-3$ results in a Hamiltonian-connected graph.*
- *For every choice of x and U with $|U| \leq k-1$ and $x \notin U$, the graph G has a spanning k-(x, U)-fan.*

AN EXTENSION RESULT

Theorem 10

Let G be an interval graph and let S be a proper subset of $V(G)$. Suppose that both G and $G[S]$ have spanning typical k-rails. Then, there is a vertex $x \in V(G) \setminus S$ such that $G[S \cup \{x\}]$ also has a spanning typical k-rail.

Setting $k = 2$ in the above theorem leads to the known result that an interval graph is cycle extendable if and only if it is Hamiltonian [1, 6].

DIFFICULT PROBLEMS FOR INTERVAL GRAPHS?

It seems that 1HP and 2HP keep their difficulties even in the small subclass of split interval graphs – no polynomial algorithm is known. Till now there are no "natural" graph problems known to be NP-complete for interval graphs, except ACHROMATIC NUMBER. May 1HP and 2HP be further candidates? Only by very superficial consideration this seems to be unlikely. – Peter Damaschke (1993) [9]

1-Fixed-Endpoint Path Cover Problem

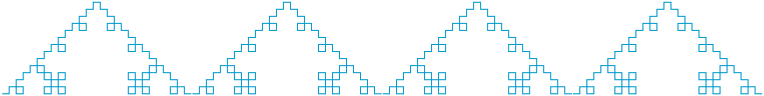
k-PC: Given a set of *k* vertices, find a minimum number of disjoint paths that cover all vertices of the graph and contain those given *k* vertices as endpoints.

Theorem 11

1PC for interval graphs can be solved in linear time.

Earlier result of Asdre and Nikolopoulos says that 1PC for interval graphs can be solved in cubic time [2].

THE ROAD NOT TAKEN ...



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