Incidence matrix and cover matrix of a poset

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November 27, 2010
Joint work with Andreas Dress and Shizhen Zhao

- Yaokun Wu, Shizhen Zhao, Incidence matrix and cover matrix of a nested interval poset, submitted.
Outline

1. Poset and its incidence algebra

2. Some observations and questions on $n$ and $C$

3. Answers and hints
A poset \((P, <)\) is just an acyclic transitive simple digraph \(^1\) on the vertex set \(P\) where the partial order relationship \(x < y\) denotes the existence of an arc from \(x\) to \(y\) in the digraph \(P\).

An ideal (upset) \(S\) of a poset \(P\) is one of its subposets such that

\[
x \in S, x < y \Rightarrow y \in S.
\]

The complement of an ideal is a filter (downset).

\(^1\)Being simple means having no multiple arcs and no loops.
Given any set $S$, $2^S$ is naturally a poset under the set inclusion relationship: $X < Y$ if and only if $X$ is a proper subset of $Y$. We call this poset a power set poset, or a Boolean algebra $^2$.

A simplicial complex is just a filter/downset of a powerset poset, that is, a set system closed under the taking subset operation.

A relative simplicial complex is an ideal of a simplicial complex. In other words, a set system $S$ is a relative simplicial complex provided for any $X, Y \in S$, all the elements from $\{Z : X \subseteq Z \subseteq Y\}$ lie in $S$ as well.

$^2$All finite Boolean algebras are powerset posets.
Interval order

Let $P$ be a collection of nonempty intervals $I_t = [a_t, b_t]$ in the real line. Define $I_t < I_r$ whenever $I_t$ is totally to the left of $I_r$, namely $b_t < a_r$. The resulting poset is usually called an interval order.

If the set of intervals can be chosen to be nested, namely $I_t \cap I_r \in \{\emptyset, I_t, I_r\}$ for any $I_t, I_r \in P$, the interval order is a nested interval order.

If the set of intervals can be chosen to have no containment relationship, namely $I_t \cap I_r \notin \{I_t, I_r\}$ for any $I_t, I_r \in P$, the interval order is said to be a proper interval order. ³

³Any proper interval order is isomorphic to an interval order in which all intervals are of unit length and hence it is also known as a unit interval order or a semiorder.
Interval poset for a given poset

Let \((P, <)\) be any poset. For \(a, b \in P\), \([a, b] = \{ c \in P : a \leq c \leq b \}\) is called an interval of \(P\).

Given a set \(S\) of nonempty intervals of \(P\), we can introduce a partial order on \(S\) so that there is an arc from \([a, b]\) to \([c, d]\) if and only if \(b < c\) in \(P\) and we thus obtain an interval poset for \(P\). The interval orders are just interval posets for the real line.

We can generalize the definition of nested interval order and proper interval order in an obvious way and define nested interval poset and proper interval poset for any given poset.
Hasse diagram

For any \( x, y \in P \), we say that \( y \) covers \( x \) provided \( |[x, y]| = 2 \) and denote this by \( x \lessdot y \).

The Hasse diagram of a poset \( P \) is the digraph \( \Gamma(P) \) with \( P \) as the vertex set and there is an arc from \( x \) to \( y \) if and only if \( x \lessdot y \).
Incidence matrix and cover matrix

The adjacency matrix of a digraph $G$ is a square matrix whose lines correspond to the vertices of $G$ and whose $(x, y)$-entry records the number of arcs going from $x$ to $y$.

The cover matrix $n_P$ is the adjacency matrix of the Hasse diagram of $P$ and the incidence matrix $C_P$ is the adjacency matrix of $P$. In other words, $n_P$ is the indicator function of $<$ and $C_P$ is the indicator function of $\prec$.

In some sense, $n_P$ is a global view of $P$ and $C_P$ is a local view of $P$; they both encode full information of $P$ but $C_P$ is more sparse than $n_P$. 
The operator $C_P$ may be regarded as an instance of the finite Radon transform [Kung, Stanley] and becomes the usual boundary operator for relative simplicial homology when $P$ is a relative simplicial complex.


The zeta function (matrix) of $P$ is $\zeta_P = n_P + \delta_P$, which is the indicator function of the partial order $\leq_P$.

The key to the Möbius Inversion Theorem (MIT) is the determination of the Möbius function (matrix) of $P$, $\mu_P = \zeta_P^{-1}$, which is, considering that $n_P$ is nilpotent, $\delta_P - n_P + n_P^2 - n_P^3 + n_P^4 - \cdots$. 
Let $d_P$ be the dimension of the kernel of $n_P$. This parameter $d_P$ turns out to be a lower bound of the number of incomparable adjacent pairs in any linear extension of $P$ [Gierz & Poguntke] and an upper bound of the width of $P$ [Gansner,Saks] and for almost all posets in the uniform random poset model it is exactly the width of $P$ [Fishkind].


Main focus of this talk

Is there any interesting or useful connection between $n_P$ and $C_P$?
For any nilpotent complex matrix $A$ and any analytic function $f(x)$ such that $f(0) = 0$ and $\left. \frac{df(x)}{dx} \right|_{x=0} \neq 0$, it is clear that $A$ and $f(A)$ have the same Jordan canonical form over the complex field.
Let $P$ be a poset whose intervals are always finite Boolean algebras, say being a relative simplicial complex. Viewed as matrices over integers, it is easy to see that

$$n_P = C_P + \frac{C_P^2}{2} + \frac{C_P^3}{3} + \cdots = \exp(C_P) - \delta_P.$$ 

This tells us that $n_P$ and $C_P$ have the same Jordan canonical form.
A poset $P$ is a upp poset (poset with the unique path property) provided there exists at most one path from any vertex to any other vertex in its Hasse diagram.

If $P$ has the unique path property, then
\[
n_P = C_P + C_P^2 + \cdots = C_P(\delta_P - C_P)^{-1}.
\]
Since $C_P$ is nilpotent, it follows that $n_P$ and $C_P$ are similar in the full complex matrix algebra.
Incidence algebra

Let $R$ be any commutative ring with a unit and let $P$ be a locally finite poset, namely each interval in $P$ is finite (We always make this assumption from now on).

The incidence algebra of $P$, denoted $Inc_{R}P$, is the set of functions/matrices $f$ from $P \times P$ to $R$ such that $f(x, y) = 0$ unless $x \leq y$ in $P$. One can also think of any element of the incidence algebra as a function from the set of all intervals of $P$ to the ring $R$.

Multiplication in the incidence algebra is just the usual matrix multiplication (function convolution):

$$fg(x, z) = \sum_{y \in [x, z]} f(x, y)g(y, z).$$

The incidence algebras of posets are important computational devices for many enumeration problem on posets and their algebraic properties have been intensively studied.

Stanley shows that the poset $P$ is uniquely determined by the structure of its incidence algebra.

Two elements $A, B \in \text{Inc}(P)$ are conjugate if there is $\alpha \in \text{Inc}(P)$ such that $A = \alpha B \alpha^{-1}$. These two matrices $A$ and $B$ are similar if they are conjugate in the full matrix algebra. Surely, $A$ and $B$ are similar over an algebraically closed field just means that they have the same Jordan canonical form or the same Jordan invariants (the sizes of its Jordan blocks) over that field.
Let
\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0 
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 
\end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 
\end{bmatrix}.
\]

It is easy to check that both \(B\) and \(C\) are the Jordan canonical forms of \(A\) in the full complex matrix algebra but there is no upper triangular matrix \(\alpha\) such that \(\alpha A \alpha^{-1} \in \{B, C\}\) \(^4\).

Note that the set of \(n \times n\) upper triangular matrices can be identified with the incidence algebra of the linear order on \(n\) elements.

\(^4\)Just compare the \((2, 2)\)-entries of \(\alpha A\), \(B\alpha\) and \(C\alpha\).
Prompted by the theory of Jordan canonical form in the full matrix algebra over an algebraically closed field, Stanley asks if there is any reasonable criterion for determining whether or not two elements of the incidence algebra of a poset are conjugate to each other. Marenich finds some interesting results in her effort to tackling the problem of Stanley and she proposes to view $\lambda \delta_{P'} + C_{P'}$, where $P'$ is a subposet of $P$, as a “Jordan block" for $Inc(P)$.

Let $P$ be a linear order, namely a poset of dimension 1, on $n$ elements, say $1 < 2 < \cdots < n$. Then,

$$
\mathbf{C}_P = \begin{bmatrix}
0 & 1 \\
0 & 1 \\
\vdots & \vdots \\
0 & 1 \\
0 & 0
\end{bmatrix}_{n \times n},
$$

$$
\mathbf{n}_P = \begin{bmatrix}
0 & 1 & 1 & \cdots & 1 \\
0 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{bmatrix}_{n \times n}.
$$

$\mathbf{C}_P$ is clearly the Jordan canonical form of $\mathbf{n}_P$. 
Are they conjugate in the incidence algebra?

\[ n_P = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix} \quad \rightarrow \quad \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} = C_P?
\]

Let \( P \) be a linear order. Let \( f \) be any nilpotent matrix in \( \text{Inc}(P) \) which coincides with \( C_P \), hence takes value 1, on the support of \( C_P \).

It is easy to illustrate that \( f \) is always conjugate to \( C_P \) in \( \text{Inc}(P) \).
\[
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 1 \\
0 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 5 & 0 \\
0 & 1 & 1 & 1 & 4 & 0 \\
0 & 1 & 1 & 3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

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\[
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 5 & 0 \\
0 & 1 & 1 & 1 & 4 & 0 \\
0 & 1 & 1 & 3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 1 & 0 \\
0 & 1 \\
0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
0 & 1 & 1 & 1 & 10 & 0 & 0 \\
0 & 1 & 1 & 6 & 0 & 0 \\
0 & 1 & 3 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 \\
0
\end{bmatrix}
\]
$$\begin{bmatrix}
0 & 1 & 1 & 1 & 10 & 0 & 0 \\
0 & 1 & 1 & 6 & 0 & 0 & 0 \\
0 & 1 & 3 & 0 & 0 & 0 \\
0 & 1 & 3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \Rightarrow \begin{bmatrix}
0 & 1 & 1 & 10 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$
\[
\begin{bmatrix}
0 & 1 & 1 & 10 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 \\
0 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
0 & 1 & 5 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 \\
0 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & 1 & 5 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 \\
0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 \\
0 & 1 \\
0
\end{bmatrix}
\]
A poset $P$ is *graded* if there is a rank function $\rho$ from $P$ to integers such that if $y$ covers $x$ then $\rho(y) = \rho(x) + 1$.

This graded poset $P$ is *homogenous* provided for any $n \leq k \leq \ell$, any $x \leq y, x \in \rho^{-1}(n), y \in \rho^{-1}(\ell)$, the set $[x, y] \cap \rho^{-1}(k)$ has a size $t_{n,k,\ell}$, which is independent of $x$ and $y$. Homogenous posets include the lattices of linear (affine) subspaces of a finite vector space and the posets of relative simplicial complexes.
Let $P$ be a homogenous poset as defined above for which $t_{n,n+1,\ell} \neq 0$ for any $n < \ell$. Suppose that $\{\rho(x) : x \in P\} = \{0, 1, 2, \ldots, m\}$. Put $T_a = \{(i,j) : 0 \leq i \leq j \leq m, j - i = a\}$ and $T = \bigcup_{a=0}^{m} T_a$.

Let $g$ be a map from $T$ to $F$ satisfying $g^{-1}(0) \supseteq T_0$ and $g^{-1}(0) \cap T_1 = \emptyset$. Let $f \in Inc_F(P)$ be a function such that $f(x,y) = g(\rho(x), \rho(y))$ for any $x \leq_P y$. Note that when $g$ takes constant value 1, $f$ is nothing but $n_P$. Marenich finds that $f$ is always conjugate to $C_P$ in $Inc_F(P)$.

Outline

1. Poset and its incidence algebra

2. Some observations and questions on $n$ and $C$

3. Answers and hints
Let $P$ be the poset with the Hasse diagram as shown above. Note that $P$ is a poset of dimension 2. Simple calculation leads to

$$\text{rank}_\mathbb{Z}(n_P) = \text{rank}_{\mathbb{F}_2}(n_P) = 5 > 4 = \text{rank}_\mathbb{Z}(C_P) = \text{rank}_{\mathbb{F}_2}(C_P).$$
From a chain to a product of many chains of two elements

The powerset poset is just the "product" of a set of 2-paths.

Observation 1 (Dress, W.)

Let $R$ be a unitary ring of characteristic 2 and let $P$ be a relative simplicial complex. Then $n_P$ and $C_P$ are conjugate in $Inc_F(P)$. 
Construction of the conjugacy: I

We assume that $P$ is a simplicial complex. Fix a linear order $\prec$ on $\bigcup_{S \in P} S$. For any $S = \{x_1 \prec x_2 \prec \cdots \prec x_t\} \in P$, set

$$E(S) = \{x_2, x_4, \ldots, x_{2\lfloor \frac{t}{2} \rfloor}\}.$$  

Specify $\Omega \in Inc_F(P)$ by letting $\Omega(R, S) = 1$ if $E(S) \subseteq R \subseteq S$ and $\Omega(R, S) = 0$ otherwise. It is not difficult to check that

$$\Omega C_P \Omega^{-1} = n_P. \tag{1}$$
Construction of the conjugacy: II

Note that for each ordering of $\cup_{S \in P} S$ the above construction gives a solution $\Omega$ to Eq. (1). To investigate if there is any other solution to Eq. (1), we are led to a special case of another general problem of Stanley, namely determining the dimension of the centralizer algebra of $C_P$ in $Inc_F(P)$.

From linear order to interval poset

If $P$ is the interval poset for a upp poset, then $n_P$ and $C_P$ have the same row space and $\text{rank}_\mathbb{Z}(n_P) = \text{rank}_\mathbb{Z}(C_P)$.

Let us remark that $\text{rank}_\mathbb{Z}(n_P) = \text{rank}_\mathbb{Z}(C_P)$ is generally not true:

Let $P$ be a simplicial complex of dimension at least 2. Then, we can suppose that $P$ contains a 2-dimensional face $\{1, 2, 3\}$ and hence the three 1-dimensional faces $\{1, 2\}$, $\{2, 3\}$, and $\{3, 1\}$. Over $\mathbb{F}_2$ we have

\[
n_P(\cdot, \{1, 2\}) + n_P(\cdot, \{2, 3\}) + n_P(\cdot, \{3, 1\}) = n_P(\cdot, \{1\})
\]

while

\[
C_P(\cdot, \{1, 2\}) + C_P(\cdot, \{2, 3\}) + C_P(\cdot, \{3, 1\}) \neq C_P(\cdot, \{1\}).
\]

This means that $n_P$ and $C_P$ have different row spaces over $\mathbb{F}_2$. 
Let $P$ be an interval order. As a special case of the observation listed in last slides, we have

$$\text{rank}(n_P) = \text{rank}(C_P)$$

and hence $n_P$ and $C_P$ have the same number of Jordan blocks.

We do not find any interval order $P$ yet for which

$$\text{rank}(n_P^2) \neq \text{rank}(C_P^2).$$

It is for about one year during which our computer experiment does not locate any interval order $P$ and any integer $i$ such that

$$\text{rank}(n_P^i) \neq \text{rank}(C_P^i).$$

This suggests us conjecture that $C_P$ and $n_P$ always have the same Jordan canonical form for any interval order $P$ and leads us to several wrong proofs.
For any digraph $\Gamma$ and any positive integer $k$, let $p_k(\Gamma)$ be the maximum size of a union of the vertex sets of $k$ vertex-disjoint (maybe empty) paths in $\Gamma$.

For any matrix $A$ we can associate a digraph $\Gamma(A)$ in which there is an arc $xy$ if and only if $A(x,y) \neq 0$.

For any poset $P$, it is clear that a longest path in $\Gamma(n_P)$ must also be a longest path in $\Gamma(C_P)$ (namely, $p_1(\Gamma(n_P)) = p_1(\Gamma(C_P))$) and so the largest Jordan block of $C_P$ has the same size with that of $n_P$. 
For any generic nilpotent matrix $A$ of order $n$, its Jordan invariants are $p_1(\Gamma(A))$, $p_2(\Gamma(A)) - p_1(\Gamma(A))$, $p_3(\Gamma(A)) - p_2(\Gamma(A))$, $\ldots$, $p_s(\Gamma(A)) - p_{s-1}(\Gamma(A))$, where $s = n - \text{rank}(A)$.

It is clear that $p_i(\Gamma(n_P)) \geq p_i(\Gamma(C_P))$. When does the equality hold?


Figure: An interval representation of a proper interval order.

\[
\begin{align*}
I_1 & \quad I_6 & \quad I_{11} & \quad I_{16} & \quad I_{21} \\
I_2 & \quad I_7 & \quad I_{12} & \quad I_{17} & \quad I_{22} \\
I_3 & \quad I_8 & \quad I_{13} & \quad I_{18} & \quad I_{23} \\
I_4 & \quad I_9 & \quad I_{14} & \quad I_{19} & \quad I_{24} \\
I_5 & \quad I_{10} & \quad I_{15} & \quad I_{20} \\
\end{align*}
\]

\[
\begin{align*}
(\text{rank}_\mathbb{Z}(n_P), \text{rank}_\mathbb{Z}(n_P^2), \ldots, \text{rank}_\mathbb{Z}(n_P^7)) &= (17, 13, 9, 6, 4, 2, 0), \\
(\text{rank}_\mathbb{Z}(C_P), \text{rank}_\mathbb{Z}(C_P^2), \ldots, \text{rank}_\mathbb{Z}(C_P^7)) &= (17, 13, 10, 7, 4, 2, 0). \\
\end{align*}
\]  

The Jordan invariants of $n_P$ and $C_P$ are $(7, 7, 4, 3, 1, 1, 1)$ and $(7, 7, 5, 2, 1, 1, 1)$, respectively. Note that $\text{rank}_\mathbb{Z}(C_P^i) \geq \text{rank}_\mathbb{Z}(n_P^i)$ for all $i$. 

---

[Figure: An interval representation of a proper interval order.]

\[
\begin{align*}
I_1 & \quad I_6 & \quad I_{11} & \quad I_{16} & \quad I_{21} \\
I_2 & \quad I_7 & \quad I_{12} & \quad I_{17} & \quad I_{22} \\
I_3 & \quad I_8 & \quad I_{13} & \quad I_{18} & \quad I_{23} \\
I_4 & \quad I_9 & \quad I_{14} & \quad I_{19} & \quad I_{24} \\
I_5 & \quad I_{10} & \quad I_{15} & \quad I_{20} \\
\end{align*}
\]

\[
\begin{align*}
(\text{rank}_\mathbb{Z}(n_P), \text{rank}_\mathbb{Z}(n_P^2), \ldots, \text{rank}_\mathbb{Z}(n_P^7)) &= (17, 13, 9, 6, 4, 2, 0), \\
(\text{rank}_\mathbb{Z}(C_P), \text{rank}_\mathbb{Z}(C_P^2), \ldots, \text{rank}_\mathbb{Z}(C_P^7)) &= (17, 13, 10, 7, 4, 2, 0). \\
\end{align*}
\]  

The Jordan invariants of $n_P$ and $C_P$ are $(7, 7, 4, 3, 1, 1, 1)$ and $(7, 7, 5, 2, 1, 1, 1)$, respectively. Note that $\text{rank}_\mathbb{Z}(C_P^i) \geq \text{rank}_\mathbb{Z}(n_P^i)$ for all $i$. 

---
The Jordan invariants of $n_P$ and $C_P$ are $(5, 4, 2, 1)$ and $(5, 3, 3, 1)$, respectively. It may be worth noting that $P$ is not any proper interval order and $\text{rank}_Z(C_P^i) \leq \text{rank}_Z(n_P^i)$ for all $i$. 

**Figure:** An interval representation of an interval graph.

\[
\begin{align*}
\{ & (\text{rank}_Z(n_P), \text{rank}_Z(n_P^2), \ldots, \text{rank}_Z(n_P^5)) = (8, 5, 3, 1, 0), \\
& (\text{rank}_Z(C_P), \text{rank}_Z(C_P^2), \ldots, \text{rank}_Z(C_P^5)) = (8, 5, 2, 1, 0), \\
\}\end{align*}
\]
Let $P$ be any proper interval order with at most 11 elements. A computer enumeration shows that $\text{rank}_{\mathbb{Z}}(n^k_P) = \text{rank}_{\mathbb{Z}}(C^k_P)$ for all positive integer $k$ and hence $n_P$ and $C_P$ have the same Jordan canonical form over the complex field.

It seems interesting to understand which kind of obstruction appeared in an interval order $P$ can cause the Jordan invariants of $C_P$ and $n_P$ different.
A one-child poset is a poset in whose Hasse diagram each vertex can have at most one out-neighbor.

**Observation 2 (W., Zhao)**

Let $P$ be a nested interval poset for some one-child poset and let $F$ be a unitary commutative ring of characteristic 0. Then $n_P$ and $C_P$ are conjugate in $Inc_F(P)$, and hence have the same Jordan canonical form when $F$ is a characteristic 0 algebraically closed field.
We can similarly define single-parent posets. What we observe for one-child posets surely apply for single-parent posets and both single-parent posets and one-child posets are upp posets.

Let $P$ be a nested interval poset for a upp poset and $F$ a characteristic 0 field. Are $n_P$ and $C_P$ always in the same conjugacy class of $Inc_F(P)$? Do they at least have the same Jordan invariants when viewed as integer matrices? Recall that if the poset is itself a upp poset, we have seen earlier that the answer is yes.

Can we relax the characteristic 0 condition in Observation 2 for one-child posets?
Observation 3 (W., Zhao)

Let $Q$ be the (strong) Bruhat order of the dihedral group of order $2m$ for $m \geq 3$. Then $n_Q$ and $C_Q$ are conjugate to each other in $Inc_R(Q)$ over any unitary commutative ring $R$.

Note that this observation means that the characteristic 0 condition can be removed in this special case.
Observation 4 (W., Zhao)

Let $Q$ be the weak order of the dihedral group of order $2m$, $m \geq 3$. Then $n_Q$ and $C_Q$ are conjugate to each other in $Inc_R(Q)$ over any unitary commutative ring $R$. 

Figure: Hasse diagram for $m = 5$
What about general Bruhat orders?

Figure: Hasse diagram of the Bruhat order of type $B_3$

This diagram is also depicted as [BB, Fig. 2.2] but one arc is missing there.

<table>
<thead>
<tr>
<th>Type of Bruhat order</th>
<th>Jordan invariants of $n$ and $C$</th>
<th>Hasse diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_2$</td>
<td>$5, 1^3$</td>
<td>[BB,Fig. 2.1]</td>
</tr>
<tr>
<td>$B_3$</td>
<td>$10, 8^2, 6^2, 4^2, 2$</td>
<td>See last slides</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$4, 1^2$</td>
<td>[BB,Fig. 2.3]</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$7, 5, 4^2, 3, 1$</td>
<td>[BB,Fig. 2.4]</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$11, 9, 8^4, 7^3, 5^5, 4^2, 3^2, 2^2, 1^4$</td>
<td></td>
</tr>
<tr>
<td>$A_5$</td>
<td>$16, 14^2, 13^4, 12^5, 11^4, 10^{12}, 9^2, 8^{16}, 7^6, 6^{18}, 5^2, 4^{15}, 3^4, 2^8, 1^6$</td>
<td></td>
</tr>
<tr>
<td>$S_6^{(3)}$</td>
<td>$10, 6, 4$</td>
<td>[BB,Fig. 2.7]</td>
</tr>
<tr>
<td>$E_6$ modulo $D_5$</td>
<td>$17, 9, 1$</td>
<td>[BB,Fig. 2.8]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Type of weak order</th>
<th>Jordan invariants of $n$ and $C$</th>
<th>Hasse diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_2(4)$</td>
<td>$5, 3$</td>
<td>[BB,Fig. 3.1]</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$7, 5^2, 3^2, 1$</td>
<td>[BB,Fig. 3.2]</td>
</tr>
<tr>
<td>$H_3$</td>
<td>$16, 14^2, 12^2, 10^2, 8^2, 6^2, 4$</td>
<td>[BB,Fig. 3.3]</td>
</tr>
</tbody>
</table>
Even for the poset $P$ whose Hasse diagram is as in [BB,Fig. 2.2], $n_P$ and $C_P$ have the same Jordan invariants and these invariants coincide with the ones arising from our calculation for the Bruhat order of type $B_3$.

Is there any general good relationship between $n_P$ and $C_P$ for a (weak) Bruhat order?
Outline

1. Poset and its incidence algebra

2. Some observations and questions on $n$ and $C$

3. Answers and hints
Appreciation of mathematics grows with the discernment of links between apparently dissimilar pieces of information and the perception of the general in the particular.

Any more general picture behind? Answers and hints?

Figure: A lane in SJTU

Please help!
Thanks! Welcome to SJTU!