Polynomials in Combinatorics: Four Examples

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Combinatorial Structures

Many facts and problems in mathematics, computer science, and engineering are most easily stated in terms of five combinatorial structures: 1. graphs, 2. directed graphs, 3. partially ordered sets, 4. simplicial complexes, 5. matroids. We will devote one chapter to each of these structures and a sixth chapter to computational complexity, in order to illuminate the relative difficulty of well-known problems. Since the definition of our five structures are simple and very closely related, it is somewhat surprising that they rarely appear together in an introductory combinatorics text. – Mark Skandera, Introduction to Combinatorics and Graph Theory, Lecture Notes, 2003.

Polynomials

Linear Algebra is concerned with polynomials of degree one.

Algebraic Geometry is the study of systems of polynomial equations in one or more variables, combining the algebraic with the geometric for the benefit of both.

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When V is some linear space consisting of polynomials, the above idea falls inside the scope of polynomial method.

Outline

Let $\mathcal{F}$ be a finite field. A Kakeya set in $\mathcal{F}^n$ is a set $E \subseteq \mathcal{F}^n$ which contains a line in each direction, namely for any $v \in \mathcal{F}^n$ there exists $x_v \in E$ such that

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Note that $\binom{|\mathcal{F}|+n-1}{n} \geq \frac{|\mathcal{F}|^n}{n!}$. Thus a Kakeya set over any finite field must be 'full dimensional'.

Zeev Dvir, A Ph.D. Student at the department of Computer Science at the Weizmann Institute of Science.
http://www.wisdom.weizmann.ac.il/~zdvir/

Let $V_d \leq \mathcal{F}[v_1, \ldots, v_n]$ be the $\mathcal{F}$-vectorspace of all polynomials in n variables $v_1, \ldots, v_n$ whose degrees are no greater than d. We regard $V_d = \{0\}$ for any $d < 0$.

Lemma 2

$\dim V_d = \binom{n+d}{n}$.

Proof.

When $n = 4, d = 7$, we construct a map which sends $v_1^2 v_2^0 v_3^1 v_4^2$ to $bbrrbrbbrbb$. □

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Exercise 3

The number of monomials in $\mathcal{F}[v_1, \ldots, v_n]$ of degree exactly d follows readily from Lemma 2. Please try to provide a bijective proof.

Lemma 4

Let E be a Kakeya set in $\mathcal{F}^n$. Then the natural map f from $V_{|\mathcal{F}|-1}$ to $\mathcal{F}^E$ which sends P to $P|_E$ is an injective linear map.

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Do you get a proof of Theorem 1 now?

A set $B \subseteq \mathcal{F}^n$ is a Nikodym set provided for each $v \notin B$ there is a line L such that $L \setminus B = \{v\}$.

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Theorem 6

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Theorem 7 (Deza-Frankl-Singhi, 1983)

Let p be a prime, n a positive integer, and L a set of s integers. Let $\mathcal{H} \subseteq 2^{[n]}$ such that for any $A \neq B \in \mathcal{H}$, we have $|A \cap B| \in L \pmod p$ and $|A| \notin L \pmod p$. Then $|\mathcal{H}| \leq \sum_{k=0}^{s} \binom{n}{k}$.

Theorem 8 (Alon-Babai-Suzuki, 1991)

Let $\mathcal{H} \subseteq 2^{[n]}$ *such that for any* $A \neq B \in \mathcal{H}$, *we have* $|A \cap B| \in \{\lambda_1, \ldots, \lambda_s\}$. *Then* $|\mathcal{H}| \leq \sum_{k=0}^{s} \binom{n}{k}$.

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Next: how shall we define the whale, by his obvious externals, so as conspicuously to label him for all time to come. To be short, then, a whale is a spouting fish with a horizontal tail. There you have him. However contracted, that definition is the result of expanded meditation. – Moby Dick; or The Whale 白鲸记, Herman Melville 赫尔曼.梅尔维尔(1819-1891)

Proof of Theorem 8

Suppose $\mathcal{H} = \{A_1, \ldots, A_m\}$ with $|A_1| \leq |A_2| \leq \cdots \leq |A_m|$.
Put X_i to be the characteristic vector of A_i, $i \in [m]$.

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$$f_i(v) = \prod_{j:\lambda_j < |A_i|} (v \cdot X_i - \lambda_j).$$

We adopt the convention that $\prod_{j \in \emptyset}(v \cdot X_i - \lambda_j) = 1$.

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We adopt the convention that $\prod_{j \in \emptyset}(v \cdot X_i - \lambda_j) = 1$. Define g_i to be the multilinear polynomial which takes the same value with f_i on $\{0,1\}^n$, $i \in [m]$. (Note that $v_1^6 v_2 v_4^2|_{\{0,1\}^4} = v_1 v_2 v_4|_{\{0,1\}^4}$.)

Proof of Theorem 8, Contd.

Observe that

$$M = \begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix} \begin{pmatrix} X_1 & \cdots & X_m \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \begin{pmatrix} X_1 & \cdots & X_m \end{pmatrix}.$$

is an upper triangular matrix whose diagonal is occupied by positive numbers.

To see this, just check that

$$f_i(X_i) = \prod_{j:\lambda_j<|A_i|} (|A_i| - \lambda_j) > 0$$

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and that

$$f_{i+t}(X_i) = \prod_{j:\lambda_j<|A_{i+t}|} (X_i \cdot X_{i+t} - \lambda_j) = 0.$$

Proof of Theorem 8, Fin.

If $\sum_{i=1}^{m} c_i g_i = 0$, then $\sum_{i=1}^{m} c_i g_i(X_j) = 0$ for all j and hence $(c_1 \ \ldots \ c_m)M = 0$ follows. Since M is nonsingular, we know that $c_1 = \cdots = c_m = 0$.

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We thus find that $A_i \longrightarrow g_i$ gives rise to an injective linear map from $R^{\mathcal{H}}$ to the linear space $\mathscr{L}$ of multilinear polynomials in $v_1, \ldots, v_n$ with degrees at most s. It is easy to see that $\dim \mathscr{L} = \sum_{k=0}^{s} \binom{n}{k}$ and so we are done.

Exercise 9

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> *The pleasure we obtain from music comes from counting, but counting unconsciously. Music is nothing but unconscious arithmetic. —G.W. Leibniz*

A set $A \subseteq R^n$ is a two distance set if at most two positive distances are determined by different points of A. Denote by $f(n, 2)$ the maximum size of a two distance set in R^n.

Theorem 10

$f(n, 2) \leq \frac{(n+1)(n+4)}{2}$.

Exercise 11

Can you improve the bound given in Theorem 10?

Proof of Theorem 10

Let $A \subseteq R^n$ be a two-distance set with $|A| = f(n, 2)$ and let d_1, d_2 be the distinct distances determined by A. For every $a = (a_1 \ \ldots \ a_n) \in A$, we define $f_a \in R[v_1, \ldots, v_n]$ by putting

$$f_a(v_1, \ldots, v_n) = (\sum_{i=1}^{n}(v_i - a_i)^2 - d_1^2)(\sum_{i=1}^{n}(v_i - a_i)^2 - d_2^2).$$

We claim that $\{f_a : \ a \in A\}$ is linearly independent. (Did we just learn any method to accomplish this?

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Teacher versus Researcher

http://www.math.rutgers.edu/people/Gelfand.html
Israil Gelfand (1978 Wolf Prize in Mathematics Laureate):

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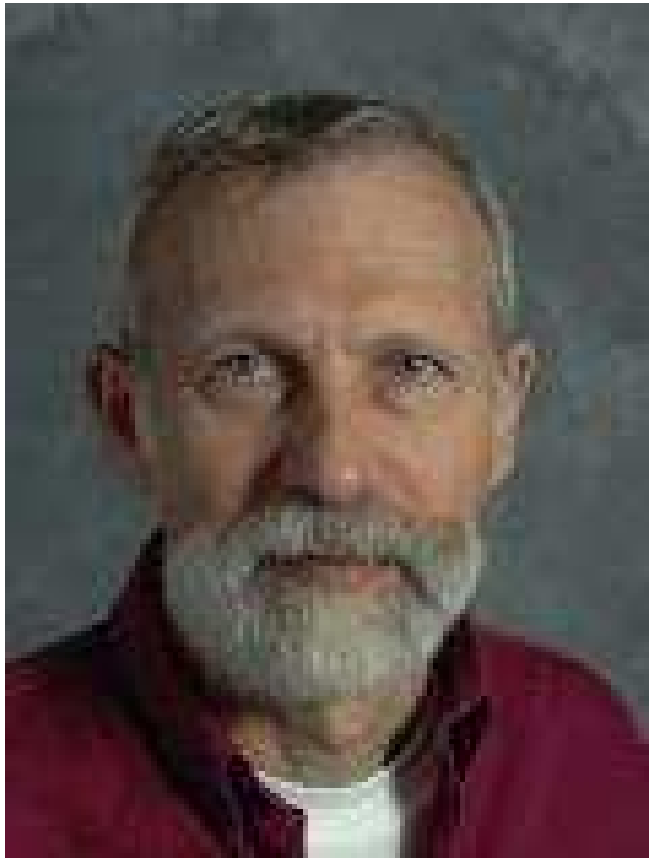
> *One of the characteristic features of Israil Moiseevic's activities has been the extremely close bond between his research work and his teaching. The formulation of new problems and unexpected questions, a tendency to look at even well known things from a new point of view characterises Gelfand as a teacher, regardless of whether at a given moment he is holding a conversation with schoolchildren or with his own colleagues.*



Hell is a boring incomprehensible math talk that goes overtime. – Harry Dym

Thank You!!! 谢谢!!!

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