Split decomposition theory and the mod 2 cohomology of finite simplicial complexes

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Outline

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1. Introduction

We begin with a famous quote of Gian-Carlo Rota*:

The lack of real contact between mathematics and biology is either a tragedy, a scandal, or a challenge, it is hard to decide which.

In his article, "Can biology lead to new theorems?"†, Bernd Sturmfels further reported:

Some scholars have begun to argue that "real contact" means being equal partners, and that meaningful intellectual contributions can, in fact, flow in both directions.

*M. Kac, G-C. Rota, J.T. Schwartz, Discrete Thoughts, Birkhäuser, Boston, 1986.

† Annual Report 2005 of the Clay Mathematics Institute, http://www.claymath.org/library/annual_report/Sturmfels.pdf

In that same paper, Bernd Sturmfels expressed clearly his opinion about "real contact" as follows:

This article argues for an affirmative answer to the question in the title. In future interactions between mathematics and biology, both fields will contribute to each other and, in particular, research in the life sciences will inspire new theorems in "pure" mathematics.

I shall present four theorems which were inspired by biology. These theorems are in algebra, geometry and combinatorics, my own areas of expertise. I leave it to others to discuss biology inspired results in dynamical systems and partial differential equations.

This talk is to report joint work with Andreas Dress on a combinatorial problem arising in computational biology and thus provide our support to the opinion of Sturmfels.

Andreas Dress developed Split Decomposition Theory in his study of "phylogenetic combinatorics". This led him to a remarkable conjecture on the mod 2 cohomology of finite simplicial complexes. His conjecture suggests examining the relationship between the boundary operator and the inclusion operator on the binary vector space over a finite simplicial complex.

There are lots of studies in algebraic combinatorics on the boundary operator and the inclusion operator, especially when the simplicial complex is just a simplex. According to Lajos Rónyai, Johannes Siemons, and Qing Xiang who are leading experts in this field, our results on the boundary operator and the inclusion operator are new and may be interesting.

In particular, it might be worth exploring whether this new framework can be used advantageously to investigate some of the specific cohomology classes and operators studied in (mod 2) cohomology.

Before turning into the definition-theorem-proof (DTP) model of mathematics, I want to say that it was quite gratifying for me to work on the relationship between the boundary operator and the inclusion operator, to a large extent just because here we have a not altogether trivial — and luckily correct — conjecture that popped up in mathematical investigations motivated by biology.*

So, this is really an example that biology can lead to new theorems!

*Andreas himself will tell you the interesting story about the birth of his conjecture at the end of this morning session. So, please keep sitting there for a while during my forthcoming DTP model presentation.

2. New Theorems

..... mathematicians can prove only trivial theorems, because every theorem that's proved is trivial. — Richard Feynman

Let K be a finite simplicial complex, namely a nonempty collection of subsets of a given finite set X such that

$$A \subset B \text{ and } B \in K \Rightarrow A \in K$$

holds for all $B \in K$.

Any set $A \in K$ with i elements is said to have dimension $i - 1$ or, simply, to be an $i-1$-face of K, and denoted $\dim(A) = i - 1$. The 0-faces of K are also called the vertices of K. Put

$$K_i = \{A \in K : \ \dim(A) = i\}.$$

Clearly, $K_{-1} = \{\emptyset\}$. The dimension of K, denoted by $\dim(K)$, is the largest i such that $K_i \neq \emptyset$. We always use n for $\dim(K)$ and will assume $n \geq 0$ hereafter.

Let $C := \mathbb{F}_2^K$ be the $\mathbb{F}_2$-vectorspace * consisting of all functions from K to the binary field $\mathbb{F}_2$. We view the elements of C as row vectors.

We are interested in two $\mathbb{F}_2$-linear maps from C to itself, the (co-)boundary operator ∂ and the inclusion operator $\mathcal{D}$, that we assume to act on the set C of row vectors as matrices from the right, mapping any $f \in C$ onto $f\mathcal{D}$ and $f\partial$, respectively. They are defined by requiring that, for any $f \in C$ and $A \in K$, we have:

$$f\mathcal{D}(A) = \sum_{B \subsetneq A} f(B); \tag{1}$$

$$f\partial(A) = \sum_{B \subset A, |B|=|A|-1} f(B). \tag{2}$$

*"C" stands for "chain complex". The role of the homomorphism is played by either $\mathcal{D}$ or ∂, as $\mathcal{D}^2 = \partial^2 = 0$.

$$
\begin{array}{ccccccc}
\longrightarrow C & \underset{\mathcal{D}}{\longrightarrow} & C & \underset{\mathcal{D}}{\longrightarrow} & C \longrightarrow \\
\partial \big\downarrow \mathcal{D} & & \partial \big\downarrow \mathcal{D} & & \partial \big\downarrow \mathcal{D} \\
\longrightarrow C & \underset{\partial}{\longrightarrow} & C & \underset{\partial}{\longrightarrow} & C \longrightarrow
\end{array}
$$

Theorem 1. *There exists an $\mathbb{F}_2$-linear map ω on C such that $\omega\partial + \mathcal{D}\omega = \mathcal{D} + \partial$ holds, implying that there is a* chain homotopy *between ∂ and $\mathcal{D}$.* *

We prove the solvability of the matrix equation $\omega\partial + \mathcal{D}\omega = \mathcal{D} + \partial$ for the unknown matrix ω by an explicit combinatorial construction.

To describe that construction, we first introduce some conventions.

*It is easy to check that $\mathcal{D}\partial = 0$ and hence both $\mathcal{D}$ and ∂ induce chain maps.

Let C^* be the dual linear space of C, which consists of all transposes of elements of C. For $A = \{a_1, \ldots, a_t\} \in K$, let $\overline{A}$ be the (square-free) polynomial $a_1 \cdots a_t$; let B be the set of $\mathbb{F}_2$-linear combinations of $\overline{A}, A \in K$, considered as a ring of polynomials in the variables K_0 with all monomials not corresponding to any face of K being set to 0–the Stanley-Reisner ring of K over $\mathbb{F}_2$. For $f \in C^*$ and $A \in K$, f_A designates the entry of f at position A.

Combining the bijection between K and $\{\overline{A}: \ A \in K\}$:

$$A \in K \qquad \longleftrightarrow \qquad \overline{A}$$

and the bijection between C^* and $\mathbb{F}_2^K = C$:

$$f \in C^* \qquad \longleftrightarrow \qquad f^* \in C : f^*(A) = f_A, \forall A \in K, \text{namely } f^* = f^\top,$$

we get a bijection between C^* and B:

$$f \in C^* \qquad \longleftrightarrow \qquad \widetilde{f} \in B : \widetilde{f}(\overline{A}) = f_A, \forall A \in K.$$

Note that, in the correspondence above, the -1-face $\emptyset \in K$ just corresponds to $1 \in B$ and the 'product' of two disjoint faces A and B of K is the (disjoint) union of A and B.

For example, the element of C^* with support $\{\{a_1, a_2\}, \{a_1\}, \{a_2\}, \emptyset\}$ corresponds to $a_1 a_2 + a_1 + a_2 + 1 = (a_1 + 1)(a_2 + 1) \in B$.

With respect to the above bijection, we identify C^* with the $\mathbb{F}_2$-vectorspace B of polynomials. The good thing about B is that we can talk about multiplications of any two elements of it provided they have no common variables.

For any $\mathbb{F}_2$-linear map (matrix) ϕ on C and any $f \in C^*$, we write $\phi(\widetilde{f})$ for ϕf, which will be viewed either as an element of B or as an element of C^*. We freely switch between these two viewpoints. To have full knowledge of ϕ, it suffices to know $\phi(\overline{A})$ for all $A \in K$.*

Notice that, using this notation, Eqs. (1) and (2) can be rewritten as

$$\mathcal{D}(\overline{A}) = \sum_{B \subsetneq A} \overline{B} \quad \text{and} \quad \partial(\overline{A}) = \sum_{B \subseteq A, |B|=|A|-1} \overline{B}. \tag{3}$$

*This is to recover ϕ column by column.

We fix a linear ordering $<$ for the vertices of K. We adopt the convention that the vertices of a face will always be ordered according to this given linear ordering. That is, whenever we write $\overline{A} = a_1 a_2 \cdots a_{i+1}$, we mean that $A = \{a_1, \cdots, a_{i+1}\}$ is an i-face of K and $a_1 < a_2 < \cdots < a_{i+1}$ holds.

Let ω be the $\mathbb{F}_2$-linear map from C to itself satisfying

$$\begin{cases} \omega(a_1 a_2 \cdots a_{2k+1}) = a_2 a_4 \cdots a_{2k} \mathcal{D}(a_1 a_3 \cdots a_{2k+1}), \\ \omega(a_1 a_2 \cdots a_{2k+2}) = a_2 a_4 \cdots a_{2k+2} \mathcal{D}(a_1 a_3 \cdots a_{2k+1}). \end{cases} \tag{4}$$

We will check in the next section that this map ω is indeed a chain homotopy between ∂ and $\mathcal{D}$, i.e., it satisfies the identity $\omega\partial + \mathcal{D}\omega = \mathcal{D} + \partial$, and hence establish Theorem 1.

Let $[j]$ stand for $\{1, 2, \ldots, j\}$. For any $S, T \subseteq [n]$, the linear operator (matrix)

$$\mathcal{D}_S^T : \oplus_{i \in S} \mathbb{F}_2^{K_i} \longrightarrow \oplus_{j \in T} \mathbb{F}_2^{K_j}$$

is obtained from $\mathcal{D}$ considered as a matrix by restricting to those rows indexed by $\cup_{i \in S} K_i$ and those columns indexed by $\cup_{j \in T} K_j$. Similar notation applies to submatrices of ∂.

We use ∂_{j-1} for $\mathcal{D}_{j-1}^j = \partial_{j-1}^j$.

With this notation, $\mathcal{D}$ and ∂ look as follows.

$$\mathcal{D}_{[j]}^{[j]} = \begin{bmatrix} 0 \\ \mathcal{D}_{[j-1]}^{[j]} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mathcal{D}_{[j-1]}^{j} & \mathcal{D}_{[j-1]}^{[j-1]} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \partial_{j-1} & 0 \\ \mathcal{D}_{[j-2]}^{j} & \mathcal{D}_{[j-2]}^{[j-1]} \end{bmatrix}, \quad (5)$$

$$\begin{aligned} \partial_{[j]}^{[j]} &= \begin{bmatrix} 0 \\ \partial_{[j-1]}^{[j]} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \partial_{j-1} & 0 \\ 0 & \partial_{[j-2]}^{[j-1]} \end{bmatrix} \\ &= \cdots = \begin{bmatrix} 0 & & & & & \\ \partial_{j-1} & 0 & & & & \\ & \partial_{j-2} & 0 & & & \\ & & \ddots & \ddots & & \\ & & & \partial_0 & 0 & \\ & & & & \partial_{-1} & 0 \end{bmatrix}. \end{aligned} \quad (6)$$

Taking $\omega^{\sharp}_{j-1} = \omega^{j}_{[j-1]}$, we easily deduce from Theorem 1 the next result.

$$\begin{array}{ccccccc}
\longrightarrow \bigoplus_{i=-1}^{j-3} \mathbb{F}_2^{K_i} & \xrightarrow[\mathcal{D}^{[j-2]}_{[j-3]}]{} & \bigoplus_{i=-1}^{j-2} \mathbb{F}_2^{K_i} & \xrightarrow[\mathcal{D}^{[j-1]}_{[j-2]}]{} & \bigoplus_{i=-1}^{j-1} \mathbb{F}_2^{K_i} \longrightarrow \\
\mathcal{D}^{j-1}_{[j-3]} \downarrow & & \mathcal{D}^{j}_{[j-2]} \downarrow & & \mathcal{D}^{j+1}_{[j-1]} \downarrow \\
\longrightarrow \mathbb{F}_2^{K_{j-1}} & \xrightarrow[\partial_{j-1}]{} & \mathbb{F}_2^{K_j} & \xrightarrow[\partial_j]{} & \mathbb{F}_2^{K_{j+1}} \longrightarrow
\end{array}$$

Theorem 2. $\omega^{\sharp}$ *is a chain homotopy between the two chain maps,* $(\mathcal{D}^{j}_{[j-2]})_j$ *and the trivial zero map, for the two easily recognizable chain complexes as depicted above. In other words, it holds for each* j *that*

$$\mathcal{D}^{j}_{[j-2]} = \omega^{\sharp}_{j-2}\partial_{j-1} + \mathcal{D}^{j-1}_{[j-2]}\omega^{\sharp}_{j-1}. \qquad (7)$$

Eq. (7) is nothing but

$$\left[\begin{array}{cc} I & 0 \\ \omega^{\sharp}_{j-2} & I \end{array}\right] \left[\begin{array}{cc} \partial_{j-1} & 0 \\ \mathcal{D}^{j}_{[j-2]} & \mathcal{D}^{j-1}_{[j-2]} \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \omega^{\sharp}_{j-1} & I \end{array}\right] = \left[\begin{array}{cc} \partial_{j-1} & 0 \\ 0 & \mathcal{D}^{j-1}_{[j-2]} \end{array}\right], \tag{8}$$

which, according to Eq. (5), immediately implies

$$\mathrm{rank}(\mathcal{D}^{[j]}_{[j]}) = \mathrm{rank}(\mathcal{D}^{[j]}_{[j-1]}) = \mathrm{rank}(\partial_{j-1}) + \mathrm{rank}(\mathcal{D}^{j-1}_{[j-2]}). \tag{9}$$

Applying Eq. (9) repeatedly yields

$$\text{rank}(\mathcal{D}) = \text{rank}(\mathcal{D}^{[n]}_{[n]}) = \text{rank}(\mathcal{D}^{[n]}_{[n-1]}) =$$
$$\text{rank}(\partial_{n-1}) + \text{rank}(\mathcal{D}^{[n-1]}_{[n-2]}) = \sum_{i=n-2}^{n-1} \text{rank}(\partial_i) + \text{rank}(\mathcal{D}^{[n-2]}_{[n-3]})$$
$$= \cdots = \sum_{i=-1}^{n-1} \text{rank}(\partial_i). \tag{10}$$

Combining Eq. (6) and Eq. (10), we obtain

$$\text{rank}(\mathcal{D}) = \text{rank}(\partial). \tag{11}$$

By the definition of reduced cohomology groups, we have $\frac{\mathsf{Ker}(\partial)}{\mathsf{Im}(\partial)} = \bigoplus_{i=0}^{n} \tilde{H}^i(K;\mathbb{F}_2)$. The next theorem readily comes from Eq. (11).

Theorem 3. $\frac{\mathsf{Ker}(\mathcal{D})}{\mathsf{Im}(\mathcal{D})} = \frac{\mathsf{Ker}(\partial)}{\mathsf{Im}(\partial)} = \bigoplus_{i=0}^{n} \tilde{H}^i(K;\mathbb{F}_2)$.

Define $\mathrm{Ker}_i(\mathcal{D})$ to be $\{f \in \mathrm{Ker}(\mathcal{D}) : f(A) = 0, \forall A \in K, \dim(A) \leq i\}$. It is surely true that

$$\{0\} = \mathrm{Ker}_n(\mathcal{D}) \leq \mathrm{Ker}_{n-1}(\mathcal{D}) \leq \cdots \leq \mathrm{Ker}_0(\mathcal{D}) \leq \mathrm{Ker}_{-1}(\mathcal{D}) = \mathrm{Ker}(\mathcal{D}), \tag{12}$$

the last equality coming from our assumption that $K_0 \neq \emptyset$. From Eq. (12), we get the following decomposition

$$\frac{\mathrm{Ker}(\mathcal{D})}{\mathrm{Im}(\mathcal{D})} = \frac{\mathrm{Ker}_{-1}(\mathcal{D}) + \mathrm{Im}(\mathcal{D})}{\mathrm{Ker}_n(\mathcal{D}) + \mathrm{Im}(\mathcal{D})} = \bigoplus_{i=-1}^{n-1} \frac{\mathrm{Ker}_i(\mathcal{D}) + \mathrm{Im}(\mathcal{D})}{\mathrm{Ker}_{i+1}(\mathcal{D}) + \mathrm{Im}(\mathcal{D})}. \tag{13}$$

By substituting $A_i = \mathsf{Ker}_i(\mathcal{D})$ and $B_i = \mathsf{Ker}_{i+1}(\mathcal{D}) + \mathsf{Im}(\mathcal{D})$ into $\frac{A_i+B_i}{B_i} = \frac{A_i}{A_i \cap B_i}$, we conclude that

$$\begin{aligned}\frac{\mathsf{Ker}_i(\mathcal{D}) + \mathsf{Im}(\mathcal{D})}{\mathsf{Ker}_{i+1}(\mathcal{D}) + \mathsf{Im}(\mathcal{D})} &= \frac{\mathsf{Ker}_i(\mathcal{D})}{\mathsf{Ker}_i(\mathcal{D}) \cap (\mathsf{Ker}_{i+1}(\mathcal{D}) + \mathsf{Im}(\mathcal{D}))} \\ &= \frac{\mathsf{Ker}_i(\mathcal{D})/\mathsf{Ker}_{i+1}(\mathcal{D})}{(\mathsf{Ker}_i(\mathcal{D}) \cap \mathsf{Im}(\mathcal{D}))/\mathsf{Ker}_{i+1}(\mathcal{D})},\end{aligned} \tag{14}$$

where we have used the shorthand $(\mathsf{Ker}_i(\mathcal{D}) \cap \mathsf{Im}(\mathcal{D}))/\mathsf{Ker}_{i+1}(\mathcal{D})$ for $((\mathsf{Ker}_i(\mathcal{D}) \cap \mathsf{Im}(\mathcal{D})) + \mathsf{Ker}_{i+1}(\mathcal{D}))/\mathsf{Ker}_{i+1}(\mathcal{D})$.

Eq. (13) along with Eq. (14) gives

$$\frac{\mathsf{Ker}(\mathcal{D})}{\mathsf{Im}(\mathcal{D})} = \bigoplus_{i=-1}^{n-1} \frac{\mathsf{Ker}_i(\mathcal{D})/\mathsf{Ker}_{i+1}(\mathcal{D})}{(\mathsf{Ker}_i(\mathcal{D}) \cap \mathsf{Im}(\mathcal{D}))/\mathsf{Ker}_{i+1}(\mathcal{D})}. \tag{15}$$

Making use of Theorem 2, we can prove the following theorem by setting up a canonical isomorphism.

Theorem 4.

$$\mathsf{Ker}_i(\mathcal{D})/\mathsf{Ker}_{i+1}(\mathcal{D}) \simeq \mathsf{Ker}(\partial_{i+1});$$

$$(\mathsf{Ker}_i(\mathcal{D}) \cap \mathsf{Im}(\mathcal{D}))/\mathsf{Ker}_{i+1}(\mathcal{D}) \simeq \mathsf{Im}(\partial_i).$$

Consequently, $\frac{\mathsf{Ker}_i(\mathcal{D})/\mathsf{Ker}_{i+1}(\mathcal{D})}{(\mathsf{Ker}_i(\mathcal{D})\cap\mathsf{Im}(\mathcal{D}))/\mathsf{Ker}_{i+1}(\mathcal{D})} \simeq \frac{\mathsf{Ker}(\partial_{i+1})}{\mathsf{Im}(\partial_i)} = \tilde{H}^{i+1}(K;\mathbb{F}_2)$*.*

Our proof of Theorem 4 and that of Eq. (15) provide a canonical isomorphism from $\frac{\mathsf{Ker}(\mathcal{D})}{\mathsf{Im}(\mathcal{D})}$ to $\bigoplus_{i=0}^{n} \tilde{H}^i(K;\mathbb{F}_2)$, thus extending Theorem 3.

3. A Proof (of Theorem 1)

A math talk without a proof is like a movie without a love scene.
— H. W. Lenstra, Jr.

If it is a Miracle, any sort of evidence will answer, but if it is a Fact, proof is necessary. — Mark Twain

Lemma 5. $\omega(a_1a_2\cdots a_{2k+1}) = a_1a_2\cdots a_{2k}+(1+a_{2k+1})\omega(a_1a_2\cdots a_{2k})$.

Proof. $LHS = a_2a_4\cdots a_{2k}\mathcal{D}(a_1a_3\cdots a_{2k+1}) = a_2a_4\cdots a_{2k}(a_1a_3\cdots a_{2k-1}$ $+\mathcal{D}(a_1a_3\cdots a_{2k-1}) + a_{2k+1}\mathcal{D}(a_1a_3\cdots a_{2k-1})) = RHS$. ∎

Lemma 6. $\omega(a_1a_2\cdots a_{2k}) = a_{2k}\omega(a_1a_2\cdots a_{2k-1})$.

Proof. $LHS = a_2a_4\cdots a_{2k}\mathcal{D}(a_1a_3\cdots a_{2k-1}) = RHS$. ∎

Lemma 7. *Suppose that X and Y are polynomials without common variables. Then $\mathcal{D}(X(Y+\mathcal{D}(Y))) = Y\mathcal{D}(X) + X\mathcal{D}(Y)$.*

Proof. $LHS = \mathcal{D}(XY) + \mathcal{D}(X\mathcal{D}(Y)) = (Y\mathcal{D}(X) + X\mathcal{D}(Y) + \mathcal{D}(X)\mathcal{D}(Y)) + (\mathcal{D}(X)\mathcal{D}(Y) + X\mathcal{D}^2(Y) + \mathcal{D}(X)\mathcal{D}^2(Y)) = RHS$. Note that to get the last equality we have used the easy fact that $\mathcal{D}^2 = 0$. ∎

Proof. (of Theorem 1) We need to show for any $A \in K$ that

$$\omega\partial(A) + \mathcal{D}\omega(A) = \mathcal{D}(A) + \partial(A). \tag{16}$$

Let us work by induction on $\dim(A)$.

If $A \in K_{-1}$, then $A = 1$ and Eq. (16) follows from $\partial(1) = \mathcal{D}(1) = \omega(1) = 0$.

Now assume that $\dim(A) = i \geq 0$ and Eq. (16) holds when $\dim(A) < i$. There are two cases to consider.

Case (i): $i = 2k - 1$ is odd.

Suppose $A = a_1 a_2 \cdots a_{2k}$. By Lemma 5, we see that

$$\omega(a_{2k}\partial(\frac{A}{a_{2k}})) = \partial(\frac{A}{a_{2k}}) + (1 + a_{2k})\omega(\partial(\frac{A}{a_{2k}})). \tag{17}$$

Making use of

$$\partial(A) = a_{2k}\partial(\frac{A}{a_{2k}}) + \frac{A}{a_{2k}}, \tag{18}$$

we deduce from Eq. (17) that

$$\omega\partial(A) = \omega(a_{2k}\partial(\frac{A}{a_{2k}})) + \omega(\frac{A}{a_{2k}}) = \partial(\frac{A}{a_{2k}}) + (1 + a_{2k})\omega(\partial(\frac{A}{a_{2k}})) + \omega(\frac{A}{a_{2k}}). \tag{19}$$

By Lemma 6, we have

$$\mathcal{D}\omega(A) = \mathcal{D}(a_{2k}\omega(\frac{A}{a_{2k}})) = \omega(\frac{A}{a_{2k}}) + (1 + a_{2k})\mathcal{D}(\omega(\frac{A}{a_{2k}})). \quad (20)$$

Moreover, our induction hypothesis gives

$$(\omega\partial + \mathcal{D}\omega)(\frac{A}{a_{2k}}) = (\mathcal{D} + \partial)(\frac{A}{a_{2k}}). \quad (21)$$

Piecing together Eqs. (19), (20) and (21) yields $(\omega\partial+\mathcal{D}\omega)(A) = \partial(\frac{A}{a_{2k}}) + (1 + a_{2k})(\mathcal{D} + \partial)(\frac{A}{a_{2k}}) = \mathcal{D}(\frac{A}{a_{2k}}) + a_{2k}(\mathcal{D} + \partial)(\frac{A}{a_{2k}}) = (\mathcal{D} + \partial)(A)$, as desired.

Case (ii): $i = 2k$ is even.

Suppose $A = a_1 a_2 \cdots a_{2k+1}$. Applying Lemma 6 and the fact that $\partial(A) = a_{2k+1}\partial(\frac{A}{a_{2k+1}}) + \frac{A}{a_{2k+1}}$, we obtain

$$\omega\partial(A) = \omega(a_{2k+1}\partial(\frac{A}{a_{2k+1}})) + \omega(\frac{A}{a_{2k+1}}). \tag{22}$$

On the other hand, in view of Lemma 5, we get

$$\mathcal{D}\omega(A) = \mathcal{D}(\frac{A}{a_{2k+1}}) + \mathcal{D}((1 + a_{2k+1})\omega(\frac{A}{a_{2k+1}})); \tag{23}$$

while from Lemma 7, we have

$$\begin{aligned}\mathcal{D}((1+a_{2k+1})\omega(\frac{A}{a_{2k+1}})) &= a_{2k+1}\mathcal{D}(\omega(\frac{A}{a_{2k+1}})) + \omega(\frac{A}{a_{2k+1}})\mathcal{D}(a_{2k+1}) \\ &= a_{2k+1}\mathcal{D}(\omega(\frac{A}{a_{2k+1}})) + \omega(\frac{A}{a_{2k+1}}).\end{aligned} \tag{24}$$

In addition, note that our induction hypothesis now reads as

$$(\omega\partial + \mathcal{D}\omega)(\frac{A}{a_{2k+1}}) = (\mathcal{D}+\partial)(\frac{A}{a_{2k+1}}). \tag{25}$$

Finally, we come to

$$
\begin{aligned}
&(\omega\partial + \mathcal{D}\omega)(A)\\
=&\omega(a_{2k+1}\partial(\frac{A}{a_{2k+1}})) + \omega(\frac{A}{a_{2k+1}}) + \mathcal{D}(\frac{A}{a_{2k+1}}) + \mathcal{D}((1 + a_{2k+1})\omega(\frac{A}{a_{2k+1}}))\\
=&\omega(a_{2k+1}\partial(\frac{A}{a_{2k+1}})) + \mathcal{D}(\frac{A}{a_{2k+1}}) + a_{2k+1}\mathcal{D}(\omega(\frac{A}{a_{2k+1}}))\\
=&a_{2k+1}\omega(\partial(\frac{A}{a_{2k+1}})) + \mathcal{D}(\frac{A}{a_{2k+1}}) + a_{2k+1}\mathcal{D}(\omega(\frac{A}{a_{2k+1}}))\\
=&a_{2k+1}((\mathcal{D} + \partial)(\frac{A}{a_{2k+1}})) + \mathcal{D}(\frac{A}{a_{2k+1}})\\
=&(\mathcal{D} + \partial)(A),
\end{aligned}
$$

where the first equality is due to Eqs. (22) and (23), the second equality follows from Eq. (24), the third equality is guaranteed by Lemma 6, and the fourth equality is a result of Eq. (25). This ends the proof. ∎

4. Final Comments

- Is there any easier proof (better understanding) to Theorem 1?

- If we replace $\mathbb{F}_2$ with other field whose characteristic is not two, the corresponding results become quite trivial and will be reported in [DWW] *.

- Some work here about simplicial complex has been generalized to the free vector space over an Eulerian poset; see [DWW]. [DWW] will also report some generalizations of other work of Andreas Dress on Split Decomposition Theory.

*A. Dress, X. Wang, Y. Wu

Thank you for your attention!

In the coming century, biology will stimulate the creation of entirely new realms of mathematics. In this sense, biology is mathematics' next physics, only better. Biology will stimulate fundamentally new mathematics because living nature is qualitatively more heterogeneous than non-living nature.

– J.E. Cohen, "Mathematics is biology's next microscope, only better; biology is mathematics' next physics, only better". PLOS Biology * 2 (2004) No.12.

`http://biology.plosjournals.org/perlserv/?request=get-document&doi=10.1371/journal.pbio.0020439`

*PLoS = Public Library of Science