From Eulerian Graph to Even Poset

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This talk is based on

Y. Wu, Even poset and a parity result for binary linear code, *Linear Algebra and its Applications*, to appear.

I shall tell you how the concept of even poset is invented and show you some examples of even posets.

By presenting this talk, I expect from you some more examples of even posets or an indication to some other mathematical objects which may be related to even posets. Any comment is welcome!

The story begins from Euler.

Recall that an Eulerian graph is a graph which has a closed walk passing through each edge of it exactly once.

In 1736 Leonhard Euler published a paper on the solution of the Königsberg Seven Bridge problem entitled ' The solution of a problem relating to the geometry of position', which is now considered as the beginnings of topology and graph theory. In this paper, Euler stated the following theorem but gave no proof, perhaps because the suitable definitions * needed for such a proof did not exist then. The first published proof was produced by Hierholzer in 1873.

Theorem 1 *A graph is Eulerian if and only if it is connected and even.*

*A definition is the enclosing of a wilderness of idea within a wall of words.
– Samuel Butler (1835-1902)

There is a wonderful three-volume book totally devoted to the topic of Eulerian graph: Herbert Fleischner, Eulerian Graphs and Related Topics, North-Holland, Vol.1, 1990, Vol. 2, 1991, Vol. 3, to appear. When I taught a Graph Theory course last year for some undergraduates in SJTU, I decided to spend about 6 hours on various results about Eulerian graphs. This leads to my awareness of the following result.

Theorem 2 [*]*A graph is Eulerian if and only if every edge of it is contained in an odd number of circuits.*

The backward direction is easy and the nontrivial part is the forward direction.

[*]S. Toida, Properties of a Euler graph, *Journal of the Franklin Institute* **295** (1973), 343–345.

Here are some other relevant papers in which Eulerian graphs, bipartite graphs, Eulerian binary matroids, and bipartite binary matroids are discussed via various approaches. Similar characterizations are established in these papers.

P.J. Wilde, The Euler circuit theorem for binary matroid, *J. Comb. Theory B* **18** (1975), 260–264.

T.A. McKee, Recharacterizing eulerian: intimations of new duality, *Discrete Mathematics* **51** (1984), 237–242.

H. Fleischner, Elementary proofs of (relatively) recent characterizations of Eulerian graphs, *Discrete Mathematics* **24** (1989), 115–119.

D.R. Woodall, A proof of McKee's Eulerian-bipartite characterization, *Discrete Mathematics* **84** (1990), 217–220.

M.M. Shikare, T.T. Raghunathan, A characterization of binary Eulerian matroids, *Indian J. Pure Appl. Math.* **27** (1996), 153–155.

P. Hoffmann, Counting maximal cycles in binary matroids, *Discrete Mathematics* **162** (1996), 291–292.

M.M. Shikare, New characterizations of Eulerian and bipartite binary matroids, *Indian J. Pure Appl. Math.* **32** (2001), 215–219.

T.A. McKee, S-minimal unions of disjoint cycles and more odd eulerian characterizations, *Congressus Numerantium* **177** (2005), 129–132.

When I tried to organize the results in the above papers into some teaching material for our undergraduates, a natural generalization of the earlier results came to my mind. To present it, we need to prepare some notations.

Consider the linear space $V = \mathbb{F}_2^n$ consisting of $1 \times n$ vectors over the binary field $\mathbb{F}_2$, which can be viewed as $\mathbb{F}_2^{[n]}$, the set of functions from $[n]$ to $\mathbb{F}_2$. Let W be a subspace of V, which is called a *binary linear code* in coding theory. Each $w \in W$ is uniquely determined by its support, denoted $\mathbf{supp}(w)$. For any $X \subseteq V$, let $\mathcal{S}(X) = \{\mathbf{supp}(w) : \ w \in X\}$. The *support poset* of W is $\mathcal{S}(W)$ ordered by the inclusion relation and we will simply refer to it also by $\mathcal{S}(W)$.

Consider a subspace W of $\mathbb{F}_2^n$. Let $\mathcal{S}'(W)$ be the *truncated support poset* of W, which is obtained from $\mathcal{S}(W)$ by removing its bottom element, namely $\emptyset$. Say that $w \in W$ is *maximal* if $\mathbf{supp}(x)$ is a maximal element in $\mathcal{S}(W)$. Say a nonzero vector $w \in W$ is minimal if $\mathbf{supp}(w)$ is *minimal* in $\mathcal{S}'(W)$. We write M(W) and m(W) for the set of maximal and minimal vectors of W, respectively.

Theorem 3 *For any $W \leq \mathbb{F}_2^n$, it happens that $\sum_{x \in m(W)} x = \sum_{x \in M(W)} x$.*

To see that it really generalizes Theorem 7, we choose W to be the cycle space of a graph and notice that for an Eulerian graph the only maximal vector in W is the all-ones vector and each minimal vector corresponds to a circuit and so we come to the forward direction of Theorem 7.

My original proof follows the idea of Woodall and is somewhat complicated.

After talking about that proof in this combinatorics seminar last year, Andreas suggested the use of deletion/contraction to find out an easier proof. In the course of searching for an easier proof, the concept of Even Poset comes out which does provide a much shorter proof of Theorem 3 and help to fit various earlier results into a unified framework. But deletion/contraction does not appear on the surface and many further questions arise subsequently.

Let $P = (X, \leq)$ be a poset. For any $x \in X$, we define $\downarrow x = \{y \in X : \ y \leq x\}$ and $\uparrow x = \{y \in X : \ y \geq x\}$, and call them a *principal ideal* and a *principal filter* of P, respectively. Note that x is *maximal* in P if and only if $|\uparrow x| = 1$ whereas x is *minimal* in P if and only if $|\downarrow x| = 1$. We say that P is an *even poset* provided every principal ideal or principal filter of it either has size 1, and hence corresponds to a minimal or maximal element of P, respectively, or has an even size. For any natural number n, $[n]$ stands for $\{1, \ldots, n\}$.

Example 4 *Let* n, m *be two natural numbers. Let* $\Theta_{n,m} = \{(A_1, A_2, \ldots, A_m) : A_i \subseteq [n], A_i \cap A_j = \emptyset, \forall i \neq j\}$ *and order it by setting* $(A_1, A_2, \ldots, A_m) \leq (B_1, B_2, \ldots, B_m)$ *if and only if* $A_i \subseteq B_i$ *for all* $i \in [m]$*. It is easy to see that the resulting poset is an even poset provided* m *is odd.*

Example 5 *A finite ranked poset with a unique top element and a unique bottom element is Eulerian if each interval of it of positive length has the same number of elements of even rank as odd rank. It is easy to see that any Eulerian poset is an even poset. Eulerian poset was introduced by Richard Stanley and enjoys many remarkable duality properties.*

Lemma 6 *For each finite even poset, the number of its maximal elements and the number of its minimal elements have the same parity.*

Proof. Let $P = (X, \leq)$ be the given even poset and $\mathcal{M}(P)$ and $\mathsf{m}(P)$ be its sets of maximal elements and minimal elements, respectively. The result is straightforward from the following double counting reasoning: $|\mathcal{M}(P)| = \sum_{x\in\mathcal{M}(P)} 1 \equiv \sum_{x\in X}\sum_{y\geq x} 1 = \sum_{y\in X}\sum_{x\leq y} 1 \equiv \sum_{y\in \mathsf{m}(P)} 1 = |\mathsf{m}(P)| \pmod 2$. ∎

We will find that the parity result for binary linear code (Theorem 3) follows from this easy duality result for even poset as long as we identify the even poset structure in binary linear codes. This is done in the next lemma.

Lemma 7 *For any binary linear code* $W \leq \mathbb{F}_2^n$ *and any* $A \subseteq [n]$*,* $\mathcal{S}_A(W)$ *is an even poset.*

Proof. Take, if any, a $B \in \mathcal{S}_A(W)$ which is neither maximal nor minimal in $\mathcal{S}_A(W)$. Our task is to show that both the principle filter $\uparrow B$ and the principal ideal $\downarrow B$ in $\mathcal{S}_A(W)$ have an even size.

Observe that $\uparrow B$ is just $\{C \cup B : C \in \mathcal{S}(q_{[n]\setminus B}(W))\}$. This tells us that it has equal size with the binary linear subspace $q_{[n]\setminus B}(W)$. But, as B is not maximal, $q_{[n]\setminus B}(W)$ is of positive dimension and thus has an even size.

We now consider $\downarrow B$. If $A = \emptyset$, then we find that $\downarrow B$ is just $\mathcal{S}(q_B(W))$. Since B is not minimal, $\mathcal{S}(q_B(W))$ is a binary space of positive dimension and so has an even size. For the remaining case $A \neq \emptyset$, we can check that the binary linear space $\mathcal{S}(q_B(W_A))$ is a disjoint union of $\downarrow B = \{C \in \mathcal{S}(q_B(W_A)) : \ A \subseteq C\}$ and $Z = \{C \in \mathcal{S}(q_B(W_A)) : \ A \cap C = \emptyset\}$. But the map sending C to $B \backslash C$ obviously induces a bijection from $\downarrow B$ to Z. This yields that $|\downarrow B| = \frac{|\mathcal{S}(q_B(W_A))|}{2}$. In light of the fact that B is not minimal in $\mathcal{S}_A(W)$, the dimension of $q_B(W_A)$ has to be greater than one. Consequently, we conclude that $|\downarrow B|$ is even, ending the proof.

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We conclude the story on even posets by giving some remarks. As before, we still need a new concept.

An *oriented poset* is a poset P with a sign function $r: P \to \{\pm 1\}$. Any $a \in r^{-1}(1)$ is called a *positive element* of P and any one from $r^{-1}(-1)$ a negative element of P. An oriented poset satisfies the *Euler-Poincar$\dot{e}$ relation* if each interval of it of positive length has the same number of positive elements as negative ones. A *quasi-Eulerian poset* is an oriented poset satisfying the Euler-Poincar$\acute{e}$ relation. All Eulerian posets are quasi-Eulerian.

We find that some results of Andreas on Boolean algebra can be easily generalized to results on quasi-Eulerian posets.

We remark that all constructions of posets mentioned in this talk are graded and have the same number of elements of even rank as odd rank in every interval of positive length, and hence are quasi-Eulerian. It is clear that the property of being quasi-Eulerian and the property of being an even poset are both invariant under the Cartesian product operation. Thus, a natural question to consider is to figure out the relationship between even posets and quasi-Eulerian posets. In general, we would like to know to which extent we can determine (classify) all even posets.

We know that minimal vectors in a linear code correspond to circuits in the corresponding binary matroid.

An OM-poset is a poset arising from an oriented matroid. A maximal cell of an OM-poset is called a tope. Seems that tope is the concept corresponding to maximal vectors.

Is there any deeper connection between the results presented here and matroid theory (oriented matroid theory)? Namely, can we find out the real role of deletion/contraction in this kind of work (and hence answer the question of Andreas)?

Give any simplicial complex Γ, we can define its *face poset* to be the set of faces of Γ ordered by inclusion.

Conversely, give any poset P, we can define a simplicial complex $\Delta(P)$ consisting of all chains of P, called its *order complex*.

Question 8 *Can we say something more about the structure of the support poset of a linear code? What about its Möbius function? What about its order complex? Can we say something about the simplicial complex whose face poset is a given support poset of a linear code?*

In the same vein, we can replace the support poset by the general even poset and ask the same question.

Finally, we mention that we are also interested in the high-dimensional support poset of a linear code. This type of study may be said to be a combinatorial study of linear spaces.

Linear space has very trivial algebraic structure; But there may be some nice unknown combinatorial structure (results) behind them.

Thank You!