Digraphs with Primitive Colorings

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http://www.math.sjtu.edu.cn/teacher/wuyk/WSA.pdf

Talk Outline

- The question;
- Some previous work;
- A necessary condition;
- Sufficient conditions;
- 2-Primitivity and 3-primitivity;
- Good homomorphism;
- Primitive road coloring

I: The question

A **k-coloring** δ of a digraph G is an onto map from $E(G)$ to $\{1, 2, \cdots, k\}$. For a k-tuple of **positive** integers $(n_1, n_2, \cdots, n_k)$, δ is **k-primitive** of **k-signature** $(n_1, n_2, \cdots, n_k)$ provided for any two vertices u and v of G there exists in G a walk $W_{u,v}$ directing from u to v with $\mathrm{Card}(\delta^{-1}(i) \cap W_{u,v}) = n_i$ for each $1 \leq i \leq k$. If a digraph admits a k-primitive k-coloring, we say that the digraph is **k-primitive**. The **k-exponent** of a k-primitive digraph is the minimum of the sum $\sum_{i=1}^{k} n_i$, where $(n_1, n_2, \cdots, n_k)$ runs over all k-signatures of all k-primitive colorings of it.

Every digraph has exactly one 1-coloring. If this unique 1-coloring is 1-primitive, then the corresponding digraph is usually referred to as a primitive digraph. The 1-exponent of a primitive digraph is usually called the exponent of the digraph and has been widely studied in combinatorial matrix theory. It is clear that the concept of k-primitive digraph is a natural generalization of the concept of primitive digraph. Fornasini and Valcher (1997,1998) proposed to study questions around this concept due to some background in 2-dimensional dynamical system.

Recall **Cerny Conjecture**:

Known the existence of a synchronizable road coloring (synchronizing word), study the shortest length of all synchronizing words of all synchronizable road colorings of the given digraph.

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As a counterpart of Cerny Conjecture, people are interested in **the estimation of the k-exponents of k-primitive digraphs**:

Known the existence of a k-primitive coloring (k-signature), study the minimum weight of all k-signatures of all k-primitive colorings of the given digraph.

There has been satisfiable knowledge about this question on k-exponents and we will not address it any more in this talk.

But is there any counterpart of the **Road Coloring Conjecture**?

What kind of structural properties of a digraph will be necessary or sufficient for the existence of a k-primitive k-coloring on it?

Observe that a k-primitive digraph must be k'-primitive for $k' \leq k$.

Question 0 (**The main concern of this talk**):

Is there any **good intrinsic** characterization for the class of **primitive digraphs** which admit k-primitive k-colorings?

II : Some previous work

For a k-coloring δ of n sets $S_1, \cdots, S_n$, put $P(\delta; S_1, \cdots, S_n)$ to be the matrix whose (i,j) entry is Card$(\delta^{-1}(i) \cap S_j)$. For any cycle C, use $\ell(C)$ for Card(C).

Assume that the circuits (=simple cycles) of a digraph G are enumerated as $C_1, \cdots, C_n$ and δ is any coloring of G. The **cycle matrix** of the automaton (G, δ) is $C(G, \delta) = P(\delta; C_1, \cdots, C_n)$ and the **cycle lattice** of it is the abelian group $L(G, \delta)$ generated by the columns of $C(G, \delta)$.

The kth **determinant factor** of an integer matrix A, recorded as $\mathcal{D}_k(A)$, is the greatest common divisor of all $k \times k$ minors of it.

Theorem 1 (Olesky, Shader and Van den Driessche): Let δ be a k-coloring of a strongly connected digraph G. Then the fact that δ is a k-primitive k-coloring is equivalent to either of the following two statements:

(i) $\mathcal{D}_k(C(G, \delta)) = 1$;

(ii) $L(G, \delta) = \mathbb{Z}^k$.

Note that Theorem 1 characterizes k-primitive k-colorings rather than digraphs with k-primitive k-colorings, namely k-primitive digraphs. However, this result is the starting point of many subsequent work, including ours.

Beasley and Kirkland deduce the following beautiful characterization by making use of Theorem 1 along with some number theoretical techniques.

Theorem 2:
Every primitive simple digraph with at least two vertices (equivalently, with at least two circuits) is 2-primitive.

They give an example to show that the existence of four circuits cannot guarantee 4-primitivity.

In addition, they propose

Conjecture 3:
For any three distinct circuits C_1, C_2, C_3, there is a 3-coloring δ such that $P(\delta; C_1, C_2, C_3)$ has determinant $\gcd(\ell(C_1), \ell(C_2), \ell(C_3))$.

III : A necessary condition

The **cycle matrix** of a digraph G is $C(G) = C(G, \delta)$ where δ is any coloring which assigns different colors to different arcs. Note that cycle matrix of a digraph is really a graph invariant independent of any additional coloring (up to permutation of lines). Define the **primitive number** of G to be

$$\text{pri}(G) = \max\{k' | \ \ G \ \text{ is } \ k'\text{-primitive}\}.$$

Define $\Phi(G) = \max\{k | \ \mathcal{D}_k(C(G)) = 1\}$. It is easy to deduce the following necessary condition for the k-primitivity of a digraph.

Theorem 4:

If a digraph G is k-primitive, then $\mathcal{D}_k(C(G)) = 1$. In other words, $\text{pri}(G) \leq \Phi(G)$.

We conjecture that this necessary condition is indeed an **intrinsic** characterization of k-primitive digraphs.

Conjecture 5:
If G is a primitive digraph, then $\mathrm{pri}(G) = \Phi(G)$. That is to say, a primitive digraph G is k-primitive if and only if $\Phi(G) \geq k$.

Here is an equivalent form of Conjecture 5.

Conjecture 5':
A $(k-1)$-primitive digraph G is k-primitive if and only if $\Phi(G) \geq k$.

We remark that if Conjecture 5 (5') is true, we can reduce Question 0 to the computation of the minimum **dimension of the cycle spaces** (=cycle lattice with coefficients in a field) of a digraph over all prime fields. But we still do not know if the latter task can be completed **efficiently**.

Theorem 6:
For any digraph G and any integer $k \leq 3$, $\Phi(G) \geq k$ if and only if G has at least k circuits.

Due to Theorem 6, we can reformulate two special cases of Conjecture 5.

Conjecture 7:
(i) A primitive digraph is 2-primitive if and only if it has at least two circuits.
(ii) A primitive digraph is 3-primitive if and only if it has at least three circuits.

In effect, when restricting to simple digraphs Conjecture 7(i) is just the assertion of Beasley and Kirkland (Theorem 2). Using a **new approach***, we will establish some sufficient conditions for k-primitivity in the next part (Part IV) and then we can provide a much shorter proof of Conjecture 7(i) than them. We also obtain partial results for Conjecture 7(ii) in Part V and we expect that a much more careful **case by case** analysis can lead to a settlement of Conjecture 7(ii).

*If time permitted, we will give an illustration of it at the end of the talk.

IV : Sufficient conditions

Theorem 8:

A primitive digraph G is k-primitive if it has k cycles $C_1, \cdots, C_k$ of lengths $\ell_1, \cdots, \ell_k$, respectively, such that no arc appears in one C_i twice, $C_i \setminus \cap_{i=1}^k C_i$ are pairwise disjoint and Card$(\cap_{i=1}^k C_i) \leq$

$$\begin{cases} \max\{\gcd(\ell_1, \cdots, \ell_k) - 1, 1\}, & \text{if there is at most one } \quad \ell_i = 1, \\ \gcd(\ell_1, \cdots, \ell_k) - 1, & \text{otherwise.} \end{cases}$$

Corollary 9:

Suppose u and v are vertices of a primitive digraph G. If there is an arc going from u to v and at least k arc-disjoint walks going from v to u among which at most one walk is the empty walk (this may happen only when $u = v$), then G is k-primitive. Especially, a primitive digraph having a k-arc-strongly connected subgraph is k-primitive.

Theorem 10:

Let $C_1, \cdots, C_k$ be k distinct cycles in a primitive digraph G. For each $i \in [k]$, let $\mathcal{I}_i \subseteq [i-1]$ be a set such that $C_i \cap (\cup_{j=1}^{i-1} C_j) \subseteq \cup_{j \in \mathcal{I}_i} C_j$. Then G is k-primitive in case that $\sum_{j \in \mathcal{I}_i} \ell(C_j) < \ell(C_i)$.

Two immediate corollaries of Theorem 10 are presented below.

Corollary 11:

A primitive digraph G is k-primitive provided it has k cycles $C_1, \cdots, C_k$ such that

$$\sum_{j=1}^{i-1} \ell(C_j) < \ell(C_i), \quad \forall i \in [k].$$

Proof: $\mathcal{I}_i = [i-1]$, $i \in [k]$.

Corollary 12:

A primitive digraph with at least k arc-disjoint cycles is k-primitive.

Proof: $\mathcal{I}_i = \emptyset$, $i \in [k]$.

A bit more effort leads to the following two results, as consequence of Corollary 11 and Corollary 12, respectively.

Corollary 13:
Let $C_1, \cdots, C_k$ be k cycles of a primitive digraph G, the lengths of which being $\ell_1, \ell_2, \cdots, \ell_k$, respectively. Then G is k-primitive if there exists $t < k$ such that

(i) $\gcd(\ell_1, \cdots, \ell_t) = \gcd(\ell_1, \cdots, \ell_k)$;

(ii) $1 + \sum_{j=1}^{i-1} \ell_j \leq \ell_i$, $\forall i \in [t]$;

(iii) $C_i \setminus \bigcup_{j=1}^{i-1} C_j \neq \emptyset$, $\forall i \in [k] \setminus [t]$.

Corollary 14:

Let $C_1, \cdots, C_k$ be k cycles of a primitive digraph G, the lengths of which being $\ell_1, \ell_2, \cdots, \ell_k$, respectively. Then G is k-primitive if there exists $t < k$ satisfying

(i) $\gcd(\ell_1, \cdots, \ell_t) = \gcd(\ell_1, \cdots, \ell_k)$;

(ii) $C_1, \cdots, C_t$ are pairwise arc-disjoint;

(iii) $C_i \setminus \bigcup_{j=1}^{i-1} C_j \neq \emptyset, \forall i \in [k] \setminus [t]$.

Example 15:
pri$(K_3) = 4 < 7 =$ pri$(L(K_3)) =$ pri(K_3^+).

$L(K_d)$ is known as the Kautz digraph $K(d-1, 2)$, characterized by the matrix equation $A + A^2 = J$.

We enumerate all the 8 circuits of $K(2, 2)$, denoted by $C_1, C_2, \cdots, C_8$ respectively, as follows: $01 \to 12 \to 20 \to 01$, $10 \to 01 \to 12 \to 21 \to 10$, $10 \to 02 \to 21 \to 10$, $12 \to 20 \to 02 \to 21 \to 12$, $01 \to 10 \to 01$, $12 \to 21 \to 12$, $02 \to 20 \to 02$, $20 \to 01 \to 10 \to 02 \to 20$. Note that C_1 and C_2 are of relatively prime lengths and that $C_i \nsubseteq \bigcup_{j=1}^{i-1} C_j$ for $i \in [7] \setminus [2]$. So, we deduce from Corollary 13 that pri$(K(2, 2)) \geq 7$. But over $\mathbb{Z}_2$ we have $\sum_{i\in[8]} \chi(C_i) = 0$ and thus pri$(K(2, 2)) = 7$ is verified, as a result of Theorem 4.

Question 16:
Determine pri(K_n), $\Phi(K_n)$, pri$(K(d, n))$, and $\Phi(K(d, n))$.

V : 2-Primitivity and 3-primitivity

Theorem 17:
Conjecture 7(i) is true, namely a primitive digraph is 2-primitive if and only if it has at least two circuits.

Proof: Let G be primitive digraph. If G has only one vertex, then G has at least two loops which guarantees that G is 2-primitive. Otherwise, G has at least two vertices. Since a primitive digraph is strongly connected, G contains a circuit which passes through more than one vertex and hence has length, say $\ell \geq 2$. Moreover, the greatest common divisor of the lengthes of all the circuits in a primitive digraph should be one. This means that G has a circuit whose length is not equal to ℓ. So, the theorem follows from Corollary 11.

Theorem 18:
Conjecture 7(ii) follows from Conjecture 3.

Theorem 19:
A primitive digraph having a 2-regular subgraph is 3-primitive.

Theorem 20: Let G be a primitive digraph. Then G is 3-primitive provided it has three distinct cycles C_1, C_2 and C_3 such that $C_3 \setminus (C_1 \bigcup C_2) \neq \emptyset$ and $\gcd(\ell_1, \ell_2, \ell_3) = \gcd(\ell_1, \ell_2)$, where $\ell_i = \ell(C_i)$ for $i \in [3]$.

Theorem 21: Suppose G is a primitive digraph with three cycles C_1, C_2, C_3 whose lengths are ℓ_1, ℓ_2, ℓ_3, respectively. Then G is 3-primitive in each of the following four cases:

(i) $\ell_1 \leq \ell_2 = \ell_3$ and $C_3 \setminus (C_1 \bigcup C_2) \neq \emptyset$;

(ii) $\ell_1 = \ell_2 \leq \ell_3$ and $C_1 \setminus (C_2 \bigcup C_3) \neq \emptyset$;

(iii) $\ell_1 < \ell_2 < \ell_3$ and C_3 is disjoint from C_1 or C_2;

(iv) $\ell_1 < \ell_2$, $\ell_1 + \ell_2 < \ell_3$.

Theorem 22:
Suppose G is a primitive digraph with three circuits C_1, C_2, C_3 satisfying $\ell(C_1) < \ell(C_2) < \ell(C_3)$. Let

$$b = \text{Card}(C_2 \setminus (C_3 \cup C_1)),\ c = \text{Card}(C_3 \setminus (C_2 \cup C_1)),$$

$$b' = \text{Card}((C_2 \cap C_1) \setminus C_3),\ c' = \text{Card}((C_3 \cap C_1) \setminus C_2).$$

Then G is 3-primitive if any of the following three conditions holds:

(i) $c' = b'$, $1 \leq b$;

(ii) $b' < c'$, $c' \leq b + b'$;

(iii) $c' < b'$.

VI: Good homomorphism

Let G and H be two strongly connected digraphs. Consider a digraph homomorphism $f = (f_0, f_1)$ from $G = (V(G), E(G))$ to $H = (V(H), E(H))$. f will naturally induce a map from the set of k-walks in G to the set of k-walks in H in an obvious way for any nonnegative integer k . If for any circuit C in H we can find a cycle C' in G such that $f_{\ell(C)}(C') = C$, then we say that f is a **good homomorphism**.

Theorem 23:
Let G and H be two digraphs. If G is strongly connected and there exists a good homomorphism from G to H, then pri$(G) \geq$ pri(H).

Proof: Given any coloring δ of $E(H)$ and any good homomorphism f from G to H, we have a coloring δ' of $E(G)$ such that $\delta'(e) = \delta(f_1(e))$. The result then follows from Theorem 1, since $L(G, \delta') \geq L(H, \delta)$.

For any arc e in a digraph G, we use $\mathtt{i}_G(e)$ to represent the initial vertex and $\mathtt{t}_G(e)$ the terminal vertex of e in G, respectively. An **out-partition** π of a digraph G is a partition of $E(G)$ into disjoint sets $E_u^1, \cdots, E_u^{\pi_u}$, $u \in V(G)$, where $\cup_{i=1}^{\pi_u} E_u^i = E_u$ is the set of outgoing arcs at u, for each $u \in V(\Gamma)$. The **out-splitting** of G corresponding to an out-partition π, denoted by G_π, has vertex set $\{u^i : u \in V(G), i \in [\pi_u]\}$ and arc set $\{e^i : e \in E(G), i \in [\pi_{\mathtt{t}_G(e)}]\}$ and the incidence structure is given by requiring that for any $e \in E(G), i \in [\pi_{\mathtt{t}_G(e)}]$ we have $\mathtt{i}_{G_\pi}(e^i) = \mathtt{i}_G(e)^j$ and $\mathtt{t}_{G_\pi}(e^i) = \mathtt{t}_G(e)^i$, where j is chosen such that $e \in E^j_{\mathtt{i}_G(e)}$. By symmetry, we can also define the **in-splitting digraphs** of a given digraph. Splitting digraph is an important construction arising from **symbolic dynamics**. G is an **amalgamation digraph** of H if and only if H is a splitting digraph of G.

Corollary 24:
If G is a splitting digraph of H, then pri$(G) \geq$ pri(H).

Proof: It is an easy matter to see that there is a good homomorphism f from G to $H_\pi \cong H$, where f_0 assigns $E_u^i \in V(G)$ to $u \in V(H)$. Moreover, it is obvious that if H is strongly connected, then so is G. By now, the claim is immediate from Theorem 23.

Since line digraph is a special splitting digraph construction, Corollary 24 confirms our observation in Example 15 that pri$(K_3) <$ pri$(L(K_3))$.

To prove Conjecture 5 (5') or its special case Conjecture 7(ii), we need to get some good lower bound estimation of primitive numbers. Corollary 24 suggests an approach to achieving it, namely for a given digraph G, we try to obtain from G a digraph H by a series of amalgamation operation for which pri(H) has been well understood. This will then lead to an investigation of the relationship between the cycle structures of a digraph and its amalgamation digraph. Note that in general there will be a richer cycle structure in a splitting digraph than in the original digraph and we can expect that the inequality in Corollary 24 will rarely become equality. Look back to Example 15.

VII : Primitive road coloring

It is immediate from Corollary 12 that any primitive digraph which has a subgraph with minimum out-degree (in-degree) at least k must has a k-primitive k-coloring. This reminds us the following nice result of Culik II, Karhumäki and Kari.

Theorem 25:
Any primitive digraph of constant out-degree k has a commutatively synchronizable road coloring.

Theorem 25 enables us strengthen the observation made at the beginning of this part as follows.

Theorem 26:
Any primitive digraph of constant out-degree k has a k-primitive road coloring.

Proof: Let G be a primitive digraph of constant out-degree k. By Theorem 25, G has a road coloring δ which has a commutatively synchronizing word α. It suffices to show that δ is a k-primitive coloring. According to Theorem 1, we need to prove $L(G, \delta) = \mathbb{Z}^k$ instead. This further reduces to verifying $e_i \in L(G, \delta)$ for each $i \in [k]$. Take any $i \in [k]$. Let u be the synchronization state for (G, δ, α) and let $v = \mathtt{t}_G(e)$ where e is the arc starting from u and receiving color i under δ. By the definition of commutatively synchronization, there is a walk W_1 going from u to u itself whose type under δ is α and hence $\alpha \in L(G, \delta)$; there is also a walk W_2 going from v to u whose type under δ is α and hence $\alpha + e_i$ belongs to $L(G, \delta)$, as by adding e to the walk W_2 we obtain a closed walk. From these two observations, we deduce that $e_i = (e_i + \alpha) - \alpha \in L(G, \delta)$. The theorem follows.

Thank You!

Our approach

Recall that a **division matrix** is a $(0, 1)$ matrix such that each row of it has at least one 1 and each column of it has exactly one 1. For any two matrices A and B, we write $A \prec B$ to mean that there is a division matrices P satisfying $PA = B$.

Question 27:

Let $L \in \mathbb{Z}^n$ and $\mathcal{D}_1(L) = 1$. Discuss the structure of those $k \times n$ nonnegative integer matrices M with $\mathbf{1}_k M = L$ and $\mathcal{D}_k(M) = 1$.

Question 28:

Study those primitive digraphs G for which when taking $L = \mathbf{1}C(G)$ Question 27 has a solution $M = C(\mathcal{A})$ for some k-automaton $\mathcal{A}$ on G. Due to the correspondence between a division matrix and a coloring, another formulation of the question is to investigate that kind of primitive digraphs G for which we can confirm the existence of a solution M to Question 27 for $L = \mathbf{1}C(G)$ with the additional property that $C(G) \prec M$.

Roughly speaking, this approach is to tackle Conjecture 5 in two steps, first consider a preliminary matrix theoretic problem (Question 27) and then a problem (Question 28) involving digraphs or sometimes just set systems, namely the set of all cycles in a digraph.

Slides Graveyard