Lecture One

Blocking System

Yaokun Wu *†

Abstract

This note is adapted from the last section of the book of Brualdi [1] and aims to give a clear description of some basic facts about blocking system, from which Fulkerson developed his famous blocking and antiblocking polyhedra theory.

Keywords: $(0,1)$ principal; blocking system; clutter

1 $(0,1)$ principal

Let E be a set (which is not necessarily finite). A family $\mathcal{F}$ on E is a collection of subsets of E with repetitions permitted. Denote by $L(E)$ the set of functions from E to the set of real numbers. A $(0,1)$-value function is a function taking values in $\{0,1\}$.

Any pair of families $(\mathcal{R},\mathcal{S})$ on E that satisfies

$$\underset{R\in\mathcal{R}}{Inf}\ \underset{x\in R}{Sup} f(x) = \underset{S\in\mathcal{S}}{Sup}\ \underset{x\in S}{Inf} f(x), \tag{1}$$

for all $f\in L(E)$ is said to be a blocking system for E. It turns out that the so-called $(0,1)$ principal in computer science is also valid here and thus we have a better understanding of the seemingly intractable concept of blocking system defined above.

Theorem 1.1 *For any two families $\mathcal{R}$, $\mathcal{S}$ on a set E, the following three assertions are equivalent:*

(i) *$(\mathcal{R},\mathcal{S})$ is a blocking system for E;*

(ii) *Eq. (1) holds for any $(0,1)$-value function f on E (Note that for an f taking only finitely many values, just like in this case, we can also write Eq. (1) as $\underset{R\in\mathcal{R}}{Min}\ \underset{x\in R}{Max} f(x) = \underset{S\in\mathcal{S}}{Max}\ \underset{x\in S}{Min} f(x)$.);*

(iii) *for any partition of E into a pair of its subsets (E_0,E_1), namely $E_0\cup E_1 = E$ and $E_0\cap E_1=\emptyset$, either a member of $\mathcal{R}$ is contained in E_0 or a member of $\mathcal{S}$ is contained in E_1, but not both.*

*Department of Mathematics, Shanghai Jiao Tong University, 1954 Huashan Road, Shanghai, 200030, China

†Email: ykwu@mail1.sjtu.edu.cn,

Proof: The following correspondence illustrates the equivalence between (ii) and (iii).

`patition` $\{E_0, E_1\}$	$(0,1)$`-value function` f
E_0	$\{x \mid f(x) = 0\}$
E_1	$\{x \mid f(x) = 1\}$
$\exists R \in \mathcal{R}$, $R \subseteq E_0$ `and` $\forall S \in \mathcal{S}$, $S \not\subseteq E_1$	$\underset{R\in\mathcal{R}}{Min}\ \underset{x\in R}{Max} f(x) = \underset{S\in\mathcal{S}}{Max}\ \underset{x\in S}{Min} f(x) = 0$
$\forall R \in \mathcal{R}$, $R \not\subseteq E_0$ `and` $\exists S \in \mathcal{S}$,$S \subseteq E_1$	$\underset{R\in\mathcal{R}}{Min}\ \underset{x\in R}{Max} f(x) = \underset{S\in\mathcal{S}}{Max}\ \underset{x\in S}{Min} f(x) = 1$

As (i) $\Rightarrow$ (ii) is trivial, our task is to demonstrate (iii) $\Rightarrow$ (i). This follows from two easy observations:

$\underset{S\in\mathcal{S}}{Sup}\ \underset{x\in S}{Inf} f(x)$ is the unique number u such that for any $\epsilon > 0$, $\{x \mid f(x) \geq u - \epsilon\}$ is a superset of some member of $\mathcal{S}$ while no number of $\mathcal{S}$ is a subset of $\{x \mid f(x) > u + \epsilon\}$; $\underset{R\in\mathcal{R}}{Inf}\ \underset{x\in R}{Sup} f(x)$ is the unique number u such that for any $\epsilon > 0$, no member of $\mathcal{R}$ is contained in $\{x \mid f(x) < u - \epsilon\}$ while some member of $\mathcal{R}$ is covered by $\{x \mid f(x) \leq u + \epsilon\}$. □

Remark If we only know that Eq. (1) holds for some $f \in L(E)$ rather than all $f \in L(E)$, we still have the $(0, 1)$ principal which follows from almost the same argument as above. Also, we can modify Eq. (1) to give other interesting relations.

Condition (iii) listed in Theorem 1.1 has an immediate consequence.

Corollary 1.1 *$(\mathcal{R}, \mathcal{S})$ is a blocking system for E if and only if $(\mathcal{S}, \mathcal{R})$ is a blocking system for E.*

2 Structural characterization

We intend to give a clear structural characterization of the blocking system here. For this purpose, for any family $\mathcal{R}$ on E let us introduce two other families on E. We denote by $C(\mathcal{R})$ the set of nonempty minimal elements of $\mathcal{R}$ under the inclusion relation and $b(\mathcal{R})$ the set of minimal elements of 2^E under the inclusion relation which have nonempty intersection with every member of $\mathcal{R}$. We will see shortly that the operator C is just the square of the operator b.

Lemma 2.1 *Let $\mathcal{R}$ and $\mathcal{S}$ be two families on E. The following are equivalent:*

(I) *For any $f \in L(E)$, it holds $\underset{R\in\mathcal{R}}{Inf}\ \underset{x\in R}{Sup} f(x) = \underset{S\in\mathcal{S}}{Inf}\ \underset{x\in S}{Sup} f(x)$;*

(II) *For any $f \in L(E)$, it holds $\underset{R\in\mathcal{R}}{Sup}\ \underset{x\in R}{Inf} f(x) = \underset{S\in\mathcal{S}}{Sup}\ \underset{x\in S}{Inf} f(x)$;*

(III) *$C(\mathcal{R}) = C(\mathcal{S})$.*

Proof: Our strategy is to show that both (I) and (II) are equivalent to (III).

One can easily verify the truth of the following two relations,

$$\underset{R\in\mathcal{R}}{Inf}\ \underset{x\in R}{Sup} f(x) = \underset{R\in C(\mathcal{R})}{Inf}\ \underset{x\in R}{Sup} f(x)$$

and

$$\underset{R\in\mathcal{R}}{Sup}\ \underset{x\in R}{Inf} f(x) = \underset{R\in C(\mathcal{R})}{Sup}\ \underset{x\in R}{Inf} f(x),$$

which then establish the forward direction.

For the converse direction, it is enough to construct two functions $f, g \in L(E)$ such that

$$\underset{R \in C(\mathcal{R})}{Inf} \ \underset{x \in R}{Sup} f(x) \neq \underset{S \in C(\mathcal{S})}{Inf} \ \underset{x \in S}{Sup} f(x), \qquad (*)$$

and

$$\underset{R \in C(\mathcal{R})}{Sup} \ \underset{x \in R}{Inf} f(x) \neq \underset{S \in C(\mathcal{S})}{Sup} \ \underset{x \in S}{Inf} f(x) \qquad (**)$$

in case that $C(\mathcal{R}) \neq C(\mathcal{S})$. We do this in the sequel.

Since $C(\mathcal{R}) \neq C(\mathcal{S})$, we have $C(\mathcal{R}) \triangle C(\mathcal{S}) \neq \emptyset$ and hence can take a minimal element of $C(\mathcal{R}) \triangle C(\mathcal{S})$, say, w.l.o.g., $R_0 \in C(\mathcal{R}) - C(\mathcal{S})$.

Let $f(x) = \chi_{E-R_0}(x)$ be the characteristic function of $E - R_0$. Then

$$\underset{R \in C(\mathcal{R})}{Inf} \ \underset{x \in R}{Sup} f(x) = \underset{x \in R_0}{Sup} f(x) = 0.$$

But $\underset{S \in C(\mathcal{S})}{Inf} \ \underset{x \in S}{Sup} f(x) = 1$ as any member S of $C(\mathcal{S})$ cannot be a subset of R_0. This gives (*). (**) can be obtained in a similar way and thus we are done. □

Theorem 2.1 *$(\mathcal{R}, \mathcal{S})$ is a blocking system if and only if $C(\mathcal{S}) = b(\mathcal{R})$.*

Proof: We use the equivalent definition for blocking system, namely Theorem 1.1 (iii) to verify our assertion.

Suppose a subset E_0 of E does not contain any member of $\mathcal{R}$. Then $E - E_0$ has nonempty intersection with each member of $\mathcal{R}$ and hence there is a set $X \subseteq E - E_0$ which is a minimal element among those having nonempty intersection with every member of $\mathcal{R}$. Note that $X \in b(\mathcal{R})$ by definition.

By now, a simple application of Lemma 2.1 gives the result. □

Theorem 2.1 tells us for any family $\mathcal{R}$ how to generate all the families $\mathcal{S}$ such that $(\mathcal{R}, \mathcal{S})$ is a blocking system. In fact, each such $\mathcal{S}$ can be obtained by first construct $b(\mathcal{R})$ and then add in some supersets of some members of $b(\mathcal{R})$. We can also deduce from Theorem 2.1 the next interesting corollary, which we have forecasted at the beginning of this section.

Corollary 2.1 *$C = b^2$; that is, for any family $\mathcal{R}$ on a set E, $C(\mathcal{R}) = b(b(\mathcal{R}))$.*

A clutter on E is a set of subsets of E with the property that no member contains another. Observe that for any family $\mathcal{R}$ $C(\mathcal{R})$ is a clutter and Lemma 2.1 suggests that it is better and essentially equivalent to discuss blocking systems consisting of clutters. This is documented in the following neat result.

Corollary 2.2 *Let R be a clutter on E. Then*

(i) *$b(\mathcal{R})$ is the unique clutter such that $(\mathcal{R}, b(\mathcal{R}))$ is a blocking system;*

(ii) *$b(b(\mathcal{R})) = \mathcal{R}$.*

References

[1] R.A. Brualdi, Introductory Combinatorics, North Holland, 1977.