Some characterizations for the wrapped butterfly *

Yaokun Wu, Qiao Li
Department of Applied Mathematics, Shanghai Jiao Tong University
Shanghai, 200030, China
{ykwu, qiaoli}@mail1.sjtu.edu.cn

Abstract

We present some characterizations for the wrapped butterfly, a class of useful network models in parallel processing. Our characterizations are parallel to the characterization of Bermond et al. for the (ordinary) butterfly. But our approach seems more natural and provides some more insight. In the course of our study, we also give a slight improvement of a result of Beineke et al. which characterizes the iterated line digraph structure in terms of the Heuchenne's condition. Some related problems and conjectures are proposed at the end of the paper.

1991 AMS Subject Classification: 05C75, 68R10.

Key Words and Phrases: characterization; generalized cycle; Heuchenne's condition; iterated line digraph; wrapped butterfly.

1 Introduction

As usual, for any natural numbers d and n, Z_d is the additive group of integers modulo d, and $Z_d^n = \{a_1a_2\cdots a_n \;:\; a_i \in Z_d,\ \forall i \in Z_n\}$ is the set of all d-nary n-bit strings. We say that a_i is the ith bit of the string $a_1a_2\cdots a_n$ for $i \in Z_n$. For any integer $p > 1$, a *generalized p-cycle G* is a digraph whose vertex set can be expressed as a disjoint union of p subsets, say, V_i, $i \in Z_p$, and whose arcs can only go from V_i to V_{i+1} for some $i \in Z_p$. Sometimes for $i \in Z_p$, the set V_i is called the ith *stage* of G and the set of arcs leading from V_i to V_{i+1} the ith *arc-stage* of G. For some very interesting results about generalized cycles, see [13, 17] and the references therein. For any $i \in Z_p$, G can be split into a $(p+1)$-stage *multistage interconnection network (MIN) digraph* [5, 42] between the stages V_i and V_{i+1}, designated

*Supported by the NNSFC Grant No. 19971056 and the RFDP Grant No. 2000024837.

by G_i, in the following three steps: deleting all the arcs of G from V_i to V_{i+1}; then making a copy of V_{i+1}, say $U_{i+1} = \{x' \mid x \in V_{i+1}\}$, as the additional set of vertices; and finally adding an additional arc from a vertex $y \in V_i$ to a vertex $x' \in U_{i+1}$ whenever there is originally an arc in G from y to x.

This paper is focusing on a special class of generalized cycles defined on alphabets, the *wrapped butterfly* (WBY). For any two integers $d, n > 1$, the d-nary n-dimensional wrapped butterfly [31], denoted by $WBY(d, n)$, is the digraph $(\cup_{i=0}^{n-1} W_i, E)$, where $W_i = \{\langle x, i\rangle : x \in Z_d^n\}$ for $i \in Z_n$ and $\langle x, i\rangle\langle y, j\rangle \in E$ if and only if $j = i+1$ and y coincides with x in all bits except possibly the jth one. Having this easily described interconnection, WBY is found to be vertex symmetric and possess many other interesting structure properties. In fact, WBY has received much attention as a good model in network design and it is universal in the sense that it can efficiently simulate an arbitrary bounded-degree network [1, 2, 14, 26, 31, 35, 37, 38, 40].

In [38], the authors defined a family of "new" Cayley digraphs and demonstrated some very attractive properties of them for providing good connection capability. It turned out that their "new" family is nothing but the family of binary wrapped butterflies $WBY(2, n)$ [9, 10, 40]. We note that there is a similar story for the general wrapped butterflies [14, 28, 29, 39].

This type of phenomena also happens with the (ordinary) *butterfly* $WBY(d, n)_{n-1}$, which is a split version of the wrapped one and is usually denoted by $BY(d, n)$ [31]; that is, many seemingly different digraphs, used widely as models of network families under various names, are found to be isomorphic with the butterfly indeed. This has prompted much research around the problem of testing whether a digraph is isomorphic with a butterfly. See [42] for a survey. A distinguished result in this area is the $P(*,*)$ characterization for the butterfly, given by Bermond, Fourneau, and Jean-Marie [5]. The $P(*,*)$ characterization not only explains very well in a unified manner many phenomena of this kind but also provides a deeper understanding of the structure of the butterfly.

Our main result in this paper will be some characterizations for the wrapped butterfly, one form of which says that a digraph G is isomorphic with $WBY(d, n)$ if and only if it is a d-regular generalized n-cycle with nd^n vertices and $d \times \texttt{rank}\ (A^n) = \texttt{rank}\ (A^{n-1}) = dn$, where A is the adjacency matrix of G. Our characterizations are parallel to the famous $P(*,*)$ characterization for the ordinary one. However, we will adopt a line digraph approach, which is completely different from the approach of [5] but in the same line with that in the series of papers [13, 15, 42, 43, 44]. For any digraph G, the *line digraph* [20] of G is a digraph $L(G)$ such that $V(L(G)) = E(G)$ and $L(G)$ has an arc from e to f if and only if, when viewing both e and f as arcs of G, the head of e is the tail of f. Generally, for any natural number

n, the nth line digraph $L^n(G)$ [3] is defined as the digraph obtained from G by iteratively applying the line digraph operator L n times. In this paper, we will freely use some elementary results about line digraph, for which a basic reference is [21].

In section 2, we show that, in some sense, the family of generalized cycles is closed for both the line digraph operator L and its inverse L^{-1}. In the next two sections, our "old" technique in [44] together with a result of Beineke and Zamfirescu [3] is utilized to establish two local criteria for iterated line digraph structure. After these preparations, in section 5 we will give some characterizations for the wrapped butterfly and present at the same time an example to illustrate that a slight weakening of the conditions listed in our characterizations will not guarantee the wrapped butterfly any more. Finally, our paper will be concluded with some problems and conjectures.

2 Generalized cycle

As the generalized cycle structure is a prominent characteristic of the WBY, it is not surprising that we will begin with a discussion of the generalized cycles here. First, we have a very simple lemma, for which the reader can take as a warm-up.

Lemma 1 *Let G be a generalized p-cycle. Then $L(G)$ is also a generalized p-cycle.*

The next lemma is in the converse direction. Roughly speaking, it means that when there are both the generalized cycle structure and the line digraph structure, we can always efface the second structure while keep the first one. In its proof, we will use the Heuchenne's characterization theorem for line digraph [4, 21, 25], which says that a digraph is a line digraph if and only if it has no multiple arc and the out-neighbor (in-neighbor) sets of any two of its vertices are either identical or disjoint. We note that a digraph satisfying only the latter requirement, namely any two out-neighbor sets are either identical or disjoint, is called a *semi-functional* digraph by Berge [4] and we will address ourselves to the functional digraph, another concept from [4], at the end of this section.

Lemma 2 *Let G be both a line digraph and a generalized p-cycle. Then there is a generalized p-cycle of which G is the line digraph.*

Proof. Since G is a generalized p-cycle, there is a partition of $V(G)$ into pairwise disjoint sets, which is cyclically ordered as $V_1, V_2, \cdots, V_p$, such that any vertex in V_i can only be adjacent to some vertices in V_{i+1}, $i \in Z_p$.

We will explicitly construct a generalized p-cycle H of which G is the line digraph.

Define two vertices of G to be equivalent if and only if for some $u \in V(G)$, they are both elements of $N(u)$, the out-neighbor set of u in G. From the Heuchenne's characterization theorem for line digraph, we know that an equivalence class is either a set $N(u)$ for some $u \in V(G)$ or a singleton set just consisting of a source vertex. Clearly, the vertices belonging to the same equivalence class will completely fall into a unique stage V_i for some $i \in Z_p$. So for each $i \in Z_p$ the set V_i naturally corresponds to a set of equivalence classes, designated by Λ_i, the union of which is exactly V_i itself.

Observe that the line digraph characterization theorem also tells us that for any given equivalence class, say x, and for all $u \in x$, the in-neighbor set of u in G, which is denoted by $N^-(u)$, must be the same. Thus, without causing any ambiguity, we can simply write $N^-(x)$ for this common set $N^-(u), u \in x$.

Now we are ready to describe the construction of H. For each $i \in Z_p$, set $U_i = \{u' \mid u \text{ is a sink vertex of } G \text{ in } V_{i-1}\}$. Take $V(H) = \cup_{i \in Z_p}(\Lambda_i \cup U_i)$. $E(H)$ is produced as follows: for any two equivalence classes x and y in $V(H)$ and for any element in $N^-(y) \cap x \subseteq V(G)$, add an arc from x to y (Note that when $N^-(y) \cap x = \emptyset$ we add no arc.); for any sink vertex u of G, add an arc from the equivalence class containing u to the vertex u' of H. Since the arcs of H can only go from $\Lambda_i \cup U_i$ to $\Lambda_{i+1} \cup U_{i+1}$ for $i \in Z_p$, H indeed keeps the generalized p-cycle structure as we hope.

On the other hand, for any vertex $u \in V(G)$ and the equivalence class x containing u, by sending u to xu' if u has out-degree zero in G and sending u to its corresponding arc from x to $N(u)$ ($N(u)$ here should be viewed as an equivalence class, namely a vertex of H.) otherwise, we get a bijection from $V(G)$ to $E(H)$. Moreover, a routine check shows that this mapping induces in fact an isomorphism from G to $L(H)$. This finishes the proof.

In view of the dynamic behavior of the line digraph operator L, the above two simple results suggest that we can regard any generalized p-cycle as an imitation of the cycle of length p. Likewise, we know that any p-stage MIN digraph can be thought of as an imitation of the path of length $p-1$ [42]. We will deviate a little to include a generalization of these observations.

Recall that the class of *functional digraphs* [4] consists of those digraphs with maximum out-degree not greater than one and hence contains all cycles and paths. Clearly, a functional digraph must be a line digraph. Let H be any digraph. A generalized-H digraph is a digraph whose vertices can be partitioned into a disjoint union of independent sets S_v (may be an empty set) for $v \in V(H)$ and there may exist arcs from S_v to S_w only if there are arcs from v to w in H. A careful reader can check that in proving Lemma 2 the only property of the cycle we have used is its functional digraph structure. Thus, our final result in this section follows from the same argument there.

Lemma 3 *Let H be a functional digraph and H_1 be a digraph of which H is the line digraph. Then, for any generalized-H line digraph G there is a generalized-H_1 digraph of which G is the line digraph.*

3 A technical lemma

For the use of the next section, we can simply quote a result in [43]. But for the completeness of this paper, we shall prove in this section a new technical lemma, which has that result as a corollary. The idea for this forthcoming lemma arises from [44].

To present our result, we need some notations. Define $S_x = 1$ for any number $x \neq 0$ and $S_0 = 0$. In other words, $S_x = 1 - \delta_{x,0}$, where δ is the Kronecker symbol. For any matrix $A = (a_{ij})$, we write S_A for $(S_{a_{ij}})$, the $(0, 1)$ matrix obtained by applying the operator S on A componentwise. Let J denote an all one's matrix whose size will be specified in the context. For any two matrices of the same size, say A and B, we will write $A \leq B$ to mean that $B - A$ is a nonnegative matrix.

Lemma 4 *Let A, B, and C be any three nonnegative matrices such that we can form the product ABC. Suppose*

(i) $S_A S_{BC} \leq J$;

(ii) *S_A has no zero column;*

(iii) *S_C has no zero row;*

(iv) *Any two rows of S_{AB} are either identical or orthogonal;*

(v) *Any two rows of S_{BC} are either identical or orthogonal.*

Then any two rows of S_B are either identical or orthogonal as well.

Proof. For any matrix X, we use X_i for the set $\{j : X(i, j) \neq 0\}$. Assuming $B_i \cap B_j \neq \emptyset$, let us prove that $B_i = B_j$.

Take an element $k \in B_i \cap B_j$. Condition (iii) implies that $(BC)_i \cap (BC)_j \supseteq C_k \neq \emptyset$. Henceforth, condition (v) gives

$$(BC)_i = (BC)_j. \tag{1}$$

On the other hand, condition (ii) guarantees two indices q and s such that $A(q,i) > 0$ and $A(s,j) > 0$. Consequently, $(AB)_q \cap (AB)_s \supseteq B_i \cap B_j \neq \emptyset$, and thus

$$(AB)_q = (AB)_s \tag{2}$$

follows from (iv).

Let $I = A_q - \{i\}$. For any $i' \in I$, if there is an element $u \in (BC)_i \cap (BC)_{i'}$, then $(S_A S_{BC})(q,u) \geq S_A(q,i)S_{BC}(i,u) + S_A(q,i')S_{BC}(i',u) = 2$, contracting condition (i). So we must have $(BC)_i \cap (\cup_{i' \in I}(BC)_{i'}) = \emptyset$. Thus, it follows from (1) that $(BC)_j \cap (\cup_{i' \in I}(BC)_{i'}) = \emptyset$. Noticing condition (iii), this in turn leads to $B_j \cap (\cup_{i' \in I} B_{i'}) = \emptyset$. But it holds additionally

$$B_j \subseteq (AB)_s = (AB)_q = B_i \cup (\cup_{i' \in I} B_{i'}),$$

where the first inclusion relation comes from the fact $A(s,j) > 0$, the second equality is just (2), and the last equality is due to $A_q = \{i\} \cup I$. So we obtain $B_j \subseteq B_i$. Swapping the roles of i and j, we can also establish the reverse relation, $B_i \subseteq B_j$. Combining these two relations, we arrive at $B_i = B_j$, as desired.

4 Local criteria for characterizing iterated line digraph

For any positive integer n, Hemminger [22] has proposed a necessary and sufficient condition for a general digraph to be an nth line digraph, whose verification is rather long and not published yet [24]. In spite of this general result, it seems still of interest to pursue some different methods to efficiently recognize the iterated line digraph structure, especially those permitting an easy understanding. We refer to [3, 34] for some results in this direction.

A digraph is said to satisfy the nth *Heuchenne condition* [3, 23, 25] if for any of its vertices u, v, w, and x (not necessarily distinct) for which there exist n-walks from u to w, from v to w, and from v to x, there must also exist an n-walk from u to x. In terms of the Heuchenne's conditions, Beineke et al. presented among others in [3] the following characterization of the iterated line digraph structure for some restricted class of digraphs. It puts in evidence the forbidden subgraphs and thus can be viewed as a local criterion.

Theorem 5 *[3] Let G be a digraph without sinks or sources. Then G is an nth line digraph if and only if, for $k = 1, 2, \cdots, n$, the following conditions are satisfied:*

(I) *There are no multiple k-walks between any pair of vertices;*

(II) *G satisfies the kth Heuchenne condition.*

We remark that the argument in [3] for the inductive proof of Theorem 5 also appeared in [34, 44] and it can be substituted by the corresponding matrix argument used in [15, 43], both being easy to understand and generating a short proof.

Based on Theorem 5 and Lemma 4, we can establish a simpler criterion for certain digraphs.

Theorem 6 *Let G be a digraph without sinks or sources. If there are no multiple $(1+n)$-walks between any pair of vertices in G, then it is an nth line digraph if and only if it satisfies the nth Heuchenne condition.*

Proof. By virtue of Theorem 5, we need only check the two conditions therein.

Let A be the adjacency matrix of G, which has no zero row and no zero column as G has no sinks or sources. Observe that the assumption that there are no multiple $(1+n)$-walks between any pair of vertices in G just means that A^{1+n} is a $(0,1)$ matrix. For $k = 1, 2, \cdots, n$, since A has no zero row and hence A^{1+n-k} has no zero row too, we deduce from $A^{1+n} = A^k A^{1+n-k}$ that A^k is a $(0,1)$ matrix. This is the condition (I). The nth Heuchenne condition can be interpreted as that any two rows of $S_{A^n} = A^n$ are either identical or orthogonal. By putting $A^{1+n} = AA^{n-1}A$, Lemma 4 implies that G satisfies the $(n-1)$th Heuchenne condition. Continuing with $A^n = AA^{n-2}A$, Lemma 4 then gives us the $(n-2)$th Heuchenne condition. Going this way, we can get all the kth Heuchenne conditions, $k = 1, 2, \cdots, n$. This verifies the condition (II) and thus we are done.

By now the reader may find that the following slight improvement of Theorem 5 has become rather trivial. But please note that a simple result is not necessarily useless.

Theorem 7 *Let G be a digraph without sinks or sources. Then G is an nth line digraph if and only if the following conditions are satisfied:*

(I') *There are no multiple n-walks between any pair of vertices;*

(II') *G satisfies both the nth and the $(n-1)$th Heuchenne conditions.*

5 Characterizations for the wrapped butterfly

Lemma 8 *For any two integers $d, n > 1$, there is a unique digraph G which is both a d-regular generalized n-cycle with uniform stage size d^n and an nth line digraph.*

Proof. Let $G = L^n(H)$. The d-regularity of H follows from the d-regularity of G. Clearly G has $d^n n$ vertices and thus H has $\mid V(G) \mid /d^n = n$ vertices. Moreover, the Corollary 8.2 in [21] tells us that H is unique for G and henceforth we know from Lemma 2 that H is always a generalized n-cycle. But such an H can only be a digraph with the vertex set Z_n and the arc set of cardinality dn with d ones going from i to $i+1$ for $i \in Z_n$. The uniqueness of G is a result of the uniqueness of H.

At this stage, we cannot help wondering what is the unique digraph claimed in Lemma 8. Yes, it is just the WBY! To see it, one can simply make reference to Theorem 7 and our definition for the WBY. Therefore, making use of several equivalent formulations of the Heuchenne's conditions, Theorem 7 and Lemma 8 together will enable us to list some characterizations for the wrapped butterfly in the ensuing theorem. We note that all these forms have been used in characterizing the ordinary butterfly [5, 27, 42] and the equivalence among them is not difficult to prove, for a detailed analysis of which one can refer to [42]. This is why we do not bother to include a proof below. We also mention that parallel to the many results about the butterfly, the reader can still produce some more characterizations along the approach here.

To state one of our characterizations, we need to define a kind of refinement of the Heuchenne's condition, which was first introduced by Hwang in [30] for MIN digraphs. Let G be a generalized n-cycle with a proper cyclic stage partition $V(G) = \cup_{i \in Z_n} V_i$. For any $x \in V(G)$, let $N^j(x)$ represent the set of vertices that x can reach in exactly j steps. For any $k \in Z_n$ and any positive integer j, G is said to have the (k, j) *buddy property* [30] if and only if for any two of its stage-k vertices, say u and v, it holds either $N^j(u) = N^j(v)$ or $N^j(u) \cap N^j(v) = \emptyset$.

Theorem 9 *Let G be a d-regular generalized n-cycle with nd^n vertices. Then G is the $WBY(d, n)$ if and only if any one of the following conditions is fulfilled:*

(a) $d \times \textbf{rank}\ (A^n) = \textbf{rank}\ (A^{n-1}) = dn$, *where A is the adjacency matrix of G;*

(b) *There are no multiple n-paths between any two different vertices and each subgraph of G obtained by removing one of its arc-stages has exactly d weak connected components;*

(c) *For any $i \in Z_n$, G_i is the butterfly $BY(d, n)$;*

(d) *G has the $(k, n-1)$ buddy property for each $k \in Z_n$ and there is at least one path of length n between any ordered pair of (not necessarily distinct) vertices in the same stage.*

Theorem 7 strengthens Theorem 5 by reducing the number of Heuchenne's conditions to be checked. This suggests that in order to improve Theorem 9, or to recognize the wrapped butterfly more efficiently, we can try to reduce the number of buddy properties listed in the above criterion. However, we have noticed the following example which shows that there is not much room left in this direction. Let $n \geq 3, d \geq 2$, and

$$x = \langle \underbrace{00 \cdots 0}_{n-1 \text{ zeros}} 1, n-1 \rangle,$$

$$y = \langle \underbrace{00 \cdots 0}_{n-3 \text{ zeros}} 101, n-1 \rangle.$$

We write H for the digraph obtained from $WBY(d,n)$ by exchanging the out-neighbors of x and y. It is not difficult to see that H satisfies all (k,j) buddy properties, with the exception of $(n-2, j)$ ones for $2 \leq j \leq n-1$. Of course, H cannot be isomorphic with $WBY(d,n)$, as the sets of buddy properties they have, though only slightly different from each other, are indeed not the same. By splitting H into its MIN version, we [42] can refute a recent improvement ([8] Corollary 7) of the Bermond's $P(*,*)$ characterization for the butterfly.

The reader may feel strange that why our characterizations for the wrapped butterfly, and those for the butterfly [42] and the De Bruijn digraph [43], will look much alike. Here is an explanation. Recall that for any two digraphs G and H, one can define their conjunction $G \otimes H$ to be the digraph with $V(G \otimes H) = V(G) \times V(H)$ and $E(G \otimes H)$ having $(x,y)(z,w)$ as an element of multiplicity $m_1 m_2$, where m_1 is the multiplicity of (x,y) in $E(G)$ and m_2 the multiplicity of (z,w) in $E(H)$ (In the language of matrix, the adjacency matrix of $G \otimes H$ is the Kronecker product of those of G and H). As an interesting application of Theorem 9, a reader familiar with the De Bruijn digraph [7, 43] will find that $WBY(d,n)$ can also be expressed as $C_n \otimes B(d,n)$, the conjunction of the directed cycle of length n with the d-nary n-dimensional De Bruijn digraph. Note also that the ordinary butterfly $BY(d,n)$ is just $P_{1+n} \otimes B(d,n)$ [42], where P_{1+n} stands for the path of length n. This observation then reveals that the butterfly, the wrapped butterfly, and the De Bruijn digraph, are three families of highly relevant structures and their similar characterizations just reflect the common underlying structure behind their different appearances.

6 Discussion

We end this paper with a brief discussion of some related problems and conjectures.

- Let $\mathcal{B}_n$ be the set of digraphs with no multiple n-walks between any pair of vertices. Let $H(i)$ denote the ith Heuchenne condition. For a digraph G, let $f_n(G) = \{1 \leq i \leq n \mid G \text{ satisfies } H(i)\}$. To obtain Theorem 7 from Theorem 5, the key is to show that when $G \in \mathcal{B}_n$ has no sources and no sinks, we can get from $\{n-1, n\} \subseteq f_n(G)$ automatically that $f_n(G) = \{1, 2, \cdots, n\}$. Thereby, for some given family of digraphs, say $\mathcal{C}$, we are interested in discussing the subsets $\mathcal{A}$ of $\{1, 2, \cdots, n\}$ which has a digraph realization in the sense that there exists a digraph $G \in \mathcal{B}_n \cap \mathcal{C}$ such that it satisfies all $H(i)$ for $i \in \mathcal{A}$ while violates all $H(i)$ for $i \in \{1, 2, \cdots, n\} - \mathcal{A}$. Similarly, for the family of generalized cycles, we have also the realization problem for the buddy properties.

- In Theorem 9, the generalized cycle structure has been assumed. However, making use of the Perron-Frobenius theorem for nonnegative matrices, we have found that this assumption can be removed indeed. Namely, we can prove the following result stated in the language of matrix equation: For any two integers $d, n \geq 2$, let A be an irreducible $(0,1)$ matrix such that $\texttt{rank}\ (A^{n-1}) = dn$, and A^n is an $n \times n$ diagonal block matrix of the form

$$\begin{bmatrix} J & & & & \\ & J & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & J \end{bmatrix},$$

  where the blocks J are the all ones matrices of size $d^n \times d^n$, then the associated digraph of A must be $WBY(d, n)$.

- Using the Corollary 8.3 in [21] and noticing the iterated line digraph construction of the $WBY(d, n)$, we can determine its automorphism group, which has order $n \prod_{i=1}^{d} i^n$. In [27], Hotzel determined the automorphism group of $BY(2, n)$ (but not using a line digraph approach) and conjectured that $BY(2, n)$ can be characterized by its symmetric property. Motivated by his conjecture, we pose a corresponding conjecture for the wrapped butterfly: The automorphism group of any d-regular generalized n-cycle other than $WBY(d, n)$ must have order less than $n \prod_{i=1}^{d} i^n$.

- In [8], Chang et al. introduced the so-called bit permutation networks, which contains many important families of network models as its special families, including those being equivalent to the butterflies. In [41], we generalize the $P(*, *)$ characterizations of Bermond et al. to give several graph theoretical characterizations for the bit permutation networks. Some new type of characterizations obtained via the technique of layered cross product [12] are presented there as well. It is natural to consider also the wrapped version of bit permutation networks, a generalization of the concept of wrapped butterfly. The further structure on which our approach can be applied to shed some light may also include the family of generalized shuffle networks [6], which is defined using a mixed radix number system instead of a fixed d-nary number system, and the even more general extended generalized shuffle networks [36], and their wrapped version.

- Following [19, 21], we define two important categories of problems relevant to the line digraph structure:

  Determination Problem. Determine which digraphs have a given digraph as their line digraph and the relationship between the structural properties of a digraph and its line digraph.

  Characterization Problem. Characterize those digraphs that have the iterated line digraph structure. Alternatively, what simple patterns can be an indication of the iterated line digraph structure?

  In this classification, we can say that our section 2 is dealing with the determination problem and section 4 the characterization problem, while our characterizations for the wrapped butterfly come from the combination of these two types of results. We hope that this approach may also be effective in analyzing some other structures of combinatorial interest. One direction is to consider other families of digraphs. In [42], we do it for the class of MIN digraphs (which have sources and sinks) and give a unified story of the topological equivalence issue for the butterfly. But can we go further for still other interesting families of digraphs? The other direction is to generalize the concept of iterated line digraph structure. For example, we know that line digraph is a special case of the concept of in-split graph and out-split graph [33] in dynamical system and a special case of the concept of n-dimentional line digraph [32] which is developed as an attempt to generalize the structure of the De Bruijn digraph; while the concepts of divisor [11] and quotient [16] in algebraic graph theory, and the concept of voltage graph [18] in topological graph theory can be modified a little to cover the concept of line digraph too. Can we get interesting results for the determination problem or the characterization problem in these more general settings?

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