Graph Theoretical Characterizations of Bit Permutation Networks *†

Yaokun Wu ‡Xue-Wen Bao ‡, Xiaohua Jia, §Qiao Li ‡

Abstract

The bit permutation networks (BPNs), proposed by Chang, Hwang and Tong (Networks, 33 (1999) 261-267), are a class of digraphs which include the underlying topological structure of almost all the commonly used switching networks or sorting networks. Many problems about BPNs have been intensively studied. Our work here is to present several graph theoretical characterizations of BPNs, which can be naturally divided into two parts. One part follows the approach of several researchers in France and German, in which we characterize BPNs in terms of their distinguished component structure. The other part combines the techniques used by several researchers in Israel and Taiwan, where layered cross product (Networks, 29 (1997) 219-223) and channel graph play a critical role. Our work confirms the observation that a high degree of regularity is the reason why most of the networks in use have BPNs as underlying topologies. Our results have implications in many kinds of problems about BPNs, such as devising algorithms for checking topological equivalence, verifying useful network representations suiting specific need, revealing the rich intrinsic combinatorial properties of BPNs, and so on.

Keywords: Bit permutation network; channel graph; characterization; components; cross product family; layered cross product; multistage interconnection network; partition; topological equivalence

1 Introduction

1.1 Background

Multistage Interconnection Networks (MIN) are popular in switching and communication applications. Although there has been a large amount of research on them, there seems to be a surprisingly small number of basic designs that recur under many disguises. In fact, for almost all the MINs considered in the

*Supported by NNSFC Grant No. 19971056, 10001024 and RFDP Grant No. 2000024837.

†Email: ykwu@mail1.sjtu.edu.cn, xw_bao@263.net.cn, qiaoli@mail1.sjtu.edu.cn, csjia@cityu.edu.hk

‡Department of Applied Mathematics, Shanghai Jiao Tong University, Shanghai, 200030, China

§Department of Computer Science, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong, China

literature and used in parallel processing, there are some integers $d, n \geq 2$ such that each of the crossbar switches in the network has d inlets and d outlets and each stage of the network is composed of d^n crossbar switches. Moreover, a common phenomenon is that the crossbars in each stage are labeled by the set of d-nary n-bit strings and the labeling strings of any two adjacent crossbars (in adjacent stages) deviate in a regular way. This type of connection patterns and the corresponding labeling schemes play an important role in making possible good routing algorithms performed on these MINs. We note that the topological properties of a network is independent of any labeling of its crossbars, though the topological structure of the network will be embodied in the existence of a special labeling and such a labeling will often naturally introduce an efficient tag routing scheme on the network.

To capture the characteristic connection pattern of the usual MIN topologies, Chang, Hwang and Tong **[12]** proposed the so-called *bit permutation networks* (BPN), which are networks permitting some special labeling scheme on them (See Sec.1.2.). This family of networks have included most of the MINs in use, like *the Beneš network*, *the shuffle − exchange network*, *the butterfly network*, *the baseline network*, and so on. We note that some concepts similar to BPN have appeared in [18, 27].

Many BPNs were known under different names for decades before it was discovered that they are actually topologically equivalent. This "blindness" is largely due to the various representations which conceal their underlying topological structure [15]. There have been work toward comprehending the topological equivalence among several families of BPNs which make the results obtained through the investigation of one network applicable to the others and highlight the possibility to develop general algorithms for all MINs in the same equivalence class. Even if some BPNs are not topologically equivalent to each other, one still would like to extract their common underlying characteristic and have a clear simple representation by which we can easily compare their structural and functional difference. This then leads to the classification and representation problem. By establishing good topological properties and providing good representations of BPNs, some practical problems about them, like VLSI layout [2, 30] and nonblocking capability [10, 11, 27], can be efficiently addressed.

Our paper aims to provide new topological characterizations for the whole class of BPN and deepen our understanding of some related issues. We note that, this subject, for which only a theoretician would be a possible tackler, may belong to one of the three types of fundamental research questions for a technology-focused theoretician, namely the question about what really matters, according to the classification of Rosenberg [33]. That is, this is an attempt to find out some structure properties of BPNs which 1) might help to explain how BPN is distinguished from the remaining MINs; 2) might help to reveal and clarify the relationship among various seemingly unrelated BPNs and give insight into checking and recognizing algorithms; and 3) might provide a tool to verify certain representation of a BPN when we need a suitable viewpoint from which we find out some property of a BPN more clearly.

In discussing such a topic focusing on the topological properties, one can just turn to the graph model of a network which is, roughly speaking, obtained by representing a crossbar as a vertex and connecting two vertices whenever the corresponding crossbars are connected by a link in the original network.

We shall adopt this approach from now on. The necessary graph theoretical machinery and some existing results stated in this machinery will be presented in the next subsection. For the purpose of this paper we do not elaborate on the background any more. The reader can refer to [1, 4, 5, 6, 12, 17, 18, 22, 24, 28, 37] for more details about how this approach is relevant to problems arising from the communication and interconnection practice.

1.2 Graph Theoretical Approach

We begin with some basic definitions together with some notations to be reserved hereafter. An *MIN digraph* G is a simple digraph (along with its vertex partition) whose vertices are partitioned into, say m, subsets, $V_1, V_2, \cdots, V_m$ and whose arcs are partitioned accordingly into $m-1$ subsets $E_1, E_2, \cdots, E_{m-1}$, where arcs in E_i always go from V_i to V_{i+1} for $1 \leq i < m$. V_i is sometimes called the i^{th} *stage* of G and E_i the i^{th} *arc-stage* of G. Usually, such an MIN can be written as $G = (V_1, V_2, \cdots, V_m; E_1, E_2, \cdots, E_{m-1})$. Let d be an integer greater than one. We say that an m-stage MIN digraph G is of degree d if all its vertices with the exception of those in V_1 have in-degree d and all vertices other than those in V_m have out-degree d. It is not difficult to see that all stages of an MIN digraph of degree d must have a common size, which will be referred to as the *size of the digraph*. For our purpose, any digraph mentioned in this and the next section will always be an m-stage MIN digraph of degree d with size equal to a power of d, say d^n, for some integers $n > 0, d, m > 1$.

For any pair of numbers i and j, $1 \leq i \leq j \leq m$, the subdigraph of G induced by $V_i \cup \cdots \cup V_j$, denoted as $G_{i,j}$ throughout the paper, will be referred to as a *full subdigraph* of G. For brevity, we simply call a weakly connected component a *component*. We mention that any components of a full subdigraph of an MIN digraph G of degree d must also be an MIN digraph of the same degree. Clearly any MIN digraph of a constant degree can only have a unique vertex partition to fit into the definition of MIN and any isomorphism between two such digraphs must be stage preserving.

Denote by Z_d^n the set of d-nary n-bit strings, $\{x_n x_{n-1} \cdots x_1 \mid x_i = 1, 2, \cdots, d\}$. For $k = 1, 2, \cdots, n$, x_k is called the k^{th} bit of the string $x_n x_{n-1} \cdots x_1$. Chang, Hwang and Tong [12] defined a *bit-k group* in Z_d^n to be a set of d elements whose bits are identical among the group except the k^{th} one. Generally, for any subset A of $\{1, 2, \cdots n\}$ we define a bit-A group as a set of $d^{|\mathsf{A}|}$ strings the l^{th} bits of which have a common value for all l outside of A.

To describe the adjacency structure of a digraph we need to equip its vertices with labels for easy of reference. For an MIN digraph we always assume that any vertex in its i^{th} stage has a *stage label* i and an additional *position label* from Z_d^n. We require that different vertices in any single stage V_i receive different position labels (Note that this means that the labeling induces a bijection from Z_d^n to V_i). A *labelled MIN* digraph is an MIN digraph with both stage labels and position labels properly assigned to its vertices. For a labeled MIN, we can refer to a vertex with position label x and stage label i by (x, i) or just x if there is no danger of confusion. We also say that a set of vertices in V_i is a bit-k group if so is the set of their position labels and we will write $x_n x_{n-1} \cdots x_{k+1} x_0 x_{k-1} \cdots x_1$ for the bit-k group whose common ith bit, $i \neq k$, is x_i, namely we use x_0 as a special symbol which represents the set of numbers in Z_d, as was done in [12]. Let σ be any permutation on the set $\{0, 1, \cdots, n\}$

such that $\sigma(0) \neq 0$. For a labeled MIN digraph G, we say that E_i is determined by σ if any vertex $x_n x_{n-1} \cdots x_1$ in V_i has neighbor set $x_{\sigma(n)} x_{\sigma(n-1)} \cdots x_{\sigma(1)}$ in V_{i+1}. Therefore, if $\sigma(0) = k$ and $\sigma(k^{'}) = 0$, the set E_i constitutes of disjoint collections of arcs each forming a complete bipartite digraph from a bit-k group in V_i to a corresponding bit-$k^{'}$ group in V_{i+1}. In particular, when $\sigma_i = (0, k)$, we also say V_i is *bit-k connected* to V_{i+1}. We are now ready to present the definition of bit permutation networks. We remark that it is just the convention to use the term "network" here which can simply be read as "digraph".

Definition 1.1 *[12] Let $\sigma_1, \sigma_2, \cdots, \sigma_{m-1}$ be a sequence of permutations of the set $\{0, 1, \cdots, n\}$ which do not fix 0. The bit permutation network $G(d, n, m; \sigma_1, \sigma_2, \cdots, \sigma_{m-1})$ is an m-stage digraph of degree d with stage size d^n for which there exists a labeling of its vertices such that E_i is determined by σ_i for $i = 1, 2, \cdots, m-1$.*

Bit permutation networks are a class of MIN digraphs broad enough to include most of the intensively studied MIN digraphs. Chang, Hwang, and Tong initiate the study of them as a whole and give some nice results in this direction. In order to present their result, we have to introduce first the concept of *canonical sequence* [12]. A sequence of positive integers $(k_1, k_2, \cdots, k_{m-1})$ is said to be canonical provided that for each positive integer $i \leq m-1$ we have $\{1, 2, \cdots, k_i\} \subseteq \{k_t \mid t = 1, 2 \cdots, i\}$.

Theorem 1.1 *[12] For every bit permutation network G, there is a unique canonical sequence $(k_1, k_2, \cdots, k_{m-1})$ such that $G = G(d, n, m; \sigma_1, \sigma_2, \cdots, \sigma_{m-1})$ where $\sigma_i = (0, k_i)$. Any two BPNs with different canonical sequences are not isomorphic to each other.*

This result gives a sequence representation for each bit permutation network. This is a good representation as it reduces the problem of checking topological equivalence to just comparing two associated sequences. Further, given a labeled BPN, Hwang and Yen [24] demonstrate a method to find out the canonical sequence of a BPN. However, combining these two results together, we still does not get an intuitive characteristic of the underlying topology structure of BPN. It would be interesting to get results more directly relevant to the structure property of BPN rather than their representation.

A typical Omega network can be represented as $G(d, n, n+1; \sigma, \sigma, \cdots, \sigma)$, where $\sigma = (n, n-1, \cdots, 1, 0)$. The Ω-equivalent class is the set of networks having the same topological structure with Omega networks. Several topological characterizations for them have been established by Bermond, Fourneau and Jean-Marie. The properties they used are the $P(*,*)$ property and the banyan property, defined in the sequel. A $(1+n)$-stage MIN digraph G of degree d is said to satisfy property $P(i, j)$ if $G_{i,j}$ contains exactly d^{n-j+i} components. The $P(*,*)$ property stands for the union of all $P(i, j)$ for $1 \leq i \leq j \leq n+1$. Similarly, Bermond et al. introduced the self-suggestive notation $P(1,*)$ and $P(*, n+1)$. G satisfies the *banyan property* if any input/output pair of vertices is connected by at most one path.

Theorem 1.2 *[6] An MIN digraph G of degree two is topologically equivalent to an Omega network if and only if it satisfies the $P(*,*)$ property and the banyan property.*

Based on this result, Bermond et al. go further to get the next theorem which corresponds to a simpler Ω-equivalence checking algorithm.

Theorem 1.3 *[6] An MIN digraph G of degree two is topologically equivalent to an Omega network if and only if it satisfies the properties $P(1,*)$, $P(*,n+1)$, and the banyan property.*

Several other characterizations of the Ω-equivalent class were given by Hotzel [18].

Even and Litman [15] introduced the layered cross product as a technique to construct interconnection networks. Conversely, an interesting observation is that many MINs can be decomposed as the layered cross product of two simpler MINs and this decomposition can help to investigate various properties of the original network, like tight VLSI layout, network partitioning, and rearrangeability. Recently, there have appeared work on understanding the structure of MINs along this approach [9, 10, 17, 26, 31]. Paz [31] presented the concept of prime MIN and developed a theory of decomposition into prime factors for certain class of MINs. Tong et al. [35] proved that different BPNs have different channel graphs.

As is commonly agreed that the understanding or gaining insight into a problem is linked more often than not to the discovery of hidden structures or the imposition of new ones, we aim to deepen the understanding of the uniformity of BPN structure through revealing various characteristic objects associated with a BPN structure. In Section 2, we will present some characterizations of the whole class of bit permutation networks, generalizing the results of Bermond et al. and those of Hotzel. In Section 3 using the powerful tool of layered cross product we provide a clear description of the structure of channel graph for any given BPN, from which we derive again the result of Tong et al. Further, a characterization of BPN in terms of its channel graph is given which tells us that the BPN family just consists of those MINs with a uniform channel graph structure. In section 3, we also present a simpler proof for the basic theorem in the decomposition theory of Paz [31] and we show that BPN is just a special family of MINs which can be decomposed into layered cross products of prime factors. An important consequence of this work is that we can thus transforms a problem of checking equivalence between two MINs into a problem of decomposing each of them into prime factors and comparing the multiplicities of each prime factor.

We note that Hotzel [18] made a conjecture that the Omega network is the most symmetric one among a certain range of MINs, namely the Omega network is distinguished from many others because it has a biggest automorphism group. To prove this conjecture, a natural approach is to find some properties which can both characterize the Omega network and be deduced from the symmetric properties asserted in the conjecture. Thus, by giving several different characterizations of BPNs, we also intend to prepare for proving the conjecture of Hotzel and getting any possible corresponding result for a larger class of BPN.

2 Component Characterizations

This section will give two characterizations of BPNs in terms of some properties of its components, generalizing the work in [4, 5, 6, 18]. Note that these characterizations will utilize merely graph theoretical properties, rather than relying on any specific vertex labeling.

Let π be an equivalence relation on the set $\mathsf{E} = \{E_1, E_2, \cdots, E_{m-1}\}$. We use the symbol $[E_i]$ to denote the equivalence class containing E_i. The equivalence classes under relation π induce a partition of E. With a little abuse of notation, we also use π for this partition. For any $1 \leq i \leq j \leq m-1$, let $\pi_{i,j}$ denote the number of different equivalence classes appearing in $\{[E_i], [E_{i+1}], \cdots, [E_{j-1}]\}$. G is said to satisfy the $P_\pi(i,j)$ property if $G_{i,j}$ contains exactly $d^{n-\pi_{i,j}}$ components, and is said to satisfy the $P_\pi(*,*)$ property if it satisfies property $P_\pi(i,j)$ for all $1 \leq i \leq j \leq m$. When $m = n+1$ and there are exactly n equivalence classes, the $P_\pi(i,j)$ and $P_\pi(*,*)$ properties are reduced to the $P(i,j)$ and $P(*,*)$ properties introduced by Bermond et al., respectively.

G is referred to as having property $R(i,j)$ provided that each component of $G_{i,j}$ has size $d^{\tau_{i,j}}$ for some integer $\tau_{i,j}$. Similarly, we also use the shorthand notation $R(*,*)$ for the collection of all $R(i,j)$'s. When $\tau_{i,j}$ coincides with $\pi_{i,j}$ for some partition π of E, we say that G fulfills the $R_\pi(i,j)$ property, etc.. We remark that the $R(*,*)$ property reflects some sort of homogeneous property of the BPN topology, namely the components in the same full subdigraph having the same size.

Let $u \in V_i, 1 \leq i \leq j \leq m$, and $N(u,j) = \{v \mid v \in V_j$, there is a path leading from u to $v\}$. We say that G satisfies the $Q(i,j)$ property if for any two vertices $u, v \in V_i$, we have either $N(u,j) = N(v,j)$ or $N(u,j) \cap N(v,j) = \emptyset$. Correspondingly, for each $v \in V_j$, we can define $N^-(v,i) = \{u \mid u \in V_i$, there is a path leading from u to $v\}$. One can check that the $Q(i,j)$ property can also be formulated as for any two vertices $u, v \in V_j$, it holds either $N^-(u,j) = N^-(v,j)$ or $N^-(u,j) \cap N^-(v,j) = \emptyset$. G is said to have the $Q(*,*)$ property if it satisfies $Q(i,j)$ property for each feasible ordered pair (i,j). Notice that $Q(*,*)$ property has intimate connection with the appearance of iterated line digraph structure [40] and is also somewhat similar to the modified buddy property introduced by Hotzel in his work to give alternative characterizations of the Ω-equivalent class [18].

Let us present an easy observation on the topological structure of the bit permutation networks.

Theorem 2.1 *Every bit permutation network G satisfies properties $Q(*,*)$, $R_\pi(*,*)$ and $P_\pi(*,*)$ for some partition π of E into no more than n nonempty parts.*

Proof: By virtue of Theorem 1.1, the vertices of G can be labeled so that V_i is bit-k_i connected to V_{i+1} for any i, $1 \leq i < m$. Define π to be the equivalence relation on E such that E_i and E_j are equivalent if and only if $k_i = k_j$. For any $1 \leq i < j \leq m$, let $\mathsf{A}_{i,j} = \{k_i, k_{i+1}, \cdots, k_{j-1}\}$. It is easy to see that for any $u \in V_i$, $N(u,j)$ is a bit-$\mathsf{A}_{i,j}$ group whose l^{th} bit for l outside of $\mathsf{A}_{i,j}$ has value the same as that of u. Further, we can find that for each component of $G_{i,j}$, say K, the sets of position labels of $K \cap V_l$, $l = i, \cdots, j$, are the same bit-$\mathsf{A}_{i,j}$ group. These observations give the result. $\square$

The global properties $Q(*,*)$, together with the $P_\pi(*,*)$ properties or the $R_\pi(*,*)$ properties, turn out to distinguish BPN from other topologies. To illustrate it, we still need some lemmas.

Let $1 \leq i \leq j \leq m$ and K be any subdigraph of G. We wright $K_{i,j}$ for the subdigraph of K obtained by removing all vertices of K outside of $\cup_{t=i}^{j} V_t$.

Lemma 2.1 *For an MIN digraph G, if the component number of $G_{i,l}$ is equal to that of $G_{i+1,l}$, then the component number of $G_{i,k}$ is equal to that of $G_{i+1,k}$ for all $k \geq l$.*

Proof: $G_{i,l}$ ($G_{i,k}$) and $G_{i+1,l}$ ($G_{i+1,k}$) have the same number of components if and only if each vertex in V_i has all its out-neighbors in the same component of $G_{i+1,l}$ ($G_{i+1,k}$). Notice that any component of $G_{i,l}$ entirely falls in some component of $G_{i,k}$. Thus the result follows. □

We say that an MIN has the *full access* property if for each pair of vertices, one in its first stage and the other in its last stage, there is a path connecting them. The forthcoming lemma tells us that the $Q(*,*)$ property is just the full access property for each component of full subdigraphs of G.

Lemma 2.2 *Let $1 \leq i \leq j \leq m$. G satisfies the $Q(i,j)$ property if and only if each component K of $G_{i,j}$ possesses the full access property.*

Proof: The sufficiency part is easy to see and we only consider the necessity part.

Since G is of degree d and has property $Q(i,j)$, we know that there are partitions

$$V_i = \cup_{t=1}^{r} \Upsilon_t, V_j = \cup_{t=1}^{r} \Psi_t,$$

such that for each vertex $u \in \Upsilon_t$ we have $N(u,j) = \Psi_t$ (Correspondingly we have $N^-(v,i) = \Upsilon_t$ for each $v \in \Psi_t$) We will prove that $G_{i,j}$ has exactly r components, each component having a Υ_t and Ψ_t as its first stage vertices and last stage vertices, respectively. It is easy to see that the lemma follows from this statement.

Observe that for each vertex x of $G_{i,j}$, we must have $N^-(x,i) \neq \emptyset$. Thus by taking a vertex $u \in N^-(x,i)$ we find that $N(x,j) \subseteq N(u,j)$. This says that each vertex of $G_{i,j}$ corresponds to a unique set Ψ_t such that $N(x,j) \subseteq \Psi_t$. Clearly the vertices corresponding to the same Ψ_t, denoted by S_t, are in the same component.

It remains to show that there is no arc between S_t and S_l if $t \neq l$. Assume otherwise that there is $xy \in E(K)$ with $x \in S_t$ and $y \in S_l$. Then we have $N(x,j) \supseteq N(y,j)$ and thus x should correspond to the set Ψ_l. A contradiction. □

We use the notation G^T for the digraph obtained from G by reversing the direction of all arcs of it. Observe that G satisfies the $Q(i,j)$ property if and only if G^T satisfies the $Q(m+1-j, m+1-i)$ property. The next lemma provides a local picture for BPN and will be the key for getting our characterizations.

Lemma 2.3 *Let $1 \leq i < j \leq m$, and π be a partition of E. Assume that G satisfies the properties $P_\pi(i,j)$, $P_\pi(i+1,j)$, $P_\pi(i,j+1)$, $P_\pi(i+1,j+1)$ and $Q(i,j)$. Then*

(1) if $\pi_{i,j} = \pi_{i+1,j}$, all successors of any vertex in V_i belong to one component of $G_{i+1,j}$; if $\pi_{i,j} = \pi_{i,j-1}$, all predecessors of any vertex in V_j belong to one component of $G_{i,j-1}$;

(2) if $\pi_{i,j} > \pi_{i+1,j}$*, then* $\pi_{i,j} = \pi_{i+1,j} + 1$*. Moreover, the successors of any vertex in* V_i *belong to* d *different components of* $G_{i+1,j}$ *and each component of* $G_{i,j}$ *comprises* d *components of* $G_{i+1,j}$*.*

(3)if $\pi_{i+1,j+1} > \pi_{i+1,j}$*, then* $\pi_{i+1,j+1} = \pi_{i+1,j} + 1$*. Moreover,we have that the predecessors of any vertex in* V_{j+1} *belong to* d *different components of* $G_{i+1,j}$ *and each component of* $G_{i+1,j+1}$ *comprises* d *components of* $G_{i+1,j}$*;*

(4) If $\pi_{i,j} = \pi_{i+1,j+1} = \pi_{i,j+1} = \pi_{i+1,j} + 1$*, the components of* $G_{i+1,j}$ *can be divided into groups such that each group consists of* d *components and corresponding to each group, say* $K = K_1 \cup \cdots \cup K_d$*, where* $K_1, \cdots, K_d$ *are components of* $G_{i+1,j}$*, there are a component* U *of* $G_{i,j}$ *and a component* V *of* $G_{i+1,j+1}$ *with* $U_{i+1,j} = V_{i+1,j} = K$*.*

Proof: (1) This is obvious as $\pi_{i,j} = \pi_{i+1,j}$ ($\pi_{i,j} = \pi_{i,j-1}$) implies that $G_{i,j}$ and $G_{i+1,j}$ ($G_{i,j-1}$) have the same number of components.

(2) Recalling the definition of properties $P_\pi(i,j)$ and $P_\pi(i+1,j)$, $\pi_{i,j} > \pi_{i+1,j}$ means $G_{i,j}$ has $d^{n-\pi_{i,j}}$ components and $G_{i+1,j}$ has $d^{n-\pi_{i+1,j}} \geq d^{n-\pi_{i,j}+1}$ components. Let K be an arbitrary component of $G_{i,j}$ whose subdigraph $K_{i+1,j}$ comprises, say k, components of $G_{i+1,j}$. In view of Lemma 2.2, the $Q(i,j)$ property guarantees that each vertex x in $V_i \cap K$ has access to all vertices in $V_j \cap K$. We then deduce that every component of $K(i+1,j)$, which is a component of $G(i+1,j)$ too, must contain some out-neighbor of x. Consequently, we get that $k \leq d$. Since the above argument applies to all components of $G_{i,j}$, by comparing the component numbers of $G_{i,j}$ and $G_{i+1,j}$, we know that each component of $G_{i,j}$ contains exactly d components of $G_{i+1,j}$. In particular, we see the d successors of x spread through exactly d components of $G(i+1,j)$. This finishes the proof of the claim.

(3) To get the claim in this case, it suffices to consider the digraph G^T and quote the preceding result.

(4) According to what we have just obtained above, we have two ways of partitioning the components of $G_{i+1,j}$ into groups each of size d, one satisfying that each group is in the same component of $G_{i,j}$ and the other satisfying that each group is in the same component of $G_{i+1,j+1}$. We point out that this two partitions are in fact the same, from which the theorem follows. Assume otherwise, there are components K_1, K_2 of $G_{i+1,j}$ which are in different components of $G_{i+1,j+1}$, say U and V, but in the same component of $G_{i,j}$. This shows that there is a component of $G_{i,j+1}$ which contains both U and V. Observe that each component of $G_{i,j+1}$ must contain some component of $G_{i+1,j+1}$. Hence, $G_{i+1,j+1}$ has more components than $G_{i,j+1}$, contradicting the assumption that $\pi_{i+1,j+1} = \pi_{i,j+1}$. □

By carefully checking the proof of Lemma 2.3, we find out that the $R(*,*)$ property can play the role of the $P_\pi(*,*)$ property in getting the result. For our later use, we only list the following result, whose proof is similar to that of Lemma 2.3 and is left to the reader.

Lemma 2.4 *Let* $1 \leq i < j \leq m$*. Assume that* G *satisfies the properties* $R(i,j)$*,* $R(i+1,j)$*,* $R(i,j+1)$*,* $R(i+1,j+1)$ *with corresponding component size* $d^{\tau_{i,j}}$*,* $d^{\tau_{i+1,j}}$*,* $d^{\tau_{i,j+1}}$*, and* $d^{\tau_{i+1,j+1}}$*. If* G *has further the* $Q(i,j)$ *property, then the quantities* $\tau_{i,j} - \tau_{i+1,j}$ *and* $\tau_{i,j} - \tau_{i,j-1}$ *can only take values in* $\{0,1\}$*.*

After all the above preparations, we arrive at two characterizations for BPN now.

Theorem 2.2 *G is a bit permutation network if and only if there exists a partition π of E such that G satisfies properties $P_\pi(*,*)$ and $Q(*,*)$.*

Proof: The "only if" part is just Theorem 2.1.

We assume henceforth that G satisfies both the $P_\pi(*,*)$ property and the $Q(*,*)$ property and provide below the argument for the "if" part.

To prove the theorem, we need to construct a labeling of G, namely assign a d-nary n-tuple $f(v)$ as the position label for each vertex v of G, such that the adjacency relation of G is compatible with its labeling according to the definition of BPN.

Choose a bijection from $Z_d^{\pi_{1,m}}$ to the $d^{\pi_{1,m}}$ components of G. For any vertex in the component corresponding to the element $x \in Z_d^{\pi_{1,m}}$, let $f(v)$ has x as its last $(n-\pi_{1,m})$-bit substring.

We continue to assign value bit by bit for the first $\pi_{1,m}$ bits of the label of each vertex.

Enumerate the $\pi_{1,m}$ equivalent classes in E as $\mathsf{E}_1, \mathsf{E}_2, \cdots, \mathsf{E}_{\pi_{1,m}}$. For any $j \le \pi_{1,m}$, assume that $\mathsf{E}_j = \{E_{j_1}, E_{j_2}, \cdots, E_{j_s}\}$, where $j_1 < j_2 < \cdots < j_s$, and $s = s(j)$ is determined by j. Write $j_0 = 0$ and $j_{s+1} = m$. There is a natural partition of $V(G)$ into $s+1$ parts associated with E_j, where the p^{th} part consists of vertices in stages $j_p + 1$ through j_{p+1}, $p = 0, 1, \cdots, s$. We will give labels to each part of vertices separately. Notice that

$$\begin{array}{ll} \pi_{j_0+1,j_1} + 1 = \pi_{j_0+1,j_1+1}, & \pi_{j_1+1,j_2} + 1 = \pi_{j_1+1,j_2+1}, \\ \cdots, & \pi_{j_{s-1}+1,j_s} + 1 = \pi_{j_{s-1}+1,j_s+1}; \end{array} \qquad (1)$$

$$\begin{array}{ll} \pi_{j_s,j_{s+1}} = \pi_{j_s+1,j_{s+1}} + 1, & \pi_{j_{s-1},j_s} = \pi_{j_{s-1}+1,j_s} + 1, \\ \cdots, & \pi_{j_1,j_2} = \pi_{j_1+1,j_2} + 1. \end{array} \qquad (2)$$

Let $x_p = \pi_{j_p+1,j_{p+1}}$. From Lemma 2.3 we know that the d^{n-x_p} components of $G_{j_p+1,j_{p+1}}$ can be partitioned into d^{n-x_p-1} groups of size d such that each group is in the same component of $G_{j_p+1,j_{p+1}+1}$ (except the case $p=s$) and in the same component of $G_{j_p,j_{p+1}}$ (except the case $p=0$). This is depicted in Fig. 1 below, where a small box represents a component of $G_{j_p+1,j_{p+1}}$ and a large box represents a component of $G_{j_p,j_{p+1}+1}$.

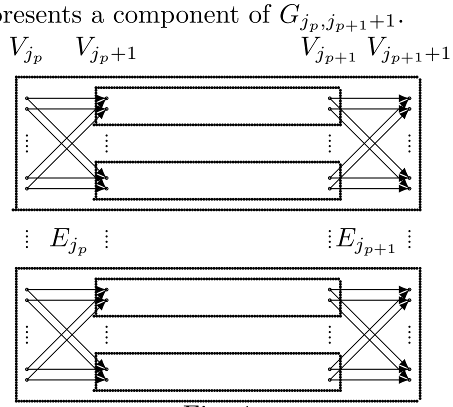


Fig. 1

Label each group of d components of $G_{j_p+1,j_{p+1}}$ from 0 through $d-1$ in an arbitrary order, and use this label as the p^{th} bit of all vertices in the corresponding components. This then finishes the labeling procedure for the vertices of G.

We are in the position to verify that this labeling does have the properties presented in the definition of BPN. To this end, we have to prove the following claims:

(i) The labeling scheme gives every vertex of G a d-nary n-bit string;

(ii) Different vertices in the same stage receive different labels;

(iii) For any $i \in \{1, 2, \cdots, m-1\}$, V_i is bit-$\theta(i)$ connected to V_{i+1}, where θ is the mapping from $\{1, 2, \cdots, m\}$ to $\pi_{1,m}$ such that $\mathsf{E}_{\theta(i)} = [E_i]$.

Claim (i) is obvious.

If Claim (ii) does not hold, then for some $l \in \{1, 2, \cdots, m\}$, there are two different vertices $u, v \in V_l$ whose labels are the same. Note that u and v must belong to the same component of $G = G_{1,m}$, for otherwise the last $n - \pi_{1,m}$ bits of their labels would be different. Choose i and k with $1 \leq i < k \leq m$ and $k - i$ being as small as possible such that u and v are in the same component of $G_{i,k}$. By the choice of i and k, we have either u and v are in two different components of $G_{i+1,k}$ or u and v are in two different components of $G_{i,k-1}$. We deal with the former case here by using relations (2) while we omit the similar analysis for the latter case which needs equations (1). Suppose that $\theta(i) = j$ and $\mathsf{E}_j = \{E_{j_1}, E_{j_2}, \cdots, E_{j_s}\}$, where $j_1 < j_2 < \cdots < j_s$. We intend to prove that the j^{th} bits of u and v are different. If $i = j_s$, Lemma 2.3, joined to the labeling process, clearly gives our claim. Otherwise, let $i = j_p$ with $p < s$. As $\pi_{j_p, j_{p+1}+1} = \pi_{j_p+1, j_{p+1}+1}$, it follows from Lemma 2.1 that $k \leq j_{p+1}$. Thus we are left to illustrate that u and v are in the same component of $G_{j_p, j_{p+1}}$ but in two different components of $G_{j_p+1, j_{p+1}}$ respectively. The first assertion also comes from Lemma 2.1. The second one can be seen from Lemma 2.3 and equations (2) as follows. By Lemma 2.3 there is a vertex $w \in V_i$ such that the out-neighbors of w intersect with each of the d components of $G_{i+1,k}$ which are in the same component of $G_{i,k}$ containing both u and v. In particular, we see that u and v are in different components of $G_{i+1,k}$, say K_1 and K_2, such that $wx_1, wx_2 \in E(G)$ for $x_1 \in K_1$ and $x_2 \in K_2$. But Lemma 2.3 tells us that there is exactly one edge from w to each of the d component of $G_{j_p+1, j_{p+1}}$ which are contained in a common component of $G_{j_p, j_{p+1}}$. This shows that K_1 and K_2 are in different components of $G_{j_p+1, j_{p+1}}$, and thus so are u and v, as desired.

Finally we are going to prove Claim (iii). For any $i \in \{1, 2, \cdots, m-1\}$, consider the subdigraph $G_{i,i+1}$, which is the disjoint union of d^{n-1} complete bipartite digraphs, each being of size $d \times d$. Let K be one of these bipartite digraphs. We only need to show that all vertices of K have a common value for each of their bits except the j^{th} one, where $j = \theta(i)$. In fact, for any $t \neq j$, suppose $\mathsf{E}_t = \{E_{t_1}, E_{t_2}, \cdots, E_{t_s}\}$, where $t_1 < t_2 < \cdots < t_s$ and $s = s(t)$. As $j \notin \{t_1, t_2, \cdots, t_s\}$, it follows that the V_i and V_{i+1} belong to the same part in the partitioning of $V(G)$ associated with E_t. Moreover, because that K is connected, K is contained in a component of the full subdigraph corresponding to that part of vertices and hence its vertices receive a common t^{th} bit in the labeling process described above. □

We note that the above theorem is a natural generalization of Theorem 1.3, since properties $P(*,*)$ and $Q(*,*)$ turn out to be equivalent for an MIN with banyan property.

Theorem 2.3 *An MIN digraph G is a bit permutation network if and only if it satisfies the properties $R(*,*)$ and $Q(*,*)$.*

Proof: In view of Theorem 2.1, we can restrict attention to the proof of the sufficiency.

For any i and j, $1 \le i \le j \le m$, let $\tau_{i,j}$ be the nonnegative integer such that each component of $G_{i,j}$ has size $d^{\tau_{i,j}}$ and thus the full subdigraph $G_{i,j}$ has exactly $d^{n-\tau_{i,j}}$ components.

Let $1 \le i < j \le m-1$. Define E_i to be adjacent to E_j if $\tau_{i,j} = \tau_{i,j+1} = \tau_{i+1,j+1} = \tau_{i+1,j} + 1$. Notice that if E_i is adjacent to E_j, then $\tau_{i,j+1} = \tau_{i+1,j+1}$, which in turn gives $\tau_{i,l} = \tau_{i+1,l}$ for all $l \ge j+1$ as a result of Lemma 2.1. Thus if there is k such that E_i is adjacent to E_k, then $\tau_{i,k} = \tau_{i+1,k} + 1$ and henceforth $k < j+1$. Clearly this says that any arc-stage of G is adjacent to at most one arc-stage. Similarly, we also can establish that any arc-stage of G can be adjacent from at most one arc-stage. This discussion enables us to endow the arc-stages of G with a partial order such that E_i covers E_j if and only if E_j is adjacent to E_i. Note that the partially ordered set thus obtained must be a disjoint sum of a set of chains. Denote by π this partition of the arc-stages of G into chains and use $[E_i]$ for the chain that E_i lies in.

To complete the proof, we appeal to Theorem 2.2 and thus need only to show that G satisfies property $P_\pi(*,*)$, or equivalently to show that $\tau_{i,j} = \pi_{i,j}$. This is done by induction on $j-i$. It is trivial for $j-i=0$ and it follows from property $Q(i,i+1)$ for $j-i=1$ (Note that we use the fact that G is simple here). Assuming the assertion for $j-i=k-1$, we consider the case when $j-i=k$.

If $\tau_{i,l} > \tau_{i+1,l}$ for all $l \in \{i+1, i+2, \cdots, j\}$, then E_i is not adjacent to any one in the set $\{E_{i+1}, \cdots, E_{j-1}\}$. Therefore, we have $[E_i] \notin \{[E_{i+1}], \cdots, [E_{j-1}]\}$, and consequently $\pi_{i,i+k} = \pi_{i+1,i+k} + 1$. Using Lemma 2.4, we see that it follows from the induction hypothesis now that $\pi_{i,i+k} = \pi_{i+1,i+k} + 1 = \tau_{i+1,i+k} + 1 = \tau_{i,i+k}$, as desired.

Otherwise, we get from Lemma 2.1 that $\tau_{i,j} = \tau_{i+1,j}$. But clearly $\tau_{i,i+1} = \tau_{i+1,i+1} + 1$. Noting Lemma 2.4 in addition, we obtain that there is an $l \in \{i+2, \cdots, j\}$, such that $\tau_{i,l} = \tau_{i+1,l}$ and $\tau_{i,l-1} - \tau_{i+1,l-1} = 1$. We use Lemma 2.4 again to deduce that it holds either $\tau_{i+1,l-1} = \tau_{i+1,l}$ or $\tau_{i+1,l-1} + 1 = \tau_{i+1,l}$. But the former case is impossible as it will give $\tau_{i,l-1} - 1 = \tau_{i+1,l-1} = \tau_{i+1,l} = \tau_{i,l}$, which contracts Lemma 2.4. Therefore, we see that E_i is adjacent to E_{l-1} and thus $\pi_{i,j} = \pi_{i+1,j}$. Using our induction hypothesis, we get $\tau_{i,j} = \pi_{i,j}$ immediately. This finishes the proof. □

We point out that the local structure reflected in Lemma 2.3 is an essential characteristic of bit permutation networks. Indeed, the proof of Theorem 2.2 shows that any MIN digraph having this kind of structure and satisfying the $Q(*,*)$ property is a bit permutation network. We do not formulate this observation as a theorem because this leads to a long-winded characterization.

Nevertheless, this observation may be helpful to produce some other variants of the characterization of BPN suiting specific application.

Theorem 2.2 and Theorem 2.3 shows that every bit permutation network corresponds to a characteristic partition of its arc-stages. In fact, they both imply that such a partition uniquely determines the structure of the MIN digraph and thus it is meaningful to refer to a BPN with associated arc-stage partition π as $G(d, n, m; \pi)$, or simply $G(\pi)$. Moreover, if two bit permutation networks, say $G(d, n, m; \pi)$ and $G(d, n, m; \pi_1)$, correspond to different partitions π and π_1, then they can not be isomorphic, since Theorem 2.2 indicates that there is a stage interval on which the component numbers of $G(d, n, m; \pi)$ and $G(d, n, m; \pi_1)$ are not equal. This then tells us that there is a one to one correspondence between all BPN and all partitions of E. Let us state this observation formally as a theorem, which is actually a reformulation of Theorem 1.1 in terms of labeling-independent language.

Theorem 2.4 *Each bit permutation network can be expressed as $G(d, n, m; \pi)$ for some partition of E and two bit permutation networks $G(d, n, m; \pi)$ and $G(d, n, m; \pi_1)$ are isomorphic if and only if $\pi = \pi_1$. Especially, we have the number of different m-stage BPNs of degree d and stage size d^n is equal to the number of partitions of the set $\{1, 2, \cdots, m-1\}$ into no more than n parts.*

A natural problem now is that given an MIN digraph how can we recognize if it is a BPN and if so how can we discover its characteristic arc-stage partition.

Corollary 2.1 *G is isomorphic to the shuffle-exchange network if and only if either one of the following conditions holds:*

(1) It satisfies the properties $P_\pi(*, *)$ and $Q(*, *)$, where π is a partition of $\{E_1, E_2, \cdots, E_{m-1}\}$ satisfying $\pi_{i,j} = min\{j - i, n\}$, $1 \le i \le j \le m$;

(2) If $m < n + 1$, each component is an Omega equivalent network, and if $m \ge n + 1$, any $n + 1$ consecutive stages induce an Omega equivalent network.

Proof: (1) The forward direction is easy from the definition of the shuffle-exchange network. The other direction comes from the fact that the condition listed uniquely determine the network topology as Theorem 2.1 asserts.

(2) The condition is obviously necessary. To show its sufficiency, we need only to show that if G satisfies the condition then it is a bit permutation network. Define π to be the partition: $E_j \in [E_i]$ if and only if $j \equiv i \ (mod\ n)$. Then it can be easily shown that G satisfies $P_\pi(*, *)$ and $Q(*, *)$. □

We remark that Corollary 2.1 can also be derived by using a generalization of a criterion of Hemminger for iterated line digraph [40].

Corollary 2.2 *$G(d, n, m; \pi)$ is isomorphic to its inverse network (obtained by reversing the direction of all arcs) if and only if π satisfies: $\pi_{i,j} = \pi_{m+1-j,m+1-i}$, $1 \le i \le j \le m$.*

3 Layered Cross Product and Channel Graph

As asserted in the introduction, this is the only section when mentioning an MIN we will not impose on it the implicit assumption that it is of degree d and all its stages have the same size. We say that an MIN is d-nary provided that the in-degrees and out-degrees of it only take values 0, 1 or d. Even and Litman [15] introduced the operation of layered cross product between two MINs with the same number of stages, which is similar to the categorical product (also called Kronecker product due to its close relationship with the Kronecker product of matrices) between two graphs [3].

Definition 3.1 *For any two m-stage MIN digraphs $G_j = (V_1^j, V_2^j, \cdots, V_m^j; E_1^j, E_2^j, \cdots, E_{m-1}^j)$, $j = 1, 2$, the layered cross product (LCP) of them, denoted by $G_1 \times G_2$, is the MIN digraph $G = (V_1, V_2, \cdots, V_m; E_1, E_2, \cdots, E_{m-1})$, where $V_i = V_i^1 \times V_i^2$ and $E_i = E_i^1 \times E_i^2$.*

Let us present two simple facts about LCP here. We say that $(a_1, \cdots, a_{m-1}; b_1, \cdots, b_{m-1})$ is the *stage-degree sequence* of an MIN provided that each of its stage i vertex has out-degree a_i for $1 \leq i \leq m-1$ and each of its stage j vertex has in-degree b_{j-1} for $j = 2, \cdots, m$.

Lemma 3.1 *Let G_1 has stage-degree sequence $(a_1, \cdots, a_{m-1}; b_1, \cdots, b_{m-1})$ and G_2 has stage-degree sequence $(c_1, \cdots, c_{m-1}; d_1, \cdots, d_{m-1})$. Then $G_1 \times G_2$ has stage-degree sequence $(a_1c_1, \cdots, a_{m-1}c_{m-1}; b_1d_1, \cdots, b_{m-1}d_{m-1})$.*

Let G_i, $i = 1, 2, 3$, be three MIN digraphs of the same stage number. We represent the disjoint union of G_1 and G_2 by $G_1 + G_2$. The next trivial result is again stated without proof.

Lemma 3.2 $(G_1 + G_2) \times G_3 = G_1 \times G_3 + G_2 \times G_3$.

A *channel graph* [23] is an acyclic digraph with a source vertex s and a target vertex t such that every vertex lies on a path from s to t. For an MIN digraph G and two of its vertices x and y, define the channel graph $C_G(x, y)$ to be the union of all paths connecting x to y in G. We make the convention that $C_G(x, y)$ is the empty graph in case that there is no path from x to y in G. In general, different MIN digraphs cannot be distinguished by the channel graphs associated with it. For example, all unique path full access (UPFA) networks of the same stage numbers look the same if we only analyze the structure of various channel graphs associated with them; while we know that there are lots of nonequivalent UPFA networks [38]. However, when restricted to BPN, those channel graphs become a kind of characteristic objects, as demonstrated by Tong, Hwang and Chang [35]. In the sequel, we will report the part of our work on characterizing BPNs motivated by their interesting discovery and the investigation of LCP by Even, Litman [15] and Paz [31].

We say that a channel graph C is the *channel graph of* G, or G has C as its channel graph, provided that C is isomorphic with $C_G(x, y)$ for each pair $x \in V_1$ and $y \in V_m$ (We assume here that the MIN in consideration has a unique stage partition.). It is well-known that the layered cross product operation is associative and commutative and thus it is meaningful to refer to the LCP of a set of MINs with the same stage number. We formalize below a simple observation about LCP and channel graph.

Lemma 3.3 *Assume that $\wp$ is a set of MIN digraphs and each member of $\wp$ has its channel graph. Then the LCP of the MIN digraphs in $\wp$ also has a channel graph which is just the LCP of the channel graphs of those in $\wp$.*

Paz considered the inverse of the LCP operation, namely the decomposition of a complex network structure into an LCP of a set of simpler "prime graphs" [31]. In fact, Paz developed a decomposition theory for it and argued that it may have wide applications [31]. We present some concepts introduced by him here in a form meeting our need.

Definition 3.2 *Let $d, m \geq 2$ be two integers. For any two integers $0 \leq i < j \leq m$, $X_{i,j}^{d,m}$ is the d-nary m-stage simple MIN digraph which is uniquely determined by the following requirements:*

(i) *it has one vertex in its stages $1, 2, \cdots, i$ and stages $j+1, \cdots, m$ and two vertices in all other stages;*

(ii) *only its stage i vertex has out-degree d and only its stage j vertex has in-degree d;*

(iii) *only its first stage vertices have in-degree 0 and only its last stage vertices have out-degree 0.*

Paz called the set of digraphs defined above *prime graphs*. For simplicity, we sometimes write $X_{i,j}$ for $X_{i,j}^{d,m}$. Several examples of prime graphs are depicted in Figure 2.

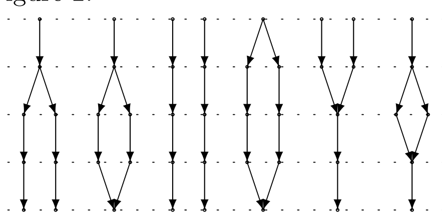


Fig. 2: several prime graphs

Definition 3.3 *[31] The m-layered d-nary cross product family (CPF) is the set of MINs which can be decomposed into the layered cross product of some d-nary m-stage prime graphs.*

CPF digraphs inherit from their prime factors some special symmetrical properties.

Lemma 3.4 *Let G be a CPF digraph. For any pair of first stage vertices (x, y) in the same component of G and any pair of last stage vertices (z, w) in the same component of G, there is an automorphism of G which swaps x and y and swaps z and w.*

Proof: We first observe that the assertion holds for prime graphs. But it is easy to see that if π_i is an automorphism of G_i for $i = 1, 2$, then the mapping $\phi_1 \times \pi_2$ on $V(G_1) \times V(G_2)$ which sends vertex (v_1, v_2) to $(\phi_1(v_1), \phi_2(v_2))$ is an automorphism of $G_1 \times G_2$. This proves the claim. □

The next lemma slightly generalizes the Theorem 5 of [35].

Lemma 3.5 *Each digraph $G \in CPF$ corresponds to a channel graph* $\mathrm{C}(G)$ *which is the channel graph of each component of G.*

Proof: Observe that the components of any prime graph are isomorphic to each other and so we have that the components of any CPF digraph must be identical too. Thus we only need to show that any connected CPF has a channel graph. It is an easy consequence of Lemma 3.4. It also follows from Lemma 3.2 and Lemma 3.3 since the components of prime graphs all have channel graphs. □

Let us recall the basic result in the decomposition theory of Paz [31]. We also include an alternative proof for completeness.

Theorem 3.1 *([31] Theorem 1) Let* $G = \prod_{0\le i<j\le m} (X_{i,j}^{d,m})^{f_{i,j}}$ *be a factorization of a CPF digraph* G*. The multiplicities of the prime factors appeared in this expression, namely the* $f_{i,j}$*'s, are uniquely determined by the topological structure of* G*.*

Proof: We say that a parameter of G is characteristic if it only relies on the topology of G. We aims at proving that all the $f_{i,j}$'s are characteristic.

By Lemma 3.2, we know that G has $d^{f_{0,m}}$ components, which means that $f_{0,m}$ is characteristic.

Let $0 < i < j < m$. Combining Lemma 3.3 and Lemma 3.2, we can write the common channel graph of any component of $G_{i,j}$ as

$$C_{i,j} = \prod_{1\le k<l\le j-i} (X_{k,l}^{d,j-i+1})^{f_{i-1+k,i-1+l}}.$$

From the above expression we can easily determine that the out-degree of the source vertex of $C_{i,j}$ is equal to

$$d^{\sum_{1<l\le j-i} f_{i,i-1+l}},$$

which in turn tells us that

$$d_{i,j} = \sum_{1<l\le j-i} f_{i,i-1+l}$$

is characteristic. The fact that $f_{i,j}$ is characteristic for $0 < i < j < m$ then results from the observation $f_{i,j} = d_{i,j} - d_{i,j-1}$ for $j-1 > i$ and $f_{i,i+1} = d_{i,i+1}$.

Finally, we deduce from Lemma 3.1 that G has stage-degree sequence (a_1, $\cdots$, a_{m-1}; $b_1, \cdots$, b_{m-1}), where $a_i = \prod_{i<j\le m} d^{f_{i,j}}$ and $b_i = \prod_{0\le j<i} d^{f_{j,i}}$ for $i = 1, \cdots, m-1$. By now we get that, for $i = 1, \cdots, m-1$, $\sum_{i<j\le m} f_{i,j}$ and $\sum_{i<j<m} f_{i,j}$, $\sum_{0\le j<i} f_{j,i}$ and $\sum_{0<j<i} f_{j,i}$ are all characteristic. Consequently, we have both $f_{i,m}$ and $f_{0,i}$ are characteristic for each $i = 1, 2, \cdots, m-1$, and thus we are done. □

The above theorem of Paz provides new insight into the problem of checking equivalence among CPF digraphs, namely it can be reduced to finding good decomposition algorithm. The next lemma points out that BPN is a subfamily of the class of CPF digraphs, which then indicates that LCP may be a powerful tool to study the equivalence relation among BPNs. See [9, 31] for some effort in this direction, where they demonstrated that several specific BPNs are CPFs.

Lemma 3.6 *BPN* $\subseteq$ *CPF.*

Proof: Given a BPN, say $G = G(d, n, m, \pi)$, where π is a partition of E into $l \leq n$ nonempty parts, say $\mathsf{E}_1, \mathsf{E}_2, \cdots, \mathsf{E}_l$, we can properly label the vertices of G so that V_i is bit-j connected to V_{i+1} when $E_i \in \mathsf{E}_j$. For any $j \in \{1, 2, \cdots, l\}$ with $\mathsf{E}_j = \{E_{t_1}, E_{t_2}, \cdots, E_{t_{h_j}}\}$, $t_1 < t_2 < \cdots < t_{h_j}$, we set $G_j = X_{0,t_1}^{d,m} \times (\prod_{i=2}^{h_j-1} X_{t_i,t_{i+1}}^{d,m}) \times X_{t_{h_j},m}^{d,m}$. For $j = l+1, \cdots, n$, we define $G_j = X_{0,m}^{d,m}$. A typical G_j for $j \leq l$ together with its LCP decomposition is shown in the following figure.

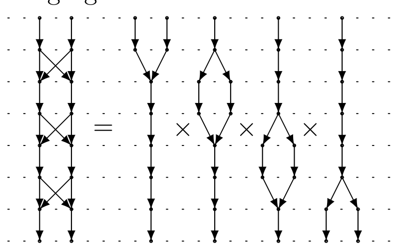


Fig. 3: A typical G_j

For each G_j, $j = 1, 2 \cdots, n$, we can label its vertices with elements from Z_d so that $(x, i)(x, i+1)$ is an arc in $E_i(G_j)$ for each $x \in Z_d$ and $i = 1, 2, \cdots, m-1$. By establishing the bijection from $V(G)$ to $V(G_1)\times V(G_2)\times \cdots \times V(G_n)$ which sends the vertex in G with label $(x_1x_2\cdots x_n, i)$ to an element of $V(G_1)\times V(G_2)\times \cdots \times V(G_n)$ with the component in $V(G_j)$ being the vertex of G_j having label (x_j, i) for each j, we can verify that $G = G_1 \times G_2 \times \cdots \times G_n$. This shows that G is in the cross product family and then finishes the proof. □

Now we arrive at our extension of the main result of [35]. We remark that each member of CPF has a stage-degree sequence as each prime graph does.

Theorem 3.2 *Assume that G and G_1 are two m-layered CPF digraph with the same degree sequence and the same number of first-stage vertices. Then G and G_1 are isomorphic if and only if $\mathrm{C}(G)$ and $\mathrm{C}(G_1)$ are isomorphic.*

Proof: We only need to prove the "if" part. We assume that $G = \prod_{0\leq i<j\leq m} (X_{i,j}^{d,m})^{f_{i,j}}$ and $G_1 = \prod_{0\leq i<j\leq m} (X_{i,j}^{d,m})^{g_{i,j}}$. By examining the proof of Theorem 3.1, we know that $\mathrm{C}(G) = \mathrm{C}(G_1)$ implies that $f_{i,j} = g_{i,j}$ for $0 < i < j < m$. Furthermore, the fact that G and G_1 possess the same degree sequence gives $f_{i,m} = g_{i,m}$ and $f_{0,i} = g_{0,i}$ for $0 < i < m$. The above arguments already tell us that any component of G is isomorphic with any component of G_1. Finally, as G and G_1 have the same number of first-stage vertices, they consist of the same number of components. This then gives the result. □

Observing the regular structure of the channel graphs associated with a BPN, one may wonder that whether such a phenomenon is actually significant enough to indicate the BPN structure. As we shall see immediately, it is indeed the case.

Theorem 3.3 *An m-stage MIN digraph G of degree d and uniform stage size d^n is a bit permutation network if and only if for any $1 \leq i < j \leq n$, there is a d-nary channel graph $\Re_{i,j}$ which is the common channel graph of each component of $G_{i,j}$.*

Proof: The forward direction follows from Lemmas 3.2, 3.3, and 3.6.

To establish the backward direction we make use of Theorem 2.3, which requires us to check the properties $R(*,*)$ and $Q(*,*)$, or $R(*,*)$ properties and the full-access property, in view of Lemma 2.2.

Clearly, for a component of a full subdigraph of G to own a channel graph it must have the full access property. Therefore, it remains to show that G has the $R(*,*)$ properties. Because G has the full access property and is of degree d, the $R(i, i+1)$ property, $i = 1, 2, \cdots, m-1$, of G is obvious.

Take a full subdigraph of G, say $G_{i,j}$, such that $j - i > 1$ and the components of $G_{i+1,j}$ have a common size d^t for some integer t. Our task in the next paragraph is to demonstrate that the components of $G_{i,j}$ also have a common size which is either d^t or d^{t+1}. Note that this argument to be presented enables us to finish the proof of the theorem by induction on $j - i$.

Let $C_1, C_2 \cdots, C_{d^{n-t}}$ be the components of $G_{i+1,j}$. For any vertex $x \in V_i$, we refer to the set of components of $G_{i+1,j}$ which intersect with the out-neighbors of x as $\Gamma(x)$. If $\Gamma(x)$ consists of a single components, say C, of $G_{i+1,j}$, then from the full access property of G we deduce that x is in a component of $G_{i,j}$ of size the same as that of C, namely d^t. Consider the other case that $\Gamma(x) = \{C_{p_1}, C_{p_2}, \cdots, C_{p_s}\}$ where $s > 1$. Note that for any $1 \le k \le s$, $I_k = \{y : \; xy \in E(G), y \in C_{p_k}\}$ must have cardinality less than d, as G is of degree d and $s > 1$. Further recall that each vertex in I_k has full access to the vertices of $V_j \cap C_{p_k}$. Now, by choosing a vertex $y \in V_j \cap C_{p_k}$ and looking at the channel graph $C_G(x, y)$, which should be d-nary as is assumed, we find that I_k is a singleton set for each k and henceforth $s = d$ as x has out-degree d. As a result, we know that there are d^{t+1} vertices in V_j to which x has access. Using the full access property of G again, we get that x is in a component of $G_{i+1,j}$ of size d^{t+1}. After the analysis of the above two cases, we come to the conclusion that each component of $G_{i,j}$ is of size either d^t or d^{t+1}. We want to show that these two cases cannot happen simultaneously which will end the proof of the required result. Assume the contrary, we have $x, x_1 \in V_i$ with $\Gamma(x) = \{C\}$ and $\Gamma(x_1) = \{C_{p_1}, \cdots, C_{p_d}\}$. Pick a vertex $y \in V_j \cap C$ and a vertex $y_1 \in V_j \cap C_{p_1}$. Considering the full access property of G, we find that x has out-degree d in $C_G(x, y)$ and out-degree 1 in $C_G(x, y_1)$, which means that $C_G(x, y)$ cannot be isomorphic to $C_G(x, y_1)$, contradicting the assumption of the theorem. □

Recalling Theorem 3.2, we know that all UPFA networks with the only exception of the Omega network are not BPNs. Further notice that the channel graphs between two vertices of a UPFA digraph appear rather regularly in the sense that they are always either an empty graph or a path. Then does this contradict with Theorem 3.3? We comment that there is not any contradiction at all, as the uniformity assumption made in Theorem 3.3 is imposed on the channel graphs of components of the full subdigraphs.

The following theorem gives us insight into the BPN family through revealing its LCP decomposition structure.

Theorem 3.4 *An MIN digraph of degree d and size d^n is a BPN if and only if it is a CPF.*

Proof: The necessity part is just Lemma 3.6. The sufficiency part follows from Lemma 3.1, Lemma 3.5, and Theorem 3.3. □

According to Theorem 3.1, a CPF is uniquely determined by the multiplicities of its prime factors. Meanwhile, Lemma 3.1 suggests a method to add suitable restrictions on these multiplicities to guarantee a CPF of degree d, namely a BPN. These considerations motivate us to list the following lemma, which together with Theorem 2.4 and Lemma 3.6 will lead to another proof (understanding) of Theorem 3.4 and from whose proof we can see clearly the connection between the characteristic partition of a BPN and its multiplicity parameters in the LCP decomposition. We mention that in Lemma 3.7 the condition (i) means the corresponding BPN has size d^n while the condition (ii) means that the BPN is of degree d.

Lemma 3.7 *Let $\mathcal{F}$ be the set of nonnegative integer vectors $(f_{i,j})_{0\leq i<j\leq m}$ such that*

(i) $\sum_{0<i\leq m} f_{0,i} = n$;

(ii) *for each* $0 < i < m$, $\sum_{i<j\leq m} f_{i,j} = \sum_{0\leq j<i} f_{j,i} = 1$.

Then the cardinality of $\mathcal{F}$ is equal to the number of partitions of $m-1$ objects into no more than n parts.

Proof: By virtue of condition (ii), there is a partial order on $\{1, 2, \cdots, m-1\}$ specified by the rule that x covers y if and only if there is z such that $f_{x,z}$ $= f_{z,y} = 1$. In fact, condition (ii) says that such a partial ordered set is the disjoint union of several chains. But condition (i) means that there can be no more than n chains. The correspondence between the set $\mathcal{F}$ and the partial ordered set gives the result. □

Theorem 3.5 *neighbour-swapping invariant property*

Corollary 3.1 *(Theorem 13 of [9]) There are totally $n\downarrow$ nonisomorphic $(2n+1)$-stage CPF digraph G of degree d and size d^n such that both $G_{1,n+1}$ and $G_{n+1,2n+1}$ are the Omega network.*

Proof: By Theorem 3.4 and our assumption on G, we know that each G thus specified is in fact a BPN and hence can be represented as $G(d, n, m; \pi)$ for a partition π of E into no more than n parts. But $G_{1,n+1} = G_{n+1,2n+1}$ is the Omega network means that both the first n edge stages and the last n edge stages are partitioned into n equivalent classes. Consequently, the number required is the number of bijections from the first n edge stages to the last n edge stages (an equivalence class under relation π consists of an edge stage in the first n edge stages and its image under the bijection.). □

4 Some Remarks

In some sense, it is a global property for a digraph to be a BPN and some confusion in the literature arose because the authors assumed carelessly that a digraph having a local structure of BPN everywhere must be a BPN. In fact we agree to the concluding remark of [22] that the topological structure within MINs is much more complicated than what is mentioned in the earlier literature.

Note that our definition of bit permutation networks is slightly different from that of Chang, Hwang and Tong. We have put emphasis on the existence of a proper labeling of all the vertices of G. This may make it clearer that the bit permutation class of networks includes all networks having a definite topology. Indeed, the permutation σ's appearing in the definition depend on the choice of a suitable labeling for the vertices. If G is a digraph with no vertex labels or the vertices are improperly labeled, then we could not readily find the permutations fitting into the definition. In such case we can not tell whether a given digraph is a bit permutation network or not. This is explained in the following figure.

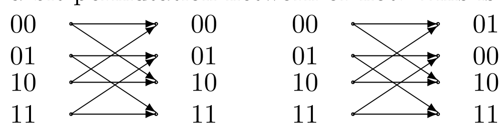


Fig. 4(a) Fig. 4(b)

Clearly the two digraphs displayed above are topologically equivalent. Figure 4(a) is a bit permutation network with $\sigma = (0, 2)$, but there is no permutation compatible with the connection scheme and the labeling depicted by Fig. 4(b) (Look at the vertex $(0, 0)$ in the first stage for this claim).

Let $l > m$ be two positive integers. For any l-stage MIN digraph, its first m stages and the last $l - m$ stages induce an m-stage MIN digraph G_1 and an $(l - m)$-stage MIN digraph G_2 respectively. We can regard G as being obtained from a concatenation of G_1 and G_2 through overlapping the last stage of G_1 and the first stage of G_2. In view of this, we see that the topology of G is determined by the topology of G_1 and G_2 and, in addition, by the overlapping relation between G_1 and G_2. In fact, the associated one to one mapping from the last stage of G_1 to the first stage of G_2, called concatenating mapping in [22], will affect a lot the topology of the concatenation of G_1 and G_2.

Hu, Shen, and Yang [22] have studied a special class of networks, namely, the $\Delta \oplus \Delta'$ networks, which are the set of networks obtained by concatenating two Omega equivalent networks. When confined to bit permutation networks, however, we should be cautious with the mappings. Since the permutation σ's in a bit permutation network are label-dependent, it is not difficult to imagine that the combination of two bit permutation networks with an arbitrary mapping is not necessarily a bit permutation networks. That is, using our notation, the concatenation of $G(d, n, m; \sigma_1, \sigma_2, \cdots, \sigma_{m-1})$ and $G(d, n, l - m + 1); \sigma_m, \sigma_{m+1}, \cdots, \sigma_{l-1})$ is not necessarily isomorphic to $G(d, n, l; \sigma_1, \sigma_2, \cdots, \sigma_{l-1})$. This suggests that the asserted improvement of a result in [22] by Chang et al. [12] does not hold. Calamoneri and Massini [9] also tried to improve the result of Hu et al.. However, they have implicitly assumed that all concatenations of two Omega-equivalent networks must be a CPF. But we have shown in Section 3 that this assumption is just equivalent to assuming that all such concatenated digraphs are BPNs, which is not true in general.

It is known that the banyan property together with the buddy property (or the $P(i, i+1)$ property) does not ensure an Omega equivalent network [1, 4]. In fact, it does not ensure even a bit permutation network, as a BPN with banyan property must be an Omega network. Therefore the proof of Corollary 7 in [12] is not correct. In fact the claim of the corollary, namely any $(n + 1)$-stage network satisfying the banyan property and $P(i, i + 1)$ and $P(1, *)$ property is in the Omega equivalent class, does not hold. For a counterexample, let H_n,

$n \geq 3$, be the digraph obtained from $G(2, n, n+1; n, n-1, \cdots, 1)$ by exchanging the successors of $(0, \cdots, 0, 0, 0, 1)$ and $(0, \cdots, 0, 1, 0, 1)$ in V_n, then H_n is not a bit permutation network and thus is not in the Omega equivalent class (use Theorem 2.2 and verify that H does not satisfy $Q(n-1, n+1)$.) The figure below is a drawing of H_3 and one can look at the two vertices with a circle to know that it does not have the $Q(2, 4)$ property.

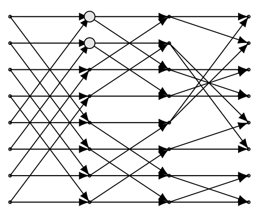


Fig. 5: H_3

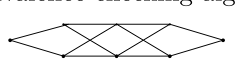
Generally, one can check that the local structure of H_n is almost the same as that of a BPN. In fact, for $n > 3$, H_n satisfies all the properties $P(i, j), Q(i, j)$ and $R(i, j)$ for $1 \leq i \leq j \leq n+1$, $(i, j) \neq (n-1, n+1)$ (but does violate $P(n-1, n+1), Q(n-1, n+1), R(n-1, n+1)$.) The result for H_3 is a bit different from that of H_n, $n > 3$. This phenomenon warns that we have to be careful in asserting that a digraph is a BPN even when we have collected many local evidence.

Calamoneri and Massini [9] used the criterion established by Bermond et al. [6] to verify their layered cross product representation of Ω-equivalent classes. As our main results generalize the criterion of Bermond et al., we can get the layered cross product representation for general BPN parallel to the approach in [9].

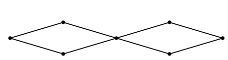
Hotzel [18] claimed that the Omega equivalent class is largely determined by their automorphisms. This is a rather vague assertion. Can we clarify it by getting more insight into the BPN structure? To what direction can we establish result to clarify this point and support this assertion? What is the counterpart of the conjecture of Hotzel for BPN or more generally for CPF?

How to extend the argument for Theorem 3.3 to get a characterization of CPF digraphs?

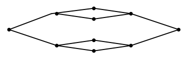
Since there have been much work on decomposing graphs into prime factors in various sense (see [25]), it is possible that some existing techniques can be adapted to study the LCP decomposition. By Theorem 3.1 and Theorem 3.4, the work in that direction can contribute to solving problems about BPN, such as understanding the complexity of checking equivalence and devising better equivalence-checking algorithms.



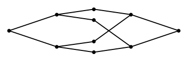
1234



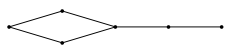
12,,34



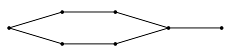
14,,23



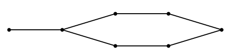
13,,24



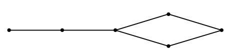
12,,3,,4



13,,2,,4



24,,1,,3



34,,1,,2

References

[1] D.P. Agrawal, Graph theoretical analysis and design of multistage interconnection networks, *IEEE Trans. Comput.* **C-32** (1983) 637-648.

[2] A. Avior, T. Calamoneri, S. Even, A. Litman, A. L. Rosenberg, A tight layout of the Butterfly network, *Theory of Computing Systems* **31(4)** (1998) 475-488.

[3] L. Babai, W. Imrich, Graph products, in : *Handbook of Combinatorics* **II** (Eds., R.L. Graham, M. Grotschel, L. Lovasz) (1995) 1463-1466.

[4] J.C. Bermond, J.M. Fourneau, A. Jean-Marie, Equivalence of multistage interconnection networks, *Info. Proc. Lett.* **26** (1987) 45-50.

[5] J.C. Bermond, J.M. Fourneau, Independent connections: An easy characterization of baseline-equivalent multistage interconnection networks, *Theor. Comput. Sci.* **64** (1989) 191-201.

[6] J.C. Bermond, J.M. Fourneau, A. Jean-Marie, A graph theoretical approach to equivalence of multistage interconnection networks, *Discr. Appl. Math.* **22** (1988/89) 201-214.

[7] G. Bilardi, Merging and sorting networks with the topology of the Omega network, *IEEE Trans. on Computers* **38** (1989) 1396-1403.

[8] M.T. Bruggencate, S. Chalasani, Equivalence between SP2 high-performance switches and three-stage Clos networks, *International Conference on Parallel Processing* **1** (1996) 1-8.

[9] T. Calamoneri, A. Massini, Efficiently checking the equivalence of multistage interconnection networks, Eleventh IASTED International Conference on Parallel and Distributed Computing and Systems (CPDC'99), pp. 23-30, 1999. (A revised version is submitted to *Theory of Computing Systems*)

[10] T. Calamoneri, A. Massini, A new approach to the rearrangeability of (2log N -1) stage MINs, IASTED International Symposium on Applied Informatics (AI 2001), pp. 365-370, 2001.

[11] H. Çam, Rearrangeability of (2n-1)-Stage shuffle-exchange networks, *SIAM Journal on Computing* to appear.

[12] G.J. Chang, F.K. Hwang, L.-D. Tong, Characterizing bit permutation networks, *Networks* **33** (1999) 261-267.

[13] D. Coudert, A. Ferreira, S. Perennes, De Bruijn isomorphisms and free space optical networks, in: IEEE IPDPS 2000, pages 769-774, IEEE Press, 2000.

[14] K. Day, A-E. Al-Ayyoub, The cross product of interconnection networks, *IEEE Transactions on Parallel and Distributed Systems* **8** (1997) 109-118.

[15] S. Even, A. Litman, Layered cross product–A technique to construct interconnection networks, *Networks* **29** (1997) 219-223.

[16] T.-Y. Feng, Multistage interconnection networks, in: *The Froehlich/Kent Encyclopedia of Telecommunications* (Eds., F.E. Froehlich, A. Kent) **vol. 12** pages 203-239, Marcel Dekker, Inc. , 1996.

[17] N. Golbandi, A. Litman, On the characterization and classification of butterfly-like networks, to appear.

[18] E. Hotzel, Components and graph automorphisms of standard 2×2-switch Banyan networks, *Networks* **27** (1996) 53-71.

[19] E. Hotzel, Binary full access networks of minimum size, GMD report 124, Dec. 2000.

[20] E. Hotzel, A catalogue of small binary banyans (I. n-stage banyans, $n \leq 4$), GMD-Studien Nr. 265, July 1995.

[21] E. Hotzel, Full access networks of small size, in: Structure, Information and Communication Complexity, 3rd Coll. SIROCCO '96, (Eds. N. Santoro, P. Spirakis) International Informatics Serries 6, Carleton University Press, 1996, pp. 210-223.

[22] Q. Hu, X. Shen, J. Yang, Topologies of combined $(2logN - 1)$-stage interconnection networks, *IEEE Trans. Comput.* **C-46** (1997) 118-124.

[23] F.K. Hwang, Superior channel graphs, in: Proc. 9th Inter. Teletraffic Cong., Torremolinos, Spain, 1979.

[24] F.K. Hwang, C.H. Yen, Characterizing the bit permutation networks obtained from the line digraphs of bit permutation networks, *Networks* **38** (2001) 1-5.

[25] W. Imrich, S. Klavzar, *Product Graphs: Structure and Recognition,* Wiley-Interscience Series in Discrete Mathematics and Optimization, New York, NY, 2000.

[26] S.D. Kaushik, S. Sharma, C.-H. Huang, An algebraic theory for modeling multistage interconnection networks, *Journal of Information Science and Engineering* **9** (1993) 1-26.

[27] M.-Y. Kim, H.Y. Yoon, S.R. Maeng, Bit-permute multistage interconnection networks, *Microprocessing and Microprogramming* **41** (1995) 449-468.

[28] C.P. Kruskal, M. Snir, A unified theory of interconnection network structure, *Theor. Comput. Sci.* **48** (1986) 75-94.

[29] C.P. Kruskal, M. Snir, Cost-performance tradeoffs for interconnection networks, *Discrete Applied Mathematics* **37-8** (1992) 359-385.

[30] S. Muthukrishnan, M.S. Paterson, S.C. Sahinalp, T. Suel, Compact grid layouts of some multi-level networks, *31st ACM Symposium on the Theory of Computing (STOC '99),* pages 455-463, 1999.

[31] A. Paz, Theory of decomposition into prime factors of layered interconnection networks, Technical Report 892, Computer Science Department, Technion-Israel Institute of Technology, 1997.

[32] Y. Perl, Better understanding of Batcher's merging networks, *Discrete Applied Mathematics* **25** (1989) 257-271.

[33] A.L. Rosenberg, Theoretical research on networks: models and methodology, (an invited talk at the 4th International Colloquium on Structural Information and Communication Complexity (SIROCCO'97)).

[34] A.L. Rosenberg, Product-shuffle networks: toward reconciling shuffles and butterflies, *Discrete Applied Mathematics* **37/38** (1992) 465-488.

[35] L.D. Tong, F.K. Hwang, G.J. Chang, Channel graphs of bit permutation networks, *Theoretical Computer Science* **263 (1-2)** (2001) 139-143.

[36] P. de la Torre, C.P. Kruskal, Upper and lower bounds for recurrent and recursively decomposable parallel processor-networks, *Communications on Pure and Applied Mathematics* **48 (9-10)** (1995) 1005-1025.

[37] C.-L. Wu, T.Y. Feng, On a class of multistage interconnection networks, *IEEE Trans. Comput.* **C-29** (1980) 694-702.

[38] Y. Wu, R. Jia, Q. Li, g-circulant solutions to the $(0, 1)$ matrix equation $A^m = J$, *Linear Algebra and its Applications* to appear.

[39] Y. Wu, R. Jia, Q. Li, A characterization of De Bruijn digraphs, submitted.

[40] Y. Wu, Q. Li, Some characterizations for the wrapped butterfly, to appear in: Proceedings of the International Symposium on Analysis, Combinatorics, and Computing in honor of Prof. L.C. Hsu's 80th birthday.