Tropical complexes of submatrices of tree metrics *

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Abstract

A zonotope is an affine projection of a hypercube and a zonotopal tiling is a subdivision of a polytope by zonotopes. Given any finite subset S in a tropical projective torus, Develin and Sturmfels defined a polyhedral complex from S and called it the tropical complex generated by S. Given a tree metric, we form a submatrix of it by removing some of its rows, then take all column vectors of that submatrix and project them into the tropical projective torus, and then move all of them by a fixed translation vector to get a set S. We show that the tropical complex generated by S, for any choice of the set of rows and the translation vector, gives a zonotopal tiling. Conversely, we prove that any zonotopal tiling realizable as a tropical complex must arise in this way. We also discuss some other old and new results about tree metrics from the point of view of tropical geometry.

Keywords: Kleene star, Lipschitz function, metric tree, tropical polytope, zonotopal tiling.

Contents

1 Introduction

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_+$ the set of nonnegative reals. Let X and Y be two sets. We often view an element of $\mathbb{R}^X$ as a column vector indexed by X. Let $\mathrm{D} \in \mathbb{R}^{X\times Y}$ be a matrix. For any $y \in Y$, we use $\mathrm{D}(y)$ for the function in $\mathbb{R}^X$ such that $\mathrm{D}(x, y) = \mathrm{D}(y)(x)$ and refer to it as the *column* of D indexed by y. Pick $U \in 2^X \setminus \{\emptyset\}$ and $V \in 2^Y \setminus \{\emptyset\}$. We denote by $\mathrm{D}(U, V)$ the restriction of D on $U \times V$, namely $\mathrm{D}(U, V)$ is a submatrix of D. If $V = \{v\}$, we just write $\mathrm{D}(U, v)$

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for $\mathrm{D}(U,V)$ and often regard it as an element in $\mathbb{R}^U$; indeed, $\mathrm{D}(U,v) = \mathrm{D}(v)|_U$. We call $\mathrm{D}(U,Y)$ a *row submatrix* of D. That is, a row submatrix of a matrix is obtained from it by removing some or none of its rows. Let $A \in \mathbb{R}^{X\times X}$ be a matrix and let $s = (s_1, \dots, s_k)$ be any sequence of elements in X. We say that A *vanishes cyclicly on* s provided $A(s_1,s_2) + \cdots + A(s_{k-1},s_k) + A(s_k,s_1) = 0$. We say that A *vanishes cyclicly* provided it vanishes cyclically on all sequences of X.

A *metric space* is a pair (X,D) where X is a set and $\mathrm{D} \in \mathbb{R}_+^{X\times X}$ is a map, called a *metric* on X, such that

$$\mathrm{D}(x,y) - \mathrm{D}(y,x) = \mathrm{D}(x,x) = 0 \le \mathrm{D}(x,y) + \mathrm{D}(y,z) - \mathrm{D}(x,z)$$

for all $x,y,z \in X$. We call a metric space (T,D) a *metric tree* [Dre84, Tit77] provided it satisfies the following two conditions:

(i) For every $x, y \in \binom{T}{2}$, there exists an isometric embedding ϕ from the interval $[0,\mathrm{D}(x,y)] \subseteq \mathbb{R}$ to (T,D) such that $\{\phi(0), \phi(\mathrm{D}(x,y))\} = \{x,y\}$.

(ii) For every injective continuous map ϕ from the interval $[0,1] \subseteq \mathbb{R}$ to (T,D) and any $t \in [0,1]$, it holds $\mathrm{D}(\phi(0),\phi(t)) + \mathrm{D}(\phi(t),\phi(1)) = \mathrm{D}(\phi(0),\phi(1))$.

A metric tree is also known as an $\mathbb{R}$-tree and possesses many equivalent definitions [AJ14, Bes01, Eva08, KS14]. We call (T,D,X,ϕ) a *metric X-tree* provided (T,D) is a metric tree, X is a set and ϕ is a map from X to T. Given a metric X-tree (T,D,X,ϕ), we let $\mathrm{D}' \in \mathbb{R}^{X\times X}$ be the matrix specified by $\mathrm{D}'(x,y) = \mathrm{D}(\phi(x),\phi(y))$. Clearly, (X,D') forms a metric space, which we refer to as the *tree metric represented by* (T,D,X,ϕ). If X is a finite set, the tree metric (X,D') is known as a *finite tree metric*. Tree structure is widely recognized as an important and basic object in mathematics [Chi01, DHK+12, Eva08, LP16, Ser03]. Can we characterize or recognize a submatrix of a tree metric? In other words, given a family of vectors in $\mathbb{R}^X$, when can they appear as column vectors of a tree metric?

Given a finite number of closed line segments in a Euclidean space, their vector sum, also called Minkowski sum, is a *zonotope*. Equivalently, a zonotope is the image of a cube under an affine transformation. When we have k linearly independent closed line segments, namely when the zonotope generated by them could not be translated into a linear subspace of dimension less than k, the resulting zonotope is known as a *parallellotope*. A polyhedral complex is *zonotopal* if all its faces/cells are zonotopes; a polyhedral complex is known as a *zonotopal tiling* if it is zonotopal and the union of its cells is convex. A main discovery of this work is a close relationship between submatrix of a finite tree metric and zonotopal tilings; see Theorem 1.2. Let us furnish some more terminology below to make the statement in Theorem 1.2 readable.

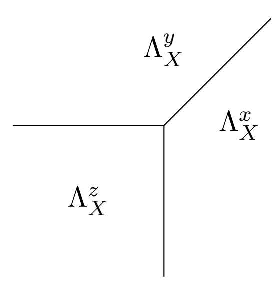

Figure 1.1: $\mathcal{F}_X$ for $X = \{x,y,z\}$.

For any two reals a and b, let

$$a \oplus b = \min\{a,b\} \quad \text{and} \quad a \otimes b = a + b.$$

We call $\oplus$ the *tropical addition* and $\otimes$ the *tropical multiplication* and will naturally extend the application of these two operations to real matrices. Note that $(\mathbb{R}, \oplus, \otimes)$ is a commutative semiring and the linear algebra over it is well-studied [But10, MS15]. We fix a set X. For each $f \in \mathbb{R}^X$, the tropical line generated by f is $f + \mathbb{R} \cdot \mathbf{1} = \{c \otimes f : c \in \mathbb{R}\}$, which is dubbed $[f]$. Accordingly, we define the *tropical projective torus* $\mathbb{TT}^X$ to be the set of all such tropical lines, that is $\mathbb{TT}^X = \mathbb{R}^X / \mathbb{R} \cdot \mathbf{1}$. For each $\alpha \in \mathbb{TT}^X$, we will write $\alpha(a, b)$ for $f(a) - f(b)$ as long as $\alpha = [f]$ and thus view α as a function on $X \times X$. With this understanding, $\mathbb{TT}^X$ can be identified with a subspace of the 2-fold exterior power of $\mathbb{R}^X$, namely the set of real functions α on $X \times X$ which vanishes cyclically on X. For any $V \subseteq \mathbb{R}^X$, we write $[V]$ for $\{[f] : f \in V\} \subseteq \mathbb{TT}^X$. For any $f \in \mathbb{R}^X$, we put $|[f]|_{\mathrm{H}} = \frac{\sup f - \inf f}{2}$ and we call $|[f]|_{\mathrm{H}}$ the *Hilbert norm* of $[f] \in \mathbb{TT}^X$ whenever it is a finite number. We call a subset U of $\mathbb{TT}^X$ *bounded* whenever $\sup_{\alpha \in U} |\alpha|_{\mathrm{H}}$ is finite. When X is finite, all norms on $\mathbb{TT}^X$ are surely equivalent to each other but the definition here allows us to discuss bounded sets in $\mathbb{TT}^X$ for any infinite set X. For each $\alpha \in \mathbb{TT}^X$, let $\mathrm{argmax}(\alpha)$ denote the set $\{x \in X : \alpha(x, y) \geq 0 \text{ for all } y \in X\}$. For each $Y \in 2^X \setminus \{\emptyset\}$, let

$$\Lambda_X^Y = \{\alpha : \alpha \in \mathbb{TT}^X, Y \subseteq \mathrm{argmax}(\alpha)\},$$

which is a cone in $\mathbb{TT}^X$. Let $\mathcal{F}_X$ be the fan in $\mathbb{TT}^X$ consisting of Λ_X^Y, $Y \in 2^X \setminus \{\emptyset\}$. We depict $\mathcal{F}_X$ in Fig. 1.1 for $X = \{x, y, z\}$ by identifying $[f] \in \mathbb{TT}^X$ with $f - f(z) \cdot \mathbf{1} \in \{g \in \mathbb{R}^X : g(z) = 0\}$. The translation of $\mathcal{F}_X$ through a vector $\alpha \in \mathbb{TT}^X$ is denoted by $\alpha + \mathcal{F}_X$. For any $S \subseteq \mathbb{TT}^X$, we write $\mathcal{CD}_S$ for the common refinement of the complexes $\alpha + \mathcal{F}_X$, $\alpha \in S$, and use $\mathcal{C}_S$ to denote the complex of bounded cells in $\mathcal{CD}_S$. We call $\mathcal{CD}_S$ the *covector decomposition* generated by S [AD09, FR15] and call $\mathcal{C}_S$ the *tropical complex* generated by S [DS04b]. Let Y be a set and let $\mathrm{D} \in \mathbb{R}^{X \times Y}$. We use $[\mathrm{D}]$ to represent $\{[\mathrm{D}(y)] : y \in Y\} \subseteq \mathbb{TT}^X$. We often write $\mathcal{CD}_{\mathrm{D}}$ for $\mathcal{CD}_{[\mathrm{D}]}$, write $\mathcal{C}_{\mathrm{D}}$ for $\mathcal{C}_{[\mathrm{D}]}$, and call them the *covector decomposition generated by* D and the *tropical complex generated by* D, respectively.

Let T and X be two sets, $\mathrm{D} \in \mathbb{R}^{T \times T}$ and ϕ a map from X to T. We say that $f \in \mathbb{R}^X$ is *compatible* with (T, D, X, ϕ) whenever there exists $p \in T$ such that $f(x) = \mathrm{D}(\phi(x), p)$ for all $x \in X$. We adopt the notation $\mathscr{C}(T, \mathrm{D}, X, \phi)$ for the set of functions in $\mathbb{R}^X$ which are compatible with (T, D, X, ϕ). If $X \subseteq T$ and ϕ is the inclusion map, we abbreviate $\mathscr{C}(T, \mathrm{D}, X, \phi)$ as $\mathscr{C}(T, \mathrm{D}, X)$. Here is a simple local-global properties of tree structures, which says that a vector is a column vector of a row submatrix of a tree metric if and only if so are all its length-3 subvectors for the same tree metric.

Theorem 1.1. *Let X be a finite set, (T, D, X, ϕ) be a metric X-tree and $f \in \mathbb{R}^X$. Then $\alpha \in [\mathscr{C}(T, \mathrm{D}, X, \phi)] \subseteq \mathbb{TT}^X$ if and only if $\alpha|_{U \times U} \in [\mathscr{C}(T, \mathrm{D}, U, \phi|_U)] \subseteq \mathbb{TT}^U$ for all $U \in \binom{X}{\leq 3}$.*

We say that $A \in \mathbb{R}^{X \times Y}$ and $B \in \mathbb{R}^{X \times Z}$ are *tropically equivalent* provided there exists $\alpha \in \mathbb{TT}^X$ such that $[A] = [B] + \alpha$ as subsets of $\mathbb{TT}^X$. For any $A \in \mathbb{R}^{X \times Y}$, we write $\langle A \rangle$ for the class of matrices which are tropically equivalent with A.

Theorem 1.2. *The tropical complex generated by a finite matrix is a zonotopal tiling if and only if that matrix is tropically equivalent with a row submatrix of a finite tree metric.*

Let X be a set. For any $a \in X$, we use δ_a for the function in $\mathbb{R}^X$ that takes value 1 at x and takes value 0 elsewhere, expecting that the reader can recognize what is the set X from the context.

Example 1.3. Let (T, D) be the metric tree as shown on the left of Fig. 1.2 in which we mark the length of the three segments. Let $X = \{a, b, c\}$ and let $Y = \{a, b, c, d\}$. By identifying $\mathbb{TT}^X$ with $\{f \in \mathbb{R}^X : f(c) = 0\}$, we draw $\mathcal{CD}_{\mathrm{D}(X,Y)}$ on the right of Fig. 1.2, where $v_a = -\delta_a, v_b = -\delta_b, v_c = \delta_a + \delta_b$ and $v_d = \mathbf{0}$. It is clear that the bounded faces of $\mathcal{CD}_{\mathrm{D}(X,Y)}$, namely $\mathcal{C}_{\mathrm{D}(X,Y)}$, gives rise to a zonotopal tiling.

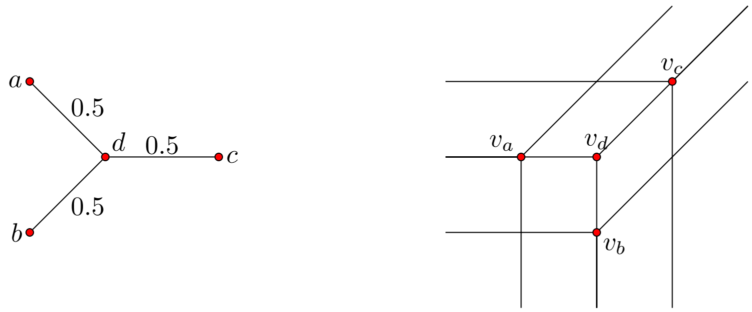

Figure 1.2: A covector decomposition from a row submatrix of a tree metric.

The remainder of the paper is organized as follows. We present a short proof of Theorem 1.1 in § 2, which will become an essential step for our proof of Theorem 1.2. We collect some simple facts on tropical geometry in § 3 for those readers who are not familiar with this subject. To tie up a few loose ends in connecting tree metric and zonotopal tiling, we prepare several more results on tree structures in § 4. Some simple facts about trees displayed there, say Propositions 4.3 and 4.6, are not necessary for our work in § 5 but may be of some independent interest. In § 5, we are ready to finish a proof of Theorem 1.2. In the course of working towards that, we obtain a sufficient condition for a tropical complex to be zonotopal; see Proposition 5.6. We close the paper with some research problems in § 6.

2 Three-point criterion

Let (T, D) be a metric tree. For any two points $x, y \in T$, we denote by $\mathrm{g}^T(xy)$ the unique geodesic connecting them in T; for every $x, y, z \in T$, $\mathrm{g}^T(xy) \cap \mathrm{g}^T(yz) \cap \mathrm{g}^T(zx)$ consists of a single point of T, which we call the *median* of x, y and z in (T, D) and record by the notation $\mathrm{m}^T(xyz)$. For any $Y \subseteq T$, let $\mathrm{conv}_T(Y) \doteq \cup_{y,z\in Y} \mathrm{g}^T(yz)$. Assume that $|Y|$ is finite. Then, for every point $r \in T$, $\mathrm{conv}_T(Y)$ contains the unique "gate" for r, namely the point $s \in \mathrm{conv}_T(Y)$ such that $\mathrm{D}(a, s) + \mathrm{D}(s, r) = \mathrm{D}(a, r)$ for all $a \in \mathrm{conv}_T(Y)$. It is noteworthy that

$$\mathrm{D}(a, r) - \mathrm{D}(b, r) = \big(\mathrm{D}(a, s) + \mathrm{D}(s, r)\big) - \big(\mathrm{D}(b, s) + \mathrm{D}(s, r)\big) = \mathrm{D}(a, s) - \mathrm{D}(b, s) \tag{2.1}$$

for all $a, b \in \mathrm{conv}_T(Y)$.

Proof of Theorem 1.1. The forward direction is trivial and so we only focus on the backward direction. Moreover, it suffices to consider the case that $X \subseteq T$ and ϕ is the inclusion map. Therefore, we now have an element $[f] \in \mathbb{TT}^X$ such that $[f|_U] \in [\mathscr{C}(T, \mathrm{D}, U)] \subseteq \mathbb{TT}^U$ for all $U \in \binom{X}{\leq 3}$ and we aim to prove that $[f] \in [\mathscr{C}(T, \mathrm{D}, X)]$.

We proceed by induction on $|X|$. When $|X| \leq 3$, the result is straightforward. We assume $|X| \geq 4$ and the result holds when $|X|$ is smaller.

For $\{u, v\} \in \binom{X}{2}$, define

$$\mathfrak{G}_{uv} \doteq \{p \in \mathrm{g}^T(uv) \setminus \{u, v\} : \ p = \mathrm{m}^T(uvw) \text{ for some } w \in X\}.$$

Let $\{y, z\} \in \binom{X}{2}$ be the one such that $|\mathfrak{G}_{yz}|$ takes the largest possible value. Since $|X| \geq 4$, $\mathfrak{G}_{yz}$ must be nonempty. Pick $u \in \mathfrak{G}_{yz}$ such that $\mathrm{D}(y, u)$ takes the minimum possible value. The definition of $\mathfrak{G}_{yz}$ guarantees the existence of $x \in X$ such that $u = \mathrm{m}^T(xyz)$.

Let $W = \{x, y, z\}$. By assumption and on account of Eq. (2.1), there exists $p_W \in \mathrm{conv}_T(W)$ such that

$$[f|_W] = [\mathrm{D}(W, p_W)]. \tag{2.2}$$

Note that $\mathrm{conv}_T(W) = \mathrm{g}^T(yz) \cup \mathrm{g}^T(xz)$. Consequently, without loss of generality, we can suppose that $p_W \in \mathrm{g}^T(yz)$.

For $X' = X \setminus \{x\}$, the induction assumption along with Eq. (2.1) ensures the existence of $p_{X'} \in \mathrm{conv}_T(X')$ such that $[f|_{X'}] = [\mathrm{D}(X', p_{X'})]$. To finish the proof, it suffices to show that $f(x) - f(y) = \mathrm{D}(x, p_{X'}) - \mathrm{D}(y, p_{X'})$. According to Eq. (2.2), we aim to show that

$$\mathrm{D}(x, p_{X'}) - \mathrm{D}(y, p_{X'}) = \mathrm{D}(x, p_W) - \mathrm{D}(y, p_W). \tag{2.3}$$

In light of $\mathrm{D}(y, p_{X'}) - \mathrm{D}(z, p_{X'}) = f(y) - f(z) = \mathrm{D}(y, p_W) - \mathrm{D}(z, p_W)$, we derive that

$$\mathbb{m}^T(yzp_{X'}) = p_W \in \mathrm{g}^T(yz).$$

We claim that $\mathrm{g}^T(p_{X'}p_W) \cap \mathrm{g}^T(xu) \subseteq \{u\}$. Otherwise, we can find $v \in X'$ such that $p_{X'} \in \mathrm{g}^T(vu)$ and $|\mathfrak{G}_{vz}| \geq |\mathfrak{G}_{yz} \cup \{\mathbb{m}^T(xvz)\}| = |\mathfrak{G}_{yz}| + 1$, which contradicts the maximality of $|\mathfrak{G}_{yz}|$. We can now see that $\mathrm{D}(x, p_W) + \mathrm{D}(p_W, p_{X'}) = \mathrm{D}(x, p_{X'})$ and $\mathrm{D}(y, p_W) + \mathrm{D}(p_W, p_{X'}) = \mathrm{D}(y, p_{X'})$, from which Eq. (2.3) follows. □

Remark 2.1. Let X be a set of size at least three and let $\mathrm{D} \in \mathbb{R}^{X \times X}$ be the tree metric such that $\mathrm{D}(x) = \mathbf{1} - \delta_x$ for all $x \in X$. Observe that D is represented by a star tree (T, D') with all edges carrying the same weight $\frac{1}{2}$ – on the left of Fig. 1.2 you find such a metric tree with $|X| = 3$. Note that $\alpha \in [\mathscr{C}(T, \mathrm{D}', X)]$ if and only if $\max_{a,b \in X} \alpha(a, b) \leq 1$ and $|\operatorname{argmax}(\alpha)| \geq |X| - 1$. Also note that, for any $f \in \mathbb{R}^X$, it holds $[f|_U] \in [\mathscr{C}(T, \mathrm{D}', U)]$ for all $U \in \binom{X}{2}$ if and only if $\max f - \min f \leq 1$. This means that we could not replace $\binom{X}{\leq 3}$ by $\binom{X}{\leq 2}$ in Theorem 1.1.

3 Polytrope and Kleene star

Let X be a finite set. For any finite set $S \subseteq \mathbb{R}^X$, the *tropical span* of S is

$$\mathrm{tspan}(S) \doteq \{\oplus_{s \in S}\, t_s \otimes s : \; t_s \in \mathbb{R}\} \subseteq \mathbb{R}^X .$$

For any finite set $S \subseteq \mathbb{TT}^X$, the *tropical polytope* generated by S is

$$\mathrm{tconv}(S) \doteq \{\oplus_{s \in S}\, t_s \otimes s : \; t_s \in \mathbb{R}\} \subseteq \mathbb{TT}^X .$$

If D is a finite matrix with rows indexed by X, with the convention that it represents the set of its columns, we can use the notation tspan(D) and tconv([D]) and talk about its tropical span and tropical polytope.

Via the use of the so-called tropical Farkas Lemma [MS15, Proposition 5.2.10], Develin and Sturmfels obtained the following basic relationship between tropical polytope and tropical complex.

Lemma 3.1. *[DS04b, Theorem 15, Proposition 16] [MS15, Proposition 5.2.11] Let X and Y be two finite sets and let $\mathrm{D} \in \mathbb{R}^{X \times Y}$. Then $\cup_{C \in \mathcal{C}_\mathrm{D}} C = \mathrm{tconv}([\mathrm{D}])$.*

For any $A \in \mathbb{R}^{X \times Y}$, its *transpose* is the matrix in $\mathbb{R}^{Y \times X}$, denoted as $A^\top$, such that $A^\top(y, x) = A(x, y)$ for all $(x, y) \in X \times Y$.

Lemma 3.2. *Let Y be a set and let $A \in \mathbb{R}^{Y \times Y}$. We assume that*

$$A(x,y) + A(y,z) \geq A(x,z) \tag{3.1}$$

for all $x, y, z \in Y$. Let X be a subset of Y such that for every $p \in Y$ and every $x \in X$ there exists $y \in X$ satisfying

$$A(x,p) + A(p,y) = A(x,y). \tag{3.2}$$

Let $B = A(X,X)$. Then $\mathbb{R}^X \supseteq \operatorname{tspan}(-B) \supseteq -\mathscr{C}(Y,A,X)$.

Proof. Take $p \in Y$ and let $g_p = A^\top(X,p)$. It is easy to check from Eqs. (3.1) and (3.2) that $(-B) \otimes g_p = -A(X,p)$ and so $-A(X,p) \in \operatorname{tspan}(-B)$, finishing the proof. $\square$

Let X be a finite set and pick $\mathrm{D} \in \mathbb{R}^{X \times Y}$. Note that $[\mathrm{D}]$ is a Y-indexed point configuration in $\mathbb{TT}^X$. For each $\alpha \in \mathbb{TT}^X$, the minimal cell of $\mathcal{CD}_\mathrm{D}$ containing α can be written as

$$\cap_{\beta \in [\mathrm{D}]} \left(\beta + \Lambda_X^{\operatorname{argmax}(\alpha - \beta)}\right). \tag{3.3}$$

We construct the $(0,1)$ matrix $\mathcal{T}_{\mathrm{D},\alpha} \in \mathbb{R}^{X \times Y}$ such that $\mathcal{T}_{\mathrm{D},\alpha}(x,y) = 1$ if and only if $x \in \operatorname{argmax}(\alpha - [\mathrm{D}(y)])$. Let $\mathrm{S} = (\mathrm{S}_x)_{x \in X} \in (2^Y)^X$ be the X-indexed set family such that $y \in \mathrm{S}_x$ if and only if $\mathcal{T}_{\mathrm{D},\alpha}(x,y) = 1$. We denote the cell as indicated in (3.3) by $\mathrm{C}_{\mathrm{S},\mathrm{D}}$, say that S is its *type* and call it a *tropical covector* of $\mathcal{CD}_\mathrm{D}$ [DS04b, p. 7] [FR15, p. 307]. We often regard the tropical covector S as the $(0,1)$ matrix $\mathcal{T}_{\mathrm{D},\alpha}$ and so it makes sense to talk about the transpose of S, which is a Y-indexed family of subsets of X. It is not hard to see that $\mathcal{T}_{\mathrm{D},\alpha}$ has no zero columns, $\cup_{x \in X} \mathrm{S}_x = Y$ and $\mathrm{C}_{\mathrm{S},\mathrm{D}}$ coincides with

$$\{\alpha \in \mathbb{TT}^X : \ \alpha(x,x') \leq [\mathrm{D}(y)](x,x') \text{ for all } x, x' \in X, y \in \mathrm{S}_{x'}\}. \tag{3.4}$$

Let us also write $\mathcal{T}_{\mathrm{D},\alpha}$ as $\mathcal{T}_{\mathrm{D},\mathrm{C}_{\mathrm{S},\mathrm{D}}}$ and view it as the characteristic matrix for the cell $\mathrm{C}_{\mathrm{S},\mathrm{D}}$ of $\mathcal{CD}_\mathrm{D}$. We mention that the collection of tropical covectors of $\mathcal{CD}_\mathrm{D}$ encodes the combinatorics of $\mathcal{CD}_\mathrm{D}$ and represents a realizable tropical oriented matroid [AD09, Theorem 3.6]. A variant of the type data, called coarse type, is considered in [DJS12].

Remark 3.3. From Eq. (3.4) one can derive easily that different cells have different characteristic matrices and a cell of type $\mathrm{S} = (\mathrm{S}_x)_{x \in X} \in (2^Y)^X$ is bounded (and hence a polytope) if and only if $\mathrm{S}_x \neq \emptyset$ for all $x \in X$ [MS15, Proposition 5.2.11]. Conversely, if $\mathrm{S} = (\mathrm{S}_x)_{x \in X} \in (2^Y)^X$ is an X-indexed set family such that $\cup_{x \in X} \mathrm{S}_x = Y$, as long as the set displayed in (3.4) is nonempty, it will be a cell of type $\mathrm{S}' = (\mathrm{S}'_x)_{x \in X}$ satisfying $\mathrm{S}'_x \supseteq \mathrm{S}_x$ for all $x \in X$.

Let S be a tropical covector of $\mathcal{CD}_\mathrm{D}$. It is known that the dimension of the cell $\mathrm{C}_{\mathrm{S},\mathrm{D}}$ of the covector decomposition $\mathcal{CD}_\mathrm{D}$ is one less than the number of connected components of the intersection graph of the set family S [MS15, Proposition 5.2.13]. This result links tropical convexity with connected components of intersection graphs and provides the basis for the forthcoming result, Theorem 3.4.

A square matrix $\mathrm{D} \in \mathbb{R}^{X \times X}$ is *tropically singular* if the minimum in

$$\operatorname{tdet}(\mathrm{D}) \doteq \oplus_{\sigma \in \mathrm{Sym}_X} \otimes_{x \in X} \mathrm{D}\left(x, \sigma(x)\right) = \min_{\sigma \in \mathrm{Sym}_X} \sum_{x \in X} \mathrm{D}\left(x, \sigma(x)\right)$$

is achieved at least twice; otherwise, we say that D is *tropically nonsingular*. The *tropical rank* of a matrix $\mathrm{D} \in \mathbb{R}^{X \times Y}$, denoted by $\operatorname{trank}(\mathrm{D})$, is the maximum positive integer k such that D contains a tropically nonsingular $k \times k$ submatrix.

Theorem 3.4. *[MS15, Theorem 5.3.23.] Let X and Y be two finite sets. The dimension of* $\operatorname{tconv}([\mathrm{D}])$ *equals* $\operatorname{trank}(\mathrm{D}) - 1$ *for any* $\mathrm{D} \in \mathbb{R}^{X\times Y}$.

A square matrix $\mathrm{D} \in \mathbb{R}^{X\times X}$ with zeros on its diagonal is a *Kleene star* if $\mathrm{D} \otimes \mathrm{D} = \mathrm{D}$, namely, if it holds

$$\mathrm{D}(x, x) = 0 \leq \mathrm{D}(x, y) + \mathrm{D}(y, z) - \mathrm{D}(x, z) \tag{3.5}$$

for all $x, y, z \in X$. We define a *directed metric* to be any nonnegative Kleene star. It is easy to see that being a metric is equivalent to being a symmetric Kleen star and also equivalent to being a symmetric directed metric.

Remark 3.5. For any $\mathrm{D} \in \mathbb{R}^{X\times X}$, we say that $f \in \mathbb{R}^X$ is a *Lipschitz function* with respect to D provided $f(x) - f(y) \leq \mathrm{D}(x, y)$ for all $x, y \in X$. Note that, for a zero-diagonal matrix D, f is Lipschitz with respect to D if and only if $f = \mathrm{D} \otimes f$. If D is further a Kleene star, we have $\mathrm{D} \otimes \mathrm{D} = \mathrm{D}$ and so $\mathrm{D} \otimes g$ is always Lipschitz with respect to D for all $g \in \mathbb{R}^X$. In summary, we now see that the set of Lipschitz functions with respect to a Kleene star D coincides with tspan(D), namely

$$\begin{aligned} \operatorname{tspan}(\mathrm{D}) &= \{f \in \mathbb{R}^X : \ f(x) - f(y) \leq \mathrm{D}(x, y) \ \text{ for all } x, y \in X\} \\ &= \{f \in \mathbb{R}^X : \ f(x) - \mathrm{D}(x, y) \leq f(y) - \mathrm{D}(y, y) \ \text{ for all } x, y \in X\}. \end{aligned}$$

For a statement which is a bit stronger than this, see Theorem 3.6 (iv).

Generally, a tropical polytope is not convex in the usual sense. Joswig and Kulas [JK10] called those tropical polytopes which are convex in usual sense *polytropes*. It turns out that polytropes are studied under the name of *L-convex sets* [Mur03] in discrete convex analysis, they were called *alcoved polytopes of type A* by Lam and Postnikov [LP07], they are recently studied by several groups in the context of Lipschitz polytopes [DH16, GP17, WX17] as an effort to answer the question posed by Vershik [Ver15] on fundamental polytopes. Note that Remark 3.5 indicates an easy link between the study of Lipschitz polytopes and tropical geometry. When D is a directed metric, tspan(D) appears in the context of a presheaf on D [Wil13, p. 703] and plays an important role in the study of the Isbell completions of directed metric spaces [KKOO12]. Furthermore, we mention that polytropes are in one-to-one correspondence with Kleene stars; see Theorem 3.6 (ii) and (iii).

Our proof towards Theorem 1.2 will utilize some well-known facts in tropical geometry, which are scattered among various sources. To make our paper self-contained, we collect them together to display in Theorem 3.6.

Theorem 3.6. (i) *Each cell of a tropical complex generated by a finite matrix is a polytrope.*

(ii) *A tropical polytope is a polytrope if and only if it can be generated by a Kleene star.*

(iii) *A polytrope cannot be generated by two different Kleene stars.*

(iv) *Let X be a finite set. For a zero diagonal matrix $\mathrm{D} \in \mathbb{R}^{X\times X}$, D is a Kleene star if and only if* tspan(D) *equals the set of Lipschitz functions with respect to* D.

(v) *Let X be a finite set and let $\mathrm{D} \in \mathbb{R}^{X\times X}$ be a Kleene star. Then $\mathcal{C}_{\mathrm{D}}$ is the polytopal complex consisting of the polytope* $\operatorname{tconv}([\mathrm{D}])$ *and all its faces.*

(vi) *Let X be a finite set and let $\mathrm{D} \in \mathbb{R}^{X\times X}$ be a Kleene star. Then* $\operatorname{tconv}(S) = \operatorname{tconv}([\mathrm{D}])$ *if and only if* $\operatorname{tconv}([\mathrm{D}]) \supseteq S \supseteq [\mathrm{D}]$.

Proof. (i) is asserted in [DS04b, Proposition 19], (ii) is just [JK15, Corollary C], while (iii) is read from [Ser07, Proposition 6]. (iv) comes from [dlP13, Theorem 2.1]; also see [JL16, Remark 25] or [But10, Theorem 2.1.1].

We now move ahead to (v). By Eq. (3.4) and Remark 3.5, we see that $\operatorname{tconv}([\mathrm{D}])$ is the cell of $\mathcal{C}_{\mathrm{D}}$ of type $\mathrm{S} = (\mathrm{S}_x = \{x\})_{x \in X}$. Item (v) thus follows as a consequence of Lemma 3.1.

Finally, we are going to prove (vi). Take $x \in X$. We assume that there exist $c_1, \ldots, c_n \in \mathbb{R}, x_1, \ldots, x_n \in X$ such that $\mathrm{D}(x) = \oplus_{i=1}^{n} c_i \otimes \mathrm{D}(x_i)$, namely

$$\mathrm{D}(y, x) = \min\{c_i + \mathrm{D}(y, x_i) : \ i = 1, \ldots, n\} \tag{3.6}$$

for all $y \in X$. Our task is to show that $[\mathrm{D}(x_i)] = [\mathrm{D}(x)]$ for some $i \in \{1, \ldots, n\}$. Setting $y = x$ in Eq. (3.6) shows that we can suppose, without loss of generality,

$$\mathrm{D}(x, x) = c_1 + \mathrm{D}(x, x_1). \tag{3.7}$$

For all $y \in X$, it then follows from Eqs. (3.5) to (3.7) that

$$c_1 + \mathrm{D}(y, x_1) \geq \mathrm{D}(y, x) \geq -\mathrm{D}(x, x_1) + \mathrm{D}(y, x_1) = c_1 - \mathrm{D}(x, x) + \mathrm{D}(y, x_1) = c_1 + \mathrm{D}(y, x_1),$$

showing that $c_1 + \mathrm{D}(x_1) = \mathrm{D}(x)$ and hence $[\mathrm{D}(x_1)] = [\mathrm{D}(x)]$. □

Let X and Y be two sets. We say that A and B from $\mathbb{R}^{X \times Y}$ are *congruent* provided there exist $\sigma \in \mathrm{Sym}_X, f \in \mathbb{R}^X, \tau \in \mathrm{Sym}_Y$, and $g \in \mathbb{R}^Y$ such that

$$A(x, y) = f(x) + B\big(\sigma(x), \tau^{-1}(y)\big) + g(y) = f(x) \otimes B\big(\sigma(x), \tau^{-1}(y)\big) \otimes g(y) \tag{3.8}$$

for all $(x, y) \in X \times Y$. We mention in passing that the Hungarian method ensures that every real square matrix is congruent with a nonnegative matrix with zero diagonal [But10, Theorem 1.6.37]. We say that A and B from $\mathbb{R}^{X \times X}$ are *similar* provided there exist $\sigma \in \mathrm{Sym}_X$ and $f \in \mathbb{R}^X$ such that

$$A(x, y) = f(x) + B\big(\sigma(x), \sigma(y)\big) - f(y) = f(x) \otimes B\big(\sigma(x), \sigma(y)\big) \otimes (-f(y)) \tag{3.9}$$

for all $(x, y) \in X \times X$. We remark that a square matrix is a Kleene star if and only if it is similar to a directed metric. The backward implication is trivial; for the other direction, we take a Kleene star $\mathrm{D} \in \mathbb{R}^{X \times X}$, take $a \in X$ and f the ath column $\mathrm{D}^\top(a)$ of $\mathrm{D}^\top$, then we can check that the matrix $\mathrm{D}' \in \mathbb{R}^{X \times X}$ that satisfies

$$\mathrm{D}'(x, y) \doteq f(x) \otimes \mathrm{D}(x, y) \otimes (-f)(y) = \mathrm{D}(a, x) \otimes \mathrm{D}(x, y) \otimes \big(-\mathrm{D}(a, y)\big) \in \mathbb{R}_+ \tag{3.10}$$

for all $x, y \in X$ is a directed metric on X. For any $A \in \mathbb{R}^{X \times X}$, the claim that A vanishes cyclicly can now be conveniently formulated by the existence of a potential function $f \in \mathbb{R}^X$ such that

$$f(x) \otimes A(x, y) \otimes \big(-f(y)\big) = 0 \tag{3.11}$$

for all $x, y \in X$. Accordingly, A is similar to the zero matrix whenever it vanishes cyclicly.

4 Tree metric

Lemma 4.1. *[WX17, Theorem 3.4] For every finite metric space* (X, D), $\operatorname{tconv}([\mathrm{D}])$ *is a zonotope if and only if* D *is a tree metric.*

Corollary 4.2. *If* D *is a finite tree metric, then* $\mathcal{C}_{\mathrm{D}}$ *is a zonotopal tiling.*

Proof. Apply Theorem 3.6 (v) and Lemma 4.1. ☐

Proposition 4.3. *Let X be a finite set and take $\mathrm{D} \in \mathbb{R}^{X \times X}$. Then D is similar to a tree metric if and only if D is a Kleene star for which $\mathrm{tconv}([\mathrm{D}])$ is a zonotope.*

Proof. If a matrix D is similar to a metric, it is surely a Kleene star. Therefore, the forward implication is direct from Lemma 4.1.

We are now going to prove the other direction. We assume that the center of $\mathrm{tconv}([\mathrm{D}])$ is $[f]$ for some $f \in \mathbb{R}^X$. Let $\mathrm{D}' \in \mathbb{R}^{X \times X}$ be the matrix such that $\mathrm{D}'(x,y) = \mathrm{D}(x,y) - f(x) + f(y)$ for all $x, y \in X$, which is similar to D. Then $\mathrm{tconv}([\mathrm{D}']) = \mathrm{tconv}([\mathrm{D}]) - [f]$, i.e., $\mathrm{tconv}([\mathrm{D}'])$ is a zonotope with $[\mathbf{0}]$ being its center.

If D' is not symmetric, say there exist $a \neq b \in X$ such that $\mathrm{D}'(a,b) > \mathrm{D}'(b,a)$, then

$$\big(-\mathrm{D}'(b)\big)(b) - \big(-\mathrm{D}'(b)\big)(a) = \mathrm{D}'(a,b) > \mathrm{D}'(b,a).$$

Hence, by virtue of Remark 3.5, $[-\mathrm{D}'(b)] \notin \mathrm{tconv}(\mathrm{D}')$. Since $\mathrm{tconv}([\mathrm{D}'])$ is centrally symmetric around $[\mathbf{0}]$, from $[-\mathrm{D}(b)] \notin \mathrm{tconv}([\mathrm{D}])$ we find that $[\mathrm{D}(b)] \notin \mathrm{tconv}([\mathrm{D}])$, which is absurd. So D' is a symmetric Kleene star and so a metric. As application of Lemma 4.1 now says that D' is a tree metric, as desired. ☐

Corollary 4.4. *A polytrope is a zonotope if and only if it is a translation of the tropical polytope generated by a tree metric.*

Proof. Use Theorem 3.6 (ii) and Proposition 4.3. ☐

Let T be a tree graph in which each edge with endpoints a and b corresponds to two arcs $\overrightarrow{ab}$ and $\overrightarrow{ba}$. We denote the set of arcs of T by $\mathrm{A}(T)$. Let w be a function from $\mathrm{A}(T)$ to nonnegative reals and let $\mathrm{D}_{T,w} \in \mathbb{R}^{\mathrm{V}(T) \times \mathrm{V}(T)}$ be the function which sends $(x,y) \in \mathrm{V}(T) \times \mathrm{V}(T)$ to the sum of the values $w(\overrightarrow{ab})$ of all arcs $\overrightarrow{ab}$ on the unique directed path on T leading from x to y. For any set $X \subseteq \mathrm{V}(T)$, we call $\mathrm{D}_{T,w}[X,X]$ the *directed tree metric* represented by (T, X, w) and denote it by $\mathrm{D}_{T,X,w}$. Surely, each directed tree metric is a directed metric and a tree metric is simply a symmetric directed tree metric. Hirai and Koichi [HK12a] found that a directed metric which is similar to a tree metric must be a directed tree metric. We present a new proof of this result in Lemma 4.5. When (X, D) is a metric space and $U \subseteq X$, McShane [McS34, Theorem 1] found that every Lipschitz function $g \in \mathbb{R}^U$ with respect to $\mathrm{D}(U,U)$ can be extended to a Lipschitz function on X with respect to D. Similar to the proof of [WX17, Theorem 3.3], the main ingredient of our proof of Lemma 4.5 is the fact that, for any Kleene star D, any Lipschitz function with respect to a submatrix of D can be extended to a Lipschitz function with respect to D.

Lemma 4.5. *[HK12a, Corollary 3.3 (a)⇔(b)] Assume that T is a tree, $w \in \mathbb{R}_+^{\mathrm{A}(T)}$, X is a finite subset of $\mathrm{V}(T)$, and $\mathrm{D} = \mathrm{D}_{T,X,w}$. Let $\mathrm{D}' \in \mathbb{R}_+^{X \times X}$ be a function which is similar to D. Then there exists $w' \in \mathbb{R}_+^{\mathrm{A}(T)}$ such that $\mathrm{D}' = \mathrm{D}_{T,X,w'}$.*

Proof. Since D' and D are similar, there exists $f \in \mathbb{R}^X$ such that $\mathrm{D}'(x,y) = \mathrm{D}(x,y) - f(x) + f(y)$. It follows from the nonnegativity of D' that f is Lipschitz with respect to D. Let $g = \mathrm{D}_{T,w}(\mathrm{V}(T), X) \otimes f$, which is Lipschitz with respect to $\mathrm{D}_{T,w}$ according to Remark 3.5. Thanks to the fact that f is Lipschitz for D, we can check that g is an extension of f, namely $g|_X = f$. Now let w' be the arc weight on T such that $w'(\overrightarrow{uv}) = \mathrm{D}_{T,w}(u,v) - g(u) + g(v) \geq 0$ for all $\overrightarrow{uv} \in \mathrm{A}(T)$. Then we obtain $\mathrm{D}_{T,X,w'}(x,y) = \mathrm{D}_{T,w}(x,y) - g(x) + g(y) = \mathrm{D}_{T,w}(x,y) - f(x) + f(y) = \mathrm{D}'(x,y)$ for all $x, y \in X$, finishing the proof. ☐

Proposition 4.6. *Let* D *be a finite directed metric. Then* tconv([D]) *is a zonotope if and only if* D *is a directed tree metric.*

Proof. Proposition 4.3 and Lemma 4.5. ☐

Let X be a finite set and let $\gamma \in \mathbb{TT}^X \setminus \{[\mathbf{0}]\}$. It is easy to see that there exists a unique positive integer r, a unique chain of sets $X \supsetneq S_1 \supsetneq S_2 \supsetneq \cdots \supsetneq S_r \supsetneq \emptyset$, and a unique sequence of positive reals $t_1, \ldots, t_r$ such that

$$\gamma = [t_1 \sum_{x \in S_1} \delta_x + t_2 \sum_{x \in S_2} \delta_x + \cdots + t_r \sum_{x \in S_r} \delta_x]. \tag{4.1}$$

We call the sequence $(S_1, \ldots S_r)$ the *slope* of α and designate it by $\mathrm{slo}(\gamma)$. Notice that $\mathrm{slo}(-\gamma)=(X \setminus S_r, \ldots, X \setminus S_1)$ whenever $\mathrm{slo}(\gamma)= (S_1, \ldots, S_r)$. In case that $r = 1$ and so $\mathrm{slo}(\gamma) = (S_1)$, we write $\mathrm{supp}(\gamma)$ for S_1. The normed linear space $(\mathbb{TT}^X, |\,|_{\mathrm{H}})$ naturally carries the *tropical metric* $\mathrm{d_H}$ given by $\mathrm{d_H}(\alpha, \beta) = |\alpha - \beta\,|_{\mathrm{H}}$ for all $\alpha, \beta \in \mathbb{TT}^X$. Every metric space (X, D) can be isometrically embedded into $(\mathbb{TT}^X, \mathrm{d_H})$ [JK14, Corollary 5.7]: Similar to the Kuratowski map [DHK+12, p. 75] or the Yoneda map [Wil13, p. 703], we just send $x \in X$ to $[-\mathrm{D}(x)] \in \mathbb{TT}^X$ for every $x \in X$. For $\mathcal{F} = \{\alpha, \beta\} \subseteq \mathbb{TT}^X$, the tropical polytope $\mathrm{tconv}(\mathcal{F})$ is called the *tropical line segment* between α and β. Develin and Sturmfels [DS04b] determined explicitly the structure of tropical line segments; we reformulate their result in Lemma 4.7 below. Note that Lemma 4.7 says that each tropical line segment in $\mathbb{TT}^X$ is a geodesic in the metric space $(\mathbb{TT}^X, \mathrm{d_H})$. Some other interesting facts about tropical line segments can be found in [DJS12, Proposition 2.11] and [dlP14, Theorem 5.4].

Lemma 4.7. *[DS04b] Take two different points $\alpha, \beta \in \mathbb{TT}^X$ such that $\mathrm{slo}(\beta - \alpha) = (S_1, \ldots, S_r)$.*

(i) *There are $r - 1$ different points $\alpha_1, \ldots, \alpha_{r-1} \in \mathbb{TT}^X$, such that the tropical segment between α and β consists of a concatenation of r ordinary segments $\alpha_0\alpha_1, \ldots, \alpha_{r-1}\alpha_r$ in $\mathbb{TT}^X$, where $\alpha_0 = \alpha$, $\alpha_r = \beta$ and the slope of $\alpha_j - \alpha_i$ is $(S_{i+1}, \ldots, S_j)$ for any $0 \le i < j \le r$. In addition, for any $i \in \{0, \ldots, r-1\}$, if γ and γ' are from the ordinary segment $\alpha_i\alpha_{i+1}$ such that $\mathrm{d_H}(\alpha_i, \gamma) < \mathrm{d_H}(\alpha_i, \gamma')$, it happens $\mathrm{supp}(\gamma' - \gamma) = S_{i+1}$.*

(ii) *For $\mathcal{F} = \{\alpha, \beta\}$, $\mathcal{C}_{\mathcal{F}}$ consists of r edges $\alpha_0\alpha_1, \ldots, \alpha_{r-1}\alpha_r$, and $r + 1$ vertices $\alpha_0, \ldots, \alpha_r$.*

(iii) *For every point γ lying in the tropical line segment connecting α and β, it holds $\mathrm{d_H}(\alpha, \beta) = \mathrm{d_H}(\alpha, \gamma) + \mathrm{d_H}(\gamma, \beta)$.*

Proof. (i). We assume that $\beta-\alpha$ is given by (4.1) and then put $\alpha_i = \alpha+[t_1 \sum_{x\in S_1} \delta_x + t_2 \sum_{x \in S_2} \delta_x + \cdots + t_i \sum_{x \in S_i} \delta_x]$ for $i = 1, \ldots, r$. We refer the reader to [DS04b, Proposition 3] for more details and end the proof of (i).

(ii). For each appropriate index i, α_i forms a vertex of type $(\mathrm{S}_x)_{x \in X}$ where

$$\mathrm{S}_x \doteq \begin{cases} \{\alpha\}, & \text{if } x \in S_{i+1}, \\ \{\alpha, \beta\}, & \text{if } x \in S_i \setminus S_{i+1}, \\ \{\beta\}, & \text{if } x \in X \setminus S_i, \end{cases}$$

and the segment $\alpha_i\alpha_{i+1}$ forms an edge of type $(\mathrm{S}_x)_{x \in X}$ where

$$\mathrm{S}_x \doteq \begin{cases} \{\alpha\}, & \text{if } x \in S_{i+1}, \\ \{\beta\}, & \text{if } x \in X \setminus S_{i+1}. \end{cases}$$

(iii). Without loss of generality, assume that $\alpha \ne \gamma$. By (i), we can assume that $\gamma = t\alpha_i + (1 - t)\alpha_{i+1}$, where $0 \le t < 1$ and $0 \le i \le r - 1$. It follows that $\mathrm{d_H}(\alpha, \beta) = \frac{t_1+\cdots+t_r}{2} = \frac{\left(t_1+\cdots+t_i+(1-t)\times t_{i+1}\right)}{2} + \frac{\left(t\times t_{i+1}+t_{i+2}+\cdots+t_r\right)}{2} = \mathrm{d_H}(\alpha, \gamma) + \mathrm{d_H}(\gamma, \beta)$, as required. ☐

In light of Theorem 3.6 (i), each edge of a tropical complex is both a tropical segment and an ordinary line segment. According to Lemma 4.7 (i), for each edge $\alpha\beta$ of a tropical complex $\mathcal{C}_{\mathcal{F}}$ in $\mathbb{TT}^X$, we can thus get two sets $\mathrm{supp}(\alpha-\beta)$ and $\mathrm{supp}(\beta-\alpha)$ which form a partition of X, and we call this pair of sets the *split of X corresponding to $\alpha\beta$*. In phylogenetic combinatorics, there are rich results on the relationship between split systems and network structures [DHK+12, Chap. 4, Chap. 7]. It is natural to believe that the set of splits associated with a tropical complex as indicated above should also carry lots of information of the tropical complex. The next example is a small hint on this.

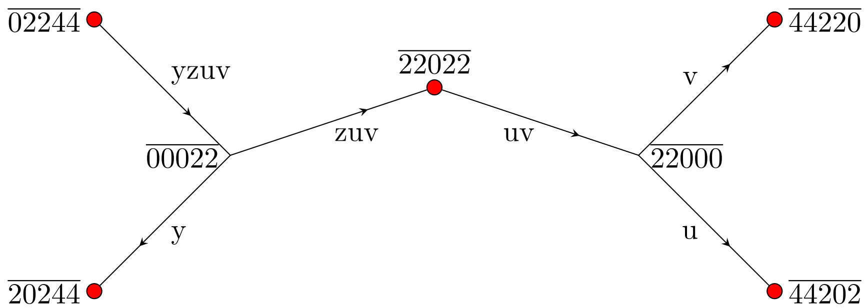

Figure 4.1: A 1-dimensional tropical polytope and its split system.

Example 4.8. Let $X = \{x, y, z, u, v\}$ and let D be the following tree metric on X:

$$\begin{matrix} x \\ y \\ z \\ u \\ v \end{matrix} \begin{pmatrix} 0 & 2 & 2 & 4 & 4 \\ 2 & 0 & 2 & 4 & 4 \\ 2 & 2 & 0 & 2 & 2 \\ 4 & 4 & 2 & 0 & 2 \\ 4 & 4 & 2 & 2 & 0 \end{pmatrix}.$$

Let $T \doteq \mathrm{tconv}([-\mathrm{D}])$. It turns out that $(T, \mathrm{d_H})$ is a metric tree, which along with one orientation of the 1-dimensional cells of $\mathcal{C}_{-\mathrm{D}}$ is displayed in Fig. 4.1. For each $f \in \mathbb{R}^X$, we use the string $\overline{f(x)f(y)f(z)f(u)f(v)}$ for $[-f] \in \mathbb{TT}^X$ and, with this convention, we mark all the vertices of $\mathcal{C}_{-\mathrm{D}}$ in Fig. 4.1; we also label each arc $\overrightarrow{\alpha\beta}$ in the orientation with $\mathrm{supp}(\beta-\alpha)$. According to the relevant discussion before Lemma 4.7, $\mathrm{D}(a, b) = \mathrm{d_H}\left([-\mathrm{D}(a)], [-\mathrm{D}(b)]\right)$ for all $a, b \in X$; that is, the tropical polytope T displays the tree metric D!

A metric space (X, D) is *proper* if $\mathrm{D}(x, y) > 0$ whenever x and y are different elements of X. From Item (i) of the definition of a metric tree, we see that every metric tree is proper.

Example 4.9. Let $X = \{a, b, c\}$ and let $\mathrm{D} \in \mathbb{R}^{X\times X}$ be a proper metric on X satisfying, without loss of generality, that $\mathrm{D}(c, a) \geq \mathrm{D}(b, c) \geq \mathrm{D}(a, b)$. We say that D is of star-type provided $\mathrm{D}(a, b) + \mathrm{D}(b, c) > \mathrm{D}(a, c)$ and we call D of path-type provided $\mathrm{D}(a, b) + \mathrm{D}(b, c) = \mathrm{D}(a, c)$. Indeed, D must be a tree metric: it can be represented by a weighted tree where the tree is a path in case that it is of path-type and it can be represented by a weighted star tree but not by any path when it is of star-type. When D is of star-type, $\mathrm{tconv}([\mathrm{D}])$ is the 2-cell surrounded by a hexagon as shown on the left of Fig. 4.2; when D is of path-type, $\mathrm{tconv}([\mathrm{D}])$ is a parallelogram as shown on the right of Fig. 4.2. In both cases, $-\mathrm{tconv}([-\mathrm{D}])$ is a 1-dimensional polytopal complex depicted with dashed lines: it is a star when D is of star-type and it is a path when D is of path-type. Evidently, for any $v \in \mathrm{tconv}([\mathrm{D}])$, $\mathcal{C}_{\mathrm{D}\cup\{v\}}$ is a zonotopal tiling if and only if $v \in -\mathrm{tconv}([-\mathrm{D}])$. In addition, for the star-type case, let v_d be the meeting point of the three dashed lines in Fig. 4.2 (see the right of Fig. 1.2) and let $v \in -\operatorname{tconv}([-\mathrm{D}]) \setminus \{v_d\}$, then $\mathcal{C}_{\mathrm{D}\cup\{v\}}$ is a zonotopal tiling which contains a non-parallellotope cell.

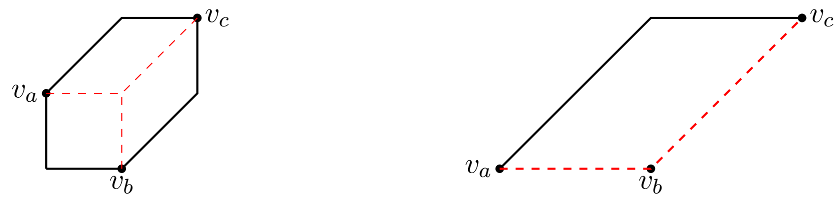

Figure 4.2: $\operatorname{tconv}([\mathrm{D}])$ and $-\operatorname{tconv}([-\mathrm{D}])$ for two kinds of proper metrics D on $\{a, b, c\}$.

The ensuing result is an effort towards understanding Examples 4.8 and 4.9. It is a minor extension of some results of Develin-Sturmfels and Hirai-Koichi by adapting a bit their arguments; see [DS04b, Theorem 28, Theorem 29], [DS04a] and [HK12a, Corollary 3.3].

Proposition 4.10. *Let X be a finite set and let $\mathrm{D} \in \mathbb{R}^{X\times X}$ be a Kleene star. The following are pairwise equivalent:*

(i) D *is tropically equivalent with a tree metric.*

(ii) D *is similar to a tree metric.*

(iii) $\mathrm{D}+\mathrm{D}^\top$ *is a tree metric and* $\mathrm{D}(a,b)+\mathrm{D}(b,c)+\mathrm{D}(c,a) = \mathrm{D}(a,c)+\mathrm{D}(c,b)+\mathrm{D}(b,a)$ *for all* $a, b, c \in X$.

(iv) $\operatorname{trank}(-\mathrm{D}) \leq 2$.

(v) $\operatorname{tconv}([-\mathrm{D}]$ *is a contractible closed subset of* $\mathbb{TT}^X$ *of dimension at most 1.*

(vi) $(\operatorname{tconv}([-\mathrm{D}]), \mathrm{d_H})$ *is a metric tree.*

Proof. (i) is surely a consequence of (ii). Let D' be a tree metric such that $\langle\mathrm{D}\rangle = \langle\mathrm{D}'\rangle$. By definition, there exists an $f \in \mathbb{R}^X$ such that $[\mathrm{D}] = [\mathrm{D}'] + [f]$. Take $\mathrm{D}'' \in \mathbb{R}^{X\times X}$ for which $\mathrm{D}''(x,y) = \mathrm{D}(x,y) - f(x) + f(y)$ for all $x, y \in X$. Then D'' is similar to D and $[\mathrm{D}''] = [\mathrm{D}']$. According to Theorem 3.6 (iii), the only possibility is that $\mathrm{D}'' = \mathrm{D}'$. This says that D is similar to D', and hence establishes the equivalence between (i) and (ii).

If D is similar to a tree metric D', then $\mathrm{D}-\mathrm{D}'$ vanishes cyclically and so (ii) implies (iii). The second condition in (iii) guarantees that $\mathrm{D}-\mathrm{D}^\top$ vanishes cyclicly and hence, by Eq. (3.11), $\mathrm{D} = \frac{\mathrm{D}+\mathrm{D}^\top}{2} + \frac{\mathrm{D}-\mathrm{D}^\top}{2}$ is similar to $\frac{\mathrm{D}+\mathrm{D}^\top}{2}$. Combined with the first condition of (iii), this tells us that (ii) is a consequence of (iii).

Observe that two matrices which are congruent to each other must have the same tropical rank. By Eq. (3.10), every Kleene star is similar to a directed metric. Recall that the equivalence of (b) and (d) in [HK12a, Corollary 3.3] says that a directed metric is similar to a tree metric if and only if its tropical rank is at most two. Taken together, we get the equivalence between (ii) and (iv).

Every tropical polytope P is contractible [DS04b, Theoem 2]. To see this, we first fix a point α in P. Then, due to Lemma 4.7, we can define a contraction map $\phi : [0,1]\times P \mapsto P$ such that $\phi(t,\beta)$ is the unique point γ on $\operatorname{tconv}(\alpha,\beta)$ such that $\mathrm{d_H}(\alpha,\gamma) = (1-t)\,\mathrm{d_H}(\alpha,\beta)$. Now the equivalence of (iv) and (v) follows from Theorem 3.4.

(vi) clearly implies (v). (v) combined with Lemma 4.7 (iii) implies (vi). ☐

Corollary 4.11. *Let A be a finite matrix such that both* $\operatorname{tconv}([A])$ *and* $\operatorname{tconv}([-A])$ *have dimension at least two. Then neither A nor $-A$ can be tropically equivalent with any submatrix of any tree metric.*

Proof. Let B be a finite submatrix of a tree metric. By the equivalence of (ii) and (iv) in Proposition 4.10, Theorem 3.4 enables us to find that $\operatorname{tconv}([B])$ has dimension at most 1. This implies the result, as claimed. □

Proposition 4.12. *Let* $\mathrm{D}' \in \mathbb{R}^{X\times X}$ *be a finite tree metric represented by a metric X-tree (T, D, X, ϕ). Then* $\operatorname{tconv}([-\mathrm{D}']) = -[\mathscr{C}(T, \mathrm{D}, X, \phi)]$.

Proof. By Lemma 3.2, $\operatorname{tconv}([-\mathrm{D}']) \supseteq -[\mathscr{C}(T, \mathrm{D}, X, \phi)]$.

For the other direction, as $-[\mathscr{C}(T, \mathrm{D}, X, \phi)] \supseteq [-\mathrm{D}']$, we only need to show that the tropical segment connecting any two points of $-[\mathscr{C}(T, \mathrm{D}, X, \phi)]$, say α and β, still lies in $-[\mathscr{C}(T, \mathrm{D}, X, \phi)]$. By the equivalence between (ii) and (v) in Proposition 4.10, $\operatorname{tconv}([-\mathrm{D}'])$ is contractible and has dimension at most 1. Therefore, we tell from Lemma 4.7 that any connected subspace of $\operatorname{tconv}([-\mathrm{D}'])$ which contains both α and β must contain $\operatorname{tconv}(\alpha, \beta)$. Note that $-[\mathscr{C}(T, \mathrm{D}, X, \phi)]$ is clearly a connected subspace of $\mathbb{TT}^X$. This means that $-[\mathscr{C}(T, \mathrm{D}, X, \phi)]$, as a subset of $\operatorname{tconv}([-\mathrm{D}'])$, must contain $\operatorname{tconv}(\alpha, \beta)$, as was to be shown. □

The concept of tight span was discovered independently by several groups of authors, including Isbell [Isb64], Dress [Dre84] and Chrobak-Larmore [CL94]. It is also known as the injective envelope or hyperconvex hull in different areas with different degrees of generality; see [DHK+12, DMT96, Lan13, Wil13] for an overview. For our purpose, for any matrix $\mathrm{D} \in \mathbb{R}^{X\times X}$ let us define the set $\{f \in \mathbb{R}^X : (-\mathrm{D}) \otimes f = -f\}$ to be the *tight span* of D and denote it by $\mathrm{T_D}$. It is straightforward from the definition that $[\mathrm{T_D}] \subseteq -\operatorname{tconv}(-[\mathrm{D}])$. Let (T, D') be a metric tree and take $X \subseteq T$. Recall the well-known fact [Dre84] that $\mathrm{T}_{\mathrm{D}'(X,X)} = \{\mathrm{D}'(X, p) : p \in \operatorname{conv}_T(X)\}$; we refer the reader to [Kåh03, p. 35] for a very friendly presentation of this fact. Combined with Proposition 4.12, we see that, for the tree metric $\mathrm{D} = \mathrm{D}'(X, X)$, $-\operatorname{tconv}(-[\mathrm{D}])$ is exactly the projection of $\mathrm{T_D}$ into the tropical projective torus; this was first observed by Develin and Sturmfels [DS04a, DS04b] via an application of some tools in tropical geometry, say Lemma 5.2 to be introduced in next section. We pointed out some more relationship between tight span and tropical geometry in [WX17] and believe that further investigation along this line should be interesting.

5 Tropical complex and zonotopal tiling

We start from a result which may be aware of by Minkowski already. The only place in which we could find its explicit statement is a recent work by Gover. Lemma 5.1 means that what are known as zonotopal subdivisions of zonotopes in the literature, say [Rei99], are all possible zonotopal tilings.

Lemma 5.1. *[Gov14, Corollary 1.7] A polytope is a zonotope if and only if it admits a zonotopal tiling.*

We continue to introduce a fundamental result of Develin and Sturmfels [DS04b] on the duality between row space and column space of a matrix in tropical geometry. For full statement of this duality beyond Lemma 5.2, we refer to [CGQ04, Theorem 42] [HK12b, §3] [JK14, §6][MS15, Theorem 5.2.21]. We remark that this duality is somehow similar to the duality of co-presheaf and presheaf of a directed metric as discussed in [Wil13, p. 715].

Lemma 5.2. *[DS04b, Lemma 22] [MS15, Lemma 5.2.18] [FR15, Proposition 6.2] Let X and Y be two finite sets and let $\mathrm{D} \in \mathbb{R}^{X\times Y}$. Let ϕ be the map defined on $\operatorname{tconv}([\mathrm{D}^\top])$ such that $\phi([g]) = [\mathrm{D} \otimes (-g)]$ and let ψ be the map defined on $\operatorname{tconv}([\mathrm{D}])$ such that $\psi([f]) = [\mathrm{D}^\top \otimes (-f)]$. Then the following hold.*

(i) *The map ϕ is a bijection from $\operatorname{tconv}([\mathrm{D}^\top])$ to $\operatorname{tconv}([\mathrm{D}])$ and the map ψ is its inverse.*

(ii) *$\{\mathcal{T}_{\mathrm{D},C} :\ C \in \mathcal{C}_\mathrm{D}\} = \{\mathcal{T}^\top_{\mathrm{D}^\top,C'} :\ C' \in \mathcal{C}_{\mathrm{D}^\top}\}$; namely, S is the type of a cell of $\mathcal{C}_\mathrm{D}$ if and only if $\mathrm{S}^\top$ is the type of a cell of $\mathcal{C}_{\mathrm{D}^\top}$.*

(iii) *For any $C \in \mathcal{C}_\mathrm{D}$, which is of type $\mathrm{S} \doteq \mathcal{T}_{\mathrm{D},C}$, ψ restricted on $C \subseteq \operatorname{tconv}([\mathrm{D}])$ is an affine isomorphism from $C = \mathrm{C}_{\mathrm{S},\mathrm{D}} \in \mathcal{C}_\mathrm{D}$ to $\mathrm{C}_{\mathrm{S}^\top,\mathrm{D}^\top} \in \mathcal{C}_{\mathrm{D}^\top}$, while ϕ restricted on $\mathrm{C}_{\mathrm{S}^\top,\mathrm{D}^\top} \subseteq \operatorname{tconv}([\mathrm{D}^\top])$ is the inverse of $\psi|_C$ and so an affine isomorphism from $\mathrm{C}_{\mathrm{S}^\top,\mathrm{D}^\top} \in \mathcal{C}_{\mathrm{D}^\top}$ to $C = \mathrm{C}_{\mathrm{S},\mathrm{D}} \in \mathcal{C}_\mathrm{D}$.*

Remark 5.3. Let X and Y be finite sets and let $\mathrm{D} \in \mathbb{R}^{X\times Y}$. Since being a zonotope is a property invariant under affine transformations, Lemma 5.2 implies that $\mathcal{C}_{\mathrm{D}^\top}$ is zonotopal if and only if $\mathcal{C}_\mathrm{D}$ is zonotopal.

Lemma 5.4. *Let X and Y be two finite sets. Let $\mathrm{D} \in \mathbb{R}^{X\times Y}$, $U \in 2^X \setminus \emptyset$ and $V \in 2^Y \setminus \emptyset$.*

(i) *If $\mathcal{C}_\mathrm{D}$ is zonotopal, then $\mathcal{C}_{\mathrm{D}(U,V)}$ is zonotopal.*

(ii) *If $\mathcal{C}_\mathrm{D}$ is a zonotopal tiling, then $\mathcal{C}_{\mathrm{D}(U,Y)}$ is a zonotopal tiling.*

Proof. (i). Recall from Remark 3.3 that each cell of a tropical complex is a polytope. Therefore, as $\mathcal{C}_\mathrm{D}$ is zonotopal, we know that each cell of $\mathcal{C}_{\mathrm{D}(X,V)}$ admits a zonotopal tiling and hence, by Lemma 5.1, $\mathcal{C}_{\mathrm{D}(X,V)}$ is zonotopal. We next apply Remark 5.3 to find that $\mathcal{C}_{\mathrm{D}^\top(V,X)}$ is zonotopal. By the same argument of showing that $\mathcal{C}_{\mathrm{D}(X,V)}$ is zonotopal, we can now find that $\mathcal{C}_{\mathrm{D}^\top(V,U)}$ is zonotopal. Finally, an application of Remark 5.3 again implies that $\mathcal{C}_{\mathrm{D}(U,V)}$ is zonotopal.

(ii). In light of (i), we only need to show that $\operatorname{tconv}\big([\mathrm{D}(U,Y)]\big)$ is a zonotope. Let p^X_U be the linear map from $\mathbb{TT}^X$ to $\mathbb{TT}^U$ sending $[f] \in \mathbb{TT}^X$ to $[f|_U] \in \mathbb{TT}^U$. It is readily true that $\operatorname{tconv}\big([\mathrm{D}(U,Y)]\big) = \mathrm{p}^X_U(\operatorname{tconv}([\mathrm{D}]))$. We know from Lemma 5.1 that $\operatorname{tconv}([\mathrm{D}])$ is a zonotope, and thus so is its projection image $\operatorname{tconv}\big([\mathrm{D}(U,Y)]\big)$. □

Remark 5.5. Lemma 5.4 (ii) is about row submatrices. Its column counterpart does not hold. For example, for the points v_a, v_b, v_c displayed on the left of Fig. 4.2, $\mathcal{C}_{\{v_a,v_b,v_c\}}$ is a zonotopal tiling but $\operatorname{tconv}(v_a, v_b)$ is not any zonotope.

Proposition 5.6. *Let X be a finite set and let $\mathrm{D} \in \mathbb{R}^{X\times X}$ be a tree metric.*

(i) *If D' is a submatrix of D or $-\mathrm{D}$, then $\mathcal{C}_{\mathrm{D}'}$ is zonotopal.*

(ii) *If D' is a row submatrix of D, then $\mathcal{C}_{\mathrm{D}'}$ is a zonotopal tiling.*

Proof. The claim about D in (i) comes from Corollary 4.2 and Lemma 5.4 (i). The remaining claim in (i) follows from the equivalence between Item (ii) and Item (v) in Proposition 4.10 as well as Lemma 5.4 (i). Corollary 4.2 together with Lemma 5.4 (ii) ensures the truth of (ii). □

Example 5.7. In Fig. 5.1, we depict some tropical complexes which are zonotopal but do not form any zonotopal tiling. Let $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $\operatorname{tconv}([A]) = \operatorname{tconv}([-A])$ can be represented as the rightmost figure of Fig. 5.1. If B is a submatrix of a tree metric, then as illustrated by Corollary 4.11, neither $\langle B \rangle$ nor $\langle -B \rangle$ could coincide with $\langle A \rangle$. Besides this rightmost one, the other three zonotopal tropical complexes in Fig. 5.1 all arise from point configurations as addressed in Proposition 5.6 (i).

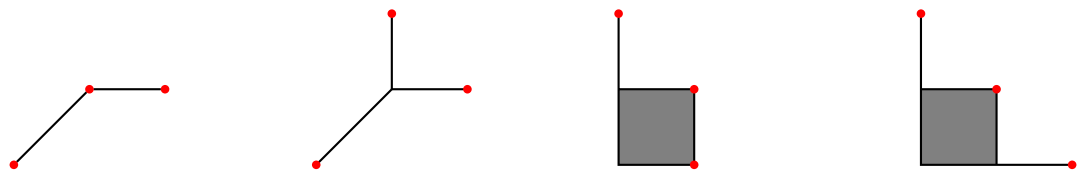

Figure 5.1: Four zonotopal tropical complexes generated by three points in $\mathbb{R}^3 / \mathbb{R} \cdot \mathbf{1}$.

Proof of Theorem 1.2. By virtue of Proposition 5.6 (ii), we only need to prove the forward direction.

Let X and Y be two finite sets and let $\mathrm{D} \in \mathbb{R}^{X \times Y}$ such that $\mathcal{C}_{\mathrm{D}}$ is a zonotopal tiling. By Lemma 5.1, $\operatorname{tconv}([\mathrm{D}])$ is a zonotope. Henceforth, by Corollary 4.4, there exists a tree metric $M \in \mathbb{R}^{X \times X}$ and a vector $f \in \mathbb{R}^X$ such that $\operatorname{tconv}([\mathrm{D}]) = [f] + \operatorname{tconv}([M])$. We may assume that M is the tree metric corresponding to the metric X-tree $(T, \mathrm{D}_T, X, \phi)$. Taking $\mathrm{D}' \in \mathbb{R}^{X \times Y}$ such that $\mathrm{D}'(y) = \mathrm{D}(y) - f$ for all $y \in Y$, we then get $\operatorname{tconv}([\mathrm{D}']) = \operatorname{tconv}([M])$. According to Theorem 3.6 (vi), this leads to $[\mathrm{D}'] \supseteq [M]$ and so, there exists a matrix $\mathrm{D}'' \in \mathbb{R}^{X \times Z}$ for some set $Z \supseteq X$ such that $M = \mathrm{D}''(X, X)$ and $[\mathrm{D}''] = [\mathrm{D}'] = [\mathrm{D}] - [f]$. To finish the proof, we should seek a tree metric $\mu \in \mathbb{R}^{Z \times Z}$ such that $[\mathrm{D}''(y)] = [\mu(X, y)]$ for all $y \in Z$.

For every $p \in T$, let $\alpha^p \in \mathbb{R}^X$ be the function such that $\alpha^p(x) = \mathrm{D}_T(\phi(x), p)$ for all $x \in X$. We intend to find a map Φ from Z to T such that $\Phi|_X = \phi$ and that

$$[\mathrm{D}''(y)] = [\alpha^{\Phi(y)}] \tag{5.1}$$

holds for all $y \in Z$. If this is done, then the required tree metric $\mu \in \mathbb{R}^{Z \times Z}$ can be obtained by putting $\mu(x, y) = \mathrm{D}_T(\Phi(x), \Phi(y))$ for all $x, y \in Z$.

When $y \in X$, considering that $\mathrm{D}''(x, y) = M(x, y) = \mathrm{D}_T(\phi(x), \phi(y))$ holds for all $x \in X$, we see that $\Phi(y) \doteq \phi(y)$ fulfils Eq. (5.1). Now assume that $y \in Z \setminus X$. Take an arbitrary $U \in \binom{X}{\leq 3}$ an denote $V = U \cup \{y\}$. Since $[\mathrm{D}''(y)] \in \operatorname{tconv}([\mathrm{D}'']) = \operatorname{tconv}([\mathrm{D}']) = \operatorname{tconv}([M])$ and M is a tree metric, from Remark 3.5 we can derive that $[\mathrm{D}''(U, y)] \in \operatorname{tconv}([M(U, U)])$. Consequently,

$$\operatorname{tconv}([\mathrm{D}''(U, V)]) = \operatorname{tconv}([M(U, U)]), \tag{5.2}$$

which is a zonotope by Corollary 4.4. Because $\mathcal{C}_{\mathrm{D}}$ is a zonotopal tiling, so is $\mathcal{C}_{\mathrm{D}''}$ and then, by Lemma 5.4 (i) and the fact that $\operatorname{tconv}([\mathrm{D}''(U, V)])$ is a zonotope, we conclude that $\mathcal{C}_{\mathrm{D}''(U,V)}$ is a zonotopal tiling. Note that

$$[M(U, U)] = [\mathrm{D}''(U, U)]. \tag{5.3}$$

We further claim that

$$[\mathrm{D}''(U, y)] \in -\operatorname{tconv}([-\mathrm{D}''(U, U)]). \tag{5.4}$$

If $|U| \leq 2$ or $\mathrm{D}''(U, U)$ is a non-proper metric, then $\operatorname{tconv}([\mathrm{D}''(U, U)]) = -\operatorname{tconv}([-\mathrm{D}''(U, U)])$ is an ordinary line segment and so, by virtue of Eqs. (5.2) and (5.3), Eq. (5.4) trivially holds. If

$|U| = 3$ and $\mathrm{D}''(U,U)$ is a proper metric, Eq. (5.4) is guaranteed by Example 4.9. Since Eqs. (5.3) and (5.4) holds for all $U \in \binom{X}{\leq 3}$, applying Theorem 1.1 and Proposition 4.12 now yields a point $\Phi(y) \in T$ which fulfils Eq. (5.1). This is the end of the proof. $\square$

6 Further research

Let $\mathrm{D} \in \mathbb{R}^{X \times X}$ be a finite tree metric and let Y be a nonempty subset of X. By Theorem 1.2 and its proof, $\mathcal{C}_{\mathrm{D}(Y,X)}$ is a zonotopal tiling of $\mathrm{tconv}(\mathrm{D}(Y,X)) = \mathrm{tconv}(\mathrm{D}(Y,Y))$. Notice that the oriented matroid of $\mathrm{tconv}(\mathrm{D}(Y,Y))$ is already determined in [WX17, Theorem 4.2]. In view of Lemma 5.1, we should mention the Bohne-Dress theorem [Boh92, Dre89, RGZ94] which provides a bijection between the zonotopal tilings of a fixed zonotope P and the single-element liftings of the oriented matroid corresponding to P. Which kind of single-element liftings of the oriented matroid of $\mathrm{tconv}(\mathrm{D}(Y,Y))$ correspond to the zonotopal tilings of the form $\mathcal{C}_{\mathrm{D}(Y,X)}$? With Theorem 1.2 in mind, let us call a zonotopal tiling which can be realized as a tropical complex a *tree-like zonotopal tiling*. How to characterize tree-like zonotopal tilings among al zonotopal tilings? When is a tree-like zonotopal tiling centrally symmetric? For any zonotopal tiling (zonotopal subdivision of a zonotope), we can define whether or not it is coherent; see [Rei99, p. 302]. What is the relationship between coherent zonotopal tilings and tree-like zonotopal tilings? Similar to the Baues problem [Rei99], how to understand the poset of all tree-like zonotopal tilings of a fixed zonotope?

With the background of tree-indexed Markov chains [LNR03], we [WXZ16, WX17] studied the average Hilbert norm of the points uniformly distributed in some special tropical polytopes related to trees. If we now have a tree-like zonotopal tiling, how to calculate the volumes of all its cells and how to estimate the average Hilbert norm of points in each cell? Which kind of tree information is encoded by them? There are some natural operations to go from one weighted tree to another weighted tree. Can we associate some of those tree operations with some natural mutations between tree-like zonotopal tilings?

Before Example 4.8, for each tropical complex in $\mathbb{TT}^X$ we assign to each edge of it a split of X and thus define a split system of the tropical complex. Which kind of split system can arise that way? How can we read the combinatorics of the point configuration $S \subseteq \mathbb{TT}^X$ from the split system of $\mathcal{C}_S$? Especially, when $\mathcal{C}_S$ is a zonotopal tiling and so we can say S is tree-like by Theorem 1.2, which information can we glean from the split system of $\mathcal{C}_S$?

Examples 4.9 and 5.7 suggest the widely open problems of understanding those point configurations whose tropical complexes are zonotopal and also those whose tropical complexes only have parallellotopes as their cells.

Let A and B be two elements of $\mathbb{R}^{X \times X}$. As we see from some arguments in § 4, sometimes we could derive Eq. (3.8) or even Eq. (3.9) from the assumption that $\mathcal{C}_A$ is a translation of $\mathcal{C}_B$. Is there any more general conditions on A and B which can guarantee this line of results?

We discuss the model of weighted trees before Lemma 4.5, where all edge/arc weights should be nonnegative. Every tree metric basically just arises from such a weighted tree and the main work in this paper is to tell the shape of the tropical complex generated by tree metrics. If we allow negative weights on some edges and define the "distance data" accordingly to generate the tree-like point configurations, what can we say on their tropical complexes?

Note that Theorem 1.1 can also be said to be about some local-global property of the projection of the tight span of finite tree metric into the tropical projective torus. Can we find any other metric spaces for which we can establish a result in analogy with Theorem 1.1?

A *zonotopal tessellation* or *zonotopal filling* is a subdivision of a whole Euclidean space into zonotopes for which we can find a number $\epsilon > 0$ such that no two different vertices (0-dimensional cells) of the subdivision can have distance less than ϵ. Zonotopal tessellation has been the subject of many interesting researches [AYP13, BDF92, CS97, dB81, Erd99].

Example 6.1. For $X = \{x, y, z\}$, if S consists of infinite many points on all three rays of $\mathcal{F}_X$ as shown in Fig. 1.1 such that the distance between any two distinct points is always bigger than a fixed positive number, then the tropical complex of S will be a zonotopal tessellation of $\mathbb{TT}^X$.

As zonotopal tessellation is a close relative of zonotopal tiling, Theorem 1.2 motivates us to ask the following question: For a finite set X, which kind of tropical complex in $\mathbb{TT}^X$ can give rise to a zonotopal tessellation? Dress and Terhalle founded the theory of tight span of valuated matroids [DT98, Ter98]. Especially, this enables us to discuss the tight span of infinite metric trees [Kåh03, §4.3]. Let us write T for the union of the three rays of $\mathcal{F}_X$ in Example 6.1 and let us mention that $\{-\alpha : \alpha \in T\}$ is the projection into the tropical projective torus of the tight span of an infinite tree with three ends. We can understand Theorem 1.2 from the viewpoint of tight span; see the discussion at the end of § 4. We believe that zonotopal tessellations which come from tropical complexes essentially all arise from the tight span of infinite metric trees, i.e., all such zonotopal tessellations are produced in a way like Example 6.1. We will investigate this issue in another paper and try to show that the tight span construction will provide a correct framework to generalize Theorem 1.2.

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