

Submodular functions and rooted trees^{*}

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Abstract. For any positive number k and for any hypergraph H with vertex set $V(H)$ and edge set $E(H) \subseteq 2^{V(H)}$, we call $U \subseteq V(H)$ a k -antimatching of H provided for every matching $F \subseteq E(H)$ it holds $\text{rank } A[U, F] \leq k$, where A is the $V(H) \times E(H)$ $(0, 1)$ matrix whose (v, e) -entry is 1 if and only if $v \in e$. Consider a finite poset P with a unique maximal element and having a rooted tree as its Hasse diagram. Let H be the hypergraph with $V(H) = P$ and with $E(H)$ being the set of all down-sets of P . Let μ be a submodular function defined on $2^{V(H)}$ such that $\mu(V(H)) \geq d + (\ell - 1)c$ for a positive integer ℓ and two nonnegative reals d and c . For any nonnegative reals d_1, \dots, d_ℓ with $\sum_{i=1}^{\ell} d_i = d$, we show that either there is a matching $\{D_1, \dots, D_\ell\}$ of H with $\mu(D_i) \geq d_i$ for all i , or there is a 1-antimatching C of H such that $\mu(C) \geq c$. We establish a countable version of this result by assuming further that μ satisfies the weak Fatou property and reverse Fatou property. We propose a conjecture on a possible extension of our result from 1-antimatching to general k -antimatching.

Keywords: (α, β) down-set · anticore · BBT spanning tree · pseudorandom · weighted hypergraph.

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1 Introduction

1.1 Matching and antimatching

Let \mathbb{F} be a field, let V and E be two sets, and let $M \in \mathbb{F}^{V \times E}$. Take a positive integer k . For any subset \mathcal{E} of 2^E , we can consider $\mathcal{V} = \{A \subseteq V : \text{rank}_{\mathbb{F}}(M[A, B]) \leq k, \forall B \in \mathcal{E}\}$. Do you think that there may exist some kind of duality between \mathcal{E} and \mathcal{V} ? If so, in which sense can you say that \mathcal{E} and \mathcal{V} correlate with each other? To make life easier, let us move to a special case where M is a zero-one matrix, namely the incidence matrix of a hypergraph.

Let H be a hypergraph, which consists of its vertex set $V(H)$ and edge set $E(H) \subseteq 2^{V(H)}$. To emphasize that we are considering a hypergraph, we often call each edge of the hypergraph H a **hyperedge** of H . When we require $\emptyset \in E(H)$, the hypergraph H is called a paved space in measure theory [13, Chapter 3]. For each positive integer k , a **k -matching** of H is a set of k disjoint hyperedges of H , while a **k -antimatching** of H is a subset C of $V(H)$ which is disjoint from at least one member of any $(k + 1)$ -matching of H [60]. Note that a k -antimatching is just a set which cannot be a transversal of any $(k + 1)$ -matching. The **matching number** of H is the largest size of a matching of H , which we denote by $\nu(H)$. For each $B \subseteq V(H)$, let $H[B]$ be the hypergraph with $V(H[B]) = B$ and $E(H[B]) = \{A \cap B : A \in E(H)\} \setminus \{\emptyset\} \subseteq 2^B$. Clearly, if $\emptyset \in E(H)$, then $\nu(H) = \nu(H[V(H)]) + 1$. Note that C is a k -antimatching of H if and only if $\nu(H[C]) \leq k$. It is also not hard to check that the set \mathcal{E} of all matchings and the set \mathcal{V} of all k -antimatchings of H addressed here form a dual pair as described in the opening paragraph.

Dualities/reciprocities/complementarities are ubiquitous in sciences and mathematics [4,6,39]. But more often we see concepts which seem to have some weakened version of duality or orthogonality [57, §2.3]. The most famous example may be various uncertainty principles between signal duration in the time domain and

signal bandwidth in the frequency domain, which have both product version and sum version [21,41,47]. Let us see if some uncertainty relation between matching and antimatching can be established in some situations.

Example 1. Let G be a graph, namely a pair $(V(G), E(G))$ where $E(G) \subseteq \binom{V(G)}{2}$. For each $v \in V(G)$, the star at v of G is the set $\{e \in E(G) : v \in e\}$, which we denote by $\text{St}_G(v)$. We construct a hypergraph \mathcal{H}_G such that $V(\mathcal{H}_G) = E(G)$ and $E(\mathcal{H}_G) = \{\text{St}_G(v) : v \in V(G)\}$. If G has at most one isolated vertex, namely a vertex v with $\text{St}_G(v) = \emptyset$, then there is a natural bijection from the family of independent sets of G to the set of matchings of \mathcal{H}_G which sends $S \subseteq V(G)$ to $\{\text{St}_G(v) : v \in S\}$. It is also easy to check that a maximum size 1-antimatching of \mathcal{H}_G corresponds to the set of all edges inside a maximum size clique of G . If G is a perfect graph, we surely know that its independence number times its clique number is no smaller than its number of vertices, which can be interpreted as some uncertainty inequality between the sizes of a matching and a 1-antimatching. Indeed, this characterizes perfect graphs [36].

A digraph Γ has a vertex set $V(\Gamma)$ and an arc set $A(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$. We say that x can reach y in a digraph Γ if there is a path from x to y in Γ . For any $x \in V(\Gamma)$, the up-set of x in Γ , denoted by $x \uparrow_\Gamma$, is the set of all vertices of Γ which can reach x ; the down-set of x in Γ , denoted by $x \downarrow_\Gamma$, is the set of all vertices of Γ which can be reached by x . Note that $x \in (x \uparrow_\Gamma) \cap (x \downarrow_\Gamma)$. In general, for any $W \subseteq V(\Gamma)$, we let $W \uparrow_\Gamma = \cup_{x \in W} x \uparrow_\Gamma$ and $W \downarrow_\Gamma = \cup_{x \in W} x \downarrow_\Gamma$; We call W an **up-set** of Γ provided $W = W \uparrow_\Gamma$ and we call it a **down-set** of Γ provided $W = W \downarrow_\Gamma$ [12, p. 20]. In graphical models, an up-set is called an ancestral set or an upper set, while a down-set is called a descendant set or a lower set [40]. In topological dynamics, an up-set of a Boolean algebra is called a Furstenberg family [3]. For each digraph Γ , we define its **down-set hypergraph**, denoted by H_Γ , to be the hypergraph with $V(H_\Gamma) = V(\Gamma)$ such that $D \in E(H_\Gamma)$ if and only if D is a down-set of Γ . Given a topological space, we can view all its open sets as hyperedges and thus get a hypergraph. In this sense, a down-set hypergraph is often known as an Alexandroff space [8], which is a topological space in which the intersection of arbitrary family of open sets is still open. In combinatorics, down-set hypergraphs appear in the context of poset antimatroid or poset greedoid or poset convex geometry, which is a fundamental class of antimatroids and convex geometries [15,34]. Finally, we mention that down-set hypergraphs play a key role in the work of lattice representations [24,48].

Recall that a poset is just a transitive acyclic digraph Γ , namely $y \in x \uparrow_\Gamma$ and $x \in y \uparrow_\Gamma$ imply that $x = y$, while $y \in x \uparrow_\Gamma$ and $z \in y \uparrow_\Gamma$ imply $z \in x \uparrow_\Gamma$. A rooted tree poset is a poset T in which 1) there is a special root vertex r with $r \downarrow_T = V(T)$, 2) for every $x \in V(T)$, the number of paths in T leading from r to x is finite, and 3) $(x \downarrow_T) \cap (y \downarrow_T)$ equals $x \downarrow_T$, $y \downarrow_T$, or \emptyset , for all $x, y \in V(T)$. A **ray** in a rooted tree poset is a maximal chain in the poset. A ray either forms a finite path which starts from the root and ends at a leaf (a minimal element in the poset), or forms an infinite path starting at the root. An **end** in a rooted tree poset is a set of the form $R \setminus U$ where R is a ray and U is an up-set. A

rooted tree [35,38] is the Hasse diagram of a rooted tree poset, which is an out-tree rooted at the maximum element of the poset, also known as the root of the rooted tree; In other words, a rooted tree poset is simply the transitive closure of a rooted tree. When drawing a rooted tree without indicating its root, we follow the convention that all arcs go downwards and so the highest vertex in the figure is regarded as the root. For example, Fig. 1 demonstrates a tree rooted at r . When we talk about a chain, a ray, or an end of a rooted tree, we mean a one in its poset.

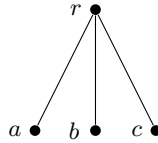


Fig. 1. A tree T rooted at r .

Example 2. For a rooted tree T and a positive integer k , a k -antimatching of H_T is a subset of a union of at most k rays, or equivalently, a union of at most k chains.

1.2 Width and height of a weighted rooted tree

According to Talagrand [54, p. 5], besides the fact that it is much easier to find the crux of the matter in a simple structure than in a complicated one, there are not so many really basic structures, so one can hope that they will remain of interest for a very long time. The simple structure of our concern is the set of rooted trees as discussed in Example 2. A **weighted hypergraph** is a pair (H, μ) where H is a hypergraph and μ is a map from $2^{V(H)}$ to \mathbb{R} which is known as a weighting function. Similarly, we define **weighted digraphs**, **weighted graphs** and their weighting functions. The most usual weighting function is the counting measure, which maps a set to its size. Due to the analogy between counting and measure, it is an active area to extend results from combinatorics to continuous combinatorics [19,32]; Moreover, as pointed out by Lovász [37], it deserves to replace linear functions by submodular functions in many combinatorial studies and see what will happen. Motivated by previous work in [9,53], we obtain an uncertainty relation for a rooted tree weighted by a signed measure [60, Theorem 1]. This paper aims to generalize that result from measures to submodular functions. Before stating our main result, we need to develop some more definitions.

For any positive integer k , we write $[k]$ for the set of the first k positive integers and we use Δ_{k-1} to designate the $(k-1)$ -dimensional probability simplex, namely $\Delta_{k-1} = \{(\delta_1, \dots, \delta_k) : \sum_{i \in [k]} \delta_i = 1, \delta_i \geq 0, \forall i \in [k]\}$. Consider a weighted hypergraph (H, μ) . A k -matching of H is a (d_1, \dots, d_k) -**matching in**

(H, μ) if its elements can be enumerated as D_1, \dots, D_k such that $\mu(D_i) \geq d_i$ for $i \in [k]$.

Definition 1. Let (H, μ) be a weighted hypergraph.

- For any positive integer k and any nonnegative real d , we say that (H, μ) is (d, k) -**fat** provided for any $(\delta_1, \dots, \delta_k) \in \Delta_{k-1}$, (H, μ) has a $(d\delta_1, \dots, d\delta_k)$ -matching.
- For any positive integer k , the k -**width** of (H, μ) is the supremum of the set of those reals d such that (H, μ) is (d, k) -fat.
- For any positive integer t and any nonnegative real c , we say that (H, μ) is (c, t) -**tall** provided we can find a t -antimatching C of H such that $\mu(C) \geq c$.
- For any positive integer t , the t -**height** of (H, μ) is the supremum of the set of those reals c such that (H, μ) is (c, t) -tall.

Example 3. Let T be the rooted tree depicted in Fig. 2 and let μ be the counting measure on $V(T)$. The set of blue vertices and the set of green vertices are both down-sets of T and hence hyperedges of \mathbb{H}_T . The 2-matching formed by them indeed gives rise to a $(4, 4)$ -matching of (\mathbb{H}_T, μ) . One can check that (\mathbb{H}_T, μ) is $(7, 2)$ -fat and $(3, 1)$ -tall.

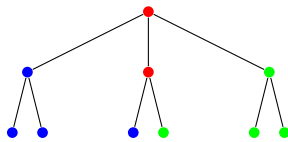


Fig. 2. A rooted tree T ; See Example 3.

Example 4. Let T be the rooted tree on three vertices with two leaves x and y . Let $d = \frac{2}{3}$ and $c = \frac{1}{3}$. Let Δ be the set of probability measures on $V(T)$ and let Leb be the Lebesgue measure on Δ . Let $X := \{\mu \in \Delta : (\mathbb{H}_T, \mu) \text{ is } (c, 1)\text{-tall}\}$ and $Y := \{\mu \in \Delta : (\mathbb{H}_T, \mu) \text{ is } (d, 2)\text{-fat}\}$. We can calculate that $\text{Leb}(X) = \text{Leb}(\Delta) - \text{Leb}(\{\mu \in \Delta : \mu(x) + \mu(r) < \frac{1}{3}, \mu(y) + \mu(r) < \frac{1}{3}\}) = \text{Leb}(\Delta)$ and that $\text{Leb}(Y) = \text{Leb}(\{\mu \in \Delta : \max\{\mu(x), \mu(y)\} \geq \frac{2}{3}, \min\{\mu(x), \mu(y)\} \geq \frac{1}{3}\}) = 0$. For each positive integer k , we are wondering what are the expected k -width and expected k -height of (\mathbb{H}_T, μ) when μ runs through all probability measures in Δ .

In mathematics and its applications, we discuss various interesting set functions, say measures, capacities and others [23]. Let X be a set and let μ be a function from 2^X to \mathbb{R} . The function μ is said to be increasing if $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$. If μ is increasing and grounded, that is, $\mu(\emptyset) = 0$, it is called

a capacity or a cooperative game [23, p. 27]. For any $t \geq 2$, we say that μ is t -alternating if for all $A_1, \dots, A_t \in 2^X$,

$$\mu\left(\bigcap_{i=1}^t A_i\right) \leq \sum_{\emptyset \neq I \subseteq [t]} (-1)^{|I|+1} \mu\left(\bigcup_{i \in I} A_i\right),$$

and we say that μ is t -monotone if $-\mu$ is t -alternating. A 2-monotone function is called **supermodular** and a 2-alternating function is **submodular**. We mention that entropy functions [20], polymatroid rank functions [16], and connectivity functions [27] are some well-known examples of submodular functions. Besides them, we also encounter submodular functions in many new contexts [25, Theorem 7]. We call μ a **subadditive function** if

$$\mu(A) + \mu(B) \geq \mu(A \cup B) \quad (1)$$

holds for all disjoint sets $A, B \subseteq X$. Taking $A = B = \emptyset$ in Eq. (1) shows that $\mu(\emptyset) \geq 0$ for all subadditive functions μ . Important classes of subadditive functions include the set of outer measures [56, §1.7] and the class of submeasures [55]. We call μ a **superadditive function** provided $-\mu$ is subadditive, and we call μ an **additive function** if it is both subadditive and superadditive. We call μ a **weakly submodular function** if

$$\mu(A) + \mu(B) \geq \mu(A \cup B) + \mu(A \cap B) = \mu(A \cup B) + \mu(\emptyset) \quad (2)$$

holds for all disjoint subsets A and B of X . If μ is weakly submodular and $\mu(\emptyset) \geq 0$, then μ is subadditive. A submodular capacity defined on a finite set is a polymatroid. We say that μ has the **weak Fatou property** provided

$$\limsup_{n \rightarrow \infty} \mu(X_n) \geq \mu(X)$$

for every sequence $X_1 \subseteq X_2 \subseteq \dots$ of subsets of X such that $X = \bigcup_{n=1}^{\infty} X_n = \lim_{n \rightarrow \infty} X_n$. We say that μ has the **reverse Fatou property** if

$$\mu(\limsup_{n \rightarrow \infty} X_n) \geq \limsup_{n \rightarrow \infty} \mu(X_n)$$

for any sequence (X_n) of subsets of X . We call μ an **F -continuous function** if it satisfies both the weak Fatou property and the reverse Fatou property. Note that a finite measure μ is basically just an F -continuous additive function – more precisely, we only need to define μ on a σ -algebra instead of the whole powerset of X .

Example 5. Let $X = \{a, b\}$ and let μ be the set function on X such that $\mu(\emptyset) = 0$, $\mu(a) = \mu(b) = 2$ and $\mu(X) = 1$. Clearly, μ is submodular. Consider a sequence $\{X_n\}_{n \in \mathbb{N}}$ of subset of X , where $X_n = \{a\}$ if n is even and $X_n = \{b\}$ otherwise. Then $\mu(\limsup_{n \rightarrow \infty} X_n) = \mu(X) = 1$ and $\limsup_{n \rightarrow \infty} (\mu(X_n)) = \limsup_{n \rightarrow \infty} 2 = 2$. This means that μ does not satisfy the reverse Fatou property.

Theorem 1. [60, Theorem 1] *Let T be a countable rooted tree and let μ be an additive and σ -additive function on $2^{\mathbf{V}(T)}$. Let k be a positive integer and let c, d be two positive reals such that*

$$\mu(\mathbf{V}(T)) \geq d + (k - 1)c. \quad (3)$$

Then (\mathbf{H}_T, μ) is either (d, k) -fat or $(c, 1)$ -tall or both.

Tightness of the bound in Theorem 1 is discussed in [60, Example 1] while the failure of generalizing Theorem 1 from rooted tree poset to general poset is reported in [60, Example 3]. How about considering other kinds of set functions?

Theorem 2. *Let T be a finite rooted tree and let μ be a submodular function on $2^{\mathbf{V}(T)}$ satisfying $\mu(\emptyset) \geq 0$. Let k be a positive integer and let c, d be two positive reals such that (3) holds.*

1. *(\mathbf{H}_T, μ) is either (d, k) -fat or $(c, 1)$ -tall or both.*
2. *If $\mu(C) < c$ for every end C of T , then (\mathbf{H}_T, μ) is (d, k) -fat.*

After presenting the above strengthening of Theorem 1 for finite trees, we should also list the following natural extension of [60, Conjecture 1].

Conjecture 1. Let t be a positive integer. If we replace (3) in Theorem 2 by the condition of $\mu(\mathbf{V}(T)) \geq d + \lceil \frac{k-1}{t} \rceil c$, then (\mathbf{H}_T, μ) is either (d, k) -fat or (c, t) -tall or both.

The next example means that $\mu(\emptyset) \geq 0$ is necessary for the truth of Theorem 2.

Example 6. Consider the rooted tree as displayed in Fig. 1. Define a function μ on $2^{\mathbf{V}(T)}$ such that $\mu(A) = \mu(A \cup \{r\}) = 10(|A| - 1)$ for all $A \subseteq \{a, b, c\}$. One can check that μ is a ∞ -alternating function but $\mu(\emptyset) < 0$. Let $c = 1$ and $d_1 = d_2 = 5$. Note that

$$\mu(\mathbf{V}(T)) = 20 \geq c + d_1 + d_2 = 11.$$

But, neither can you find a chain C with $\mu(C) \geq 1$, nor can you find two disjoint down-sets D_1 and D_2 such that $\mu(D_1) \geq 5$ and $\mu(D_2) \geq 5$.

Example 7. Let T be a rooted tree which is not a directed path, that is, T has at least two rays. Let $\mu : 2^{\mathbf{V}(T)} \rightarrow \mathbb{R}$ be the set function such that

$$\mu(X) = \begin{cases} 1 & \text{if } N \cap X \text{ is a nonempty end of } T \text{ for every nonempty end } N \text{ of } T; \\ 0 & \text{otherwise.} \end{cases}$$

Note that μ is basically a unanimity game [23, p. 42] and so is ∞ -monotone. For every positive reals d and c and integer $k \geq 2$, the weighted rooted tree (T, μ) is neither (d, k) -fat nor $(c, 1)$ -tall.

Question 1. Does Theorem 2 still hold when we replace the assumption of μ being submodular by μ being weakly submodular or even just subadditive?

There are many ways to add continuity assumptions to guarantee the transition from finite to infinite. In Theorem 1, we do not require the weighting function μ to be increasing but in the following variant we do not require the weighting function μ to be additive.

Theorem 3. *Let (T, μ) be a weighted rooted tree. Assume that $V(T)$ is countably infinite, and that μ is submodular, F -continuous and satisfies $\mu(\emptyset) \geq 0$ and $\mu(D \downarrow_T) \geq \mu(D)$ for all $D \subseteq V(T)$. Let k be a positive integer and c, d be two positive reals such that (3) holds.*

1. (H_T, μ) is either (d, k) -fat or $(c, 1)$ -tall or both.
2. If T does not have any saturated chain C with $\mu(C) \geq c$, then (H_T, μ) is (d, k) -fat.

Recall that the Hewitt-Yosida Theorem claims a unique decomposition of any additive function into a sum of a F -continuous additive function (measure) and a discontinuous additive function [33, Theorem 8.16]. Also note that the relationship between submeasures and measures can be quite complicated [55]. In view of Example 5 and Theorem 2, it is natural to wonder if we can drop/relax the continuity assumption on μ from Theorem 3. For any set X , an ultrafilter on X is a function ϕ from 2^X to $\{0, 1\}$ such that $\phi(\emptyset) = 0$, $\phi(X) = 1$ and $\phi(Y \cup Z) = \phi(Y) + \phi(Z) - \phi(Y \cap Z)$ – Here we are talking about the filters as invented by Cartan (1937) instead of the filters as introduced by Kolmogorov (1941) and Wiener (1949). It is clear that the bigness defined by ultrafilter is robust under partitioning, that is, for any partition of X into finite many parts, exactly one part will be mapped by the ultrafilter to 1. An ultrafilter on X is principal provided there is $x \in X$ such that $\phi(Y) = 1$ if and only if $x \in Y$. An easy consequence of Zorn's Lemma is that there exists a non-principal ultrafilter ϕ on any infinite set X , namely $\phi(Y) = 0$ for all finite subsets Y of X [12, p. 245, Exercise 10.8] [45, 7.5.17].

Example 8. Let X be the set of all positive integers and take $x \in X$. Let T be the rooted tree in which x is bigger than y for all $y \in X \setminus \{x\}$ and any two elements in $X \setminus \{x\}$ are incomparable. Let μ be a non-principal ultrafilter on X . Surely, μ is additive but not F -continuous: Taking $A_n = [n]$ for all n shows that the weak Fatou property fails while taking $A_n = X \setminus [n]$ for all n shows that the reverse Fatou property fails. For the weighted rooted tree (T, μ) , every chain has weight 0 and, for every two disjoint down-sets A and B in T , it must hold either $\mu(A) = 0$ or $\mu(B) = 0$. Hence, though $\mu(V(H_T)) = 1 > \frac{3}{4} = \frac{2}{4} + (2-1)\frac{1}{4}$, (H_T, μ) has neither $(\frac{1}{4}, \frac{1}{4})$ -matching nor 1-antimatching of weight at least $\frac{1}{4}$.

In Section 2 we provide a proof of Theorems 2 and 3. The proof is divided in two steps: For the first step (Section 2.1), we focus on (α, β) down-sets and use combinatorial arguments to get Theorem 4 and then Theorem 2; In the

second step (Section 2.2), combining Theorem 2 with some continuity arguments, including those preliminary work in getting Lemmas 1 and 2, we will derive Theorem 3. We remark that throughout our proof, we appeal to properties of submodular functions in Eqs. (6) to (8). So far, we do not see how to go around those arguments to get any positive result about Question 1.

In Theorem 3 we have to impose additional continuity requirement on infinite weighted trees to get our uncertainty relation. As seen in Example 5, we do not have this continuity condition in the finite case and so we are wondering if there should be some other better reason behind to guarantee the uncertainty relation. This motivates us to formulate Question 2 in Section 3. Moreover, we discuss there how the knowledge on anticores can help us go further from Theorems 2 and 3. Section 3 also reviews some simple facts on anticores for the convenience of readers. Especially, we explain how Theorem 5 there can imply Theorem 2.

Section 4 is motivated by some previous work in Ramsey theory [7,9] and connects to the existence problem for normal spanning trees and BBT spanning trees. The work there directly follows from Theorems 2 and 3. Section 5 contains two simple applications of Theorem 1 and so the weighting functions there are additive.

2 Up and down in a rooted tree

2.1 Finite tree

Definition 2. Let (Γ, μ) be a weighted digraph. For any two real numbers α and β , we say that a down-set D of Γ is an (α, β) down-set of (Γ, μ) provided $\mu(D) \geq \beta$ and $\mu(D \uparrow_\Gamma) \leq \alpha + \beta$.

Example 9. Let T and μ be what are given in Fig. 2 and Example 3. One can check that $\mu(\text{blue tree}) = 4 \geq 3$ and $\mu(\text{red tree}) = 6 \leq 3 + 3$. Therefore, blue tree is a $(3, 3)$ down-set in (T, μ) .

For each digraph Γ and $v \in V(\Gamma)$, let us use $\Gamma^+(v)$ for the set of out-neighbors of v in Γ , namely $\Gamma^+(v) = \{w \in V(\Gamma) : vw \in A(\Gamma)\}$. Of course, we have $v \downarrow_\Gamma \supseteq \Gamma^+(v)$ for all $v \in V(\Gamma)$.

Theorem 4. Let T be a finite rooted tree and let μ be a submodular function on $2^{V(T)}$ with $\mu(\emptyset) \geq 0$. Let α and β be two nonnegative reals such that $\mu(V(T)) \geq \alpha + \beta$ and $\mu(L) \leq \alpha$ for all rays L of T . Then the weighted rooted tree (T, μ) has an (α, β) down-set.

Proof. We intend to find a down-set D of T such that $\mu(D) \geq \beta$ and $\mu(D \uparrow_T) \leq \alpha + \beta$. We will demonstrate its existence by an induction on $|V(T)|$. Let r be the root of T .

If $|V(T)| = 1$, then we have $0 \geq \mu(r \uparrow_T) - \alpha = \mu(V(T)) - \alpha \geq \beta \geq 0$, which forces $\beta = 0$. Therefore, we have $\mu(r) = \mu(V(T)) \geq \alpha + \beta \geq \beta$ and $\mu(r \uparrow_T) \leq \alpha = \alpha + \beta$. This means that we can take $D = \{r\}$.

Assume now $|\mathbb{V}(T)| > 1$ and that the result holds when $|\mathbb{V}(T)|$ is smaller. Take $x \in T^+(r)$ and let $X := x \downarrow_T$. There are three cases to consider.

CASE 1. $\mu(X) \geq \beta$ and $\mu(X \cup \{r\}) \leq \alpha + \beta$.

Take $D = X$, which is a down-set of T . Then $\mu(D) = \mu(X) \geq \beta$ and $\mu(D \uparrow_T) = \mu(X \cup \{r\}) \leq \alpha + \beta$.

CASE 2. $\mu(X) \geq \beta$ and $\mu(X \cup \{r\}) > \alpha + \beta$.

Define a submodular function μ' on 2^X by putting

$$\mu'(A) = \begin{cases} \mu(A \cup \{r\}) & \text{if } x \in A \subseteq X; \\ \mu(A) & \text{if } A \subseteq X \setminus \{x\}. \end{cases}$$

Let T' be the subtree of T induced by X . For each ray L' of T' , $L' \cup \{r\}$ is a ray of T and so we have $\mu'(L') = \mu(L' \cup \{r\}) \leq \alpha$. Further note that $\mu'(\emptyset) = \mu(\emptyset) \geq 0$, and that

$$\mu'(\mathbb{V}(T')) = \mu'(X) = \mu(X \cup \{r\}) > \alpha + \beta. \quad (4)$$

By our induction hypothesis for (T', μ') , we can find a down-set D of T' such that

$$\mu'(D) \geq \beta \quad \text{and} \quad \mu'(D \uparrow_{T'}) \leq \alpha + \beta. \quad (5)$$

Comparing (5) with (4) yields $D \uparrow_{T'} \subsetneq X = x \downarrow_T$ and so $x \notin D$ follows. We now see that $D = D \downarrow_T$ satisfies $\mu(D) = \mu'(D) \geq \beta$ and $\mu(D \uparrow_T) = \mu'(D \uparrow_{T'}) \leq \alpha + \beta$, as wanted.

CASE 3. $\mu(X) < \beta$.

Let T' be the tree obtained from T by deleting X . Let μ' be the function on $2^{\mathbb{V}(T')}$ satisfying $\mu'(A) = \mu(A \cup X) - \mu(X)$ for all $A \subseteq \mathbb{V}(T')$. First of all, we have $\mu'(\emptyset) = 0$. For any $A, B \subseteq \mathbb{V}(T')$, we verify that

$$\begin{aligned} & \mu'(A \cup B) + \mu'(A \cap B) \\ &= (\mu(A \cup B \cup X) - \mu(X)) + (\mu((A \cap B) \cup X) - \mu(X)) \\ &\leq \mu(A \cup X) + \mu(B \cup X) - 2\mu(X) \\ &= \mu'(A) + \mu'(B), \end{aligned} \quad (6)$$

showing that μ' is submodular. Moreover, for any leaf u of T' , we see that

$$\begin{aligned} \mu'(u \uparrow_{T'}) &= \mu(u \uparrow_{T'} \cup X) - \mu(X) \\ &= \mu(u \uparrow_T \cup X) - \mu(X) \\ &\leq (\mu(u \uparrow_T) + \mu(X) - \mu(\emptyset)) - \mu(X) \\ &= \mu(u \uparrow_T) - \mu(\emptyset) \\ &\leq \mu(u \uparrow_T) \\ &\leq \alpha. \end{aligned} \quad (7)$$

Letting $\beta' := \beta - \mu(X) > 0$, it is clear that $\mu'(V(T')) = \mu(V(T') \cup X) - \mu(X) = \mu(V(T)) - \mu(X) \geq \alpha + \beta - \mu(X) = \alpha + \beta'$. These facts together then enables us apply the induction hypothesis on (T', μ') to get a down-set D' of T' such that

$$\mu'(D') \geq \beta' \quad \text{and} \quad \mu'(D' \uparrow_{T'}) \leq \alpha + \beta'.$$

Take $D := D' \cup X$, which is surely a down-set of T . It follows that

$$\mu(D) = \mu(D' \cup X) = \mu'(D') + \mu(X) \geq \beta' + \mu(X) = \beta$$

and

$$\mu(D \uparrow_T) = \mu(D' \uparrow_{T'} \cup X) = \mu'(D' \uparrow_{T'}) + \mu(X) \leq \alpha + \beta' + \mu(X) = \alpha + \beta.$$

This finishes the proof. \square

Proof (Proof of Theorem 2). Let r be the root of T . We will proceed with an induction on k . Surely, we only need to prove the second reading as it implies the first claim.

When $k = 1$, we can see that the down-set $V(T)$ satisfies $\mu(V(T)) \geq d$ and thus (T, μ) is $(d, 1)$ -fat. This tells us that the base case holds true.

Assume now $k \geq 2$ and the result holds for smaller k . Let d_1, \dots, d_k be k nonnegative reals such that $\sum_{i=1}^k d_i = d$. We have to find k disjoint down-sets D_1, \dots, D_k of T such that $\mu(D_i) \geq d_i$ for all $i \in [k]$.

Applying Theorem 4 for $\alpha = c$ and $\beta = d_k$ yields a down-set D of T satisfying $\mu(D) \geq \beta = d_k$ and $\mu(D \uparrow_T) \leq \alpha + \beta = c + d_k$. Consider the submodular function μ' on $2^{V(T)}$ such that $\mu'(A) := \mu(A \setminus (D \uparrow_T))$ for all $A \subseteq V(T)$. Observe that

$$\begin{aligned} (k-2)c + \sum_{i=1}^{k-1} d_i &\leq \mu(V(T)) - (c + d_k) \\ &\leq \mu(V(T)) - \mu(D \uparrow_T) \\ &\leq \mu(V(T) \setminus (D \uparrow_T)) - \mu(\emptyset) \\ &\leq \mu(V(T) \setminus (D \uparrow_T)) \\ &= \mu'(V(T)). \end{aligned} \tag{8}$$

We are ready to invoke our induction hypothesis to find $k-1$ down-sets of T , say D'_1, \dots, D'_{k-1} , such that $\mu'(D'_i) \geq d_i$ holds for all $i \in [k-1]$. For $i \in [k]$, define

$$D_i := \begin{cases} D'_i \setminus (D \uparrow_T) & \text{if } i \in [k-1]; \\ D & \text{if } i = k. \end{cases}$$

Clearly, D_1, \dots, D_k are pairwise disjoint sets. Note that $\mu(D_k) \geq d_k$, and that for $i \in [k-1]$,

$$\mu(D_i) = \mu(D'_i \setminus (D \uparrow_T)) = \mu'(D'_i) \geq d_i.$$

To verify that D_1, \dots, D_k are what we need, it remains to show that D_i is a down-set of T for every $i \in [k-1]$. Pick $i \in [k-1]$ and $x \in D_i \downarrow_T \subseteq D'_i \downarrow_T = D'_i$.

Then there exists $x' \in D_i$ such that $x \in x' \downarrow_T$. From $x' \in D_i = D'_i \setminus (D \uparrow_T)$ we derive that $(x' \downarrow_T) \cap D = \emptyset$; and so $x \in x' \downarrow_T$ implies $x \notin D \uparrow_T$. We now see that $x \in D'_i \setminus (D \uparrow_T) = D_i$. This shows that D_i is a down-set of T , as required. \square

We assume that our input is a finite rooted tree and a function $f \in \mathbb{R}^{V(T)}$. We use μ for the additive function such that $\mu(A) = \sum_{x \in A} f(x)$ for all $A \subseteq V(T)$. We further assume that (3) holds and let d_1, \dots, d_k be k nonnegative reals such that $\sum_{i=1}^k d_i = d$. The following three observations ensure that there is a linear time algorithm to find either an end C of T with $\mu(C) \geq c$ or k disjoint down-sets D_1, \dots, D_k of T such that $\mu(D_i) \geq d_i$ for all $i \in [k]$:

- (i) There is an algorithm that finds an end C of T such that $\mu(C) \geq \mu(C')$ for all ends C' in linear time.
- (ii) Write $DS(T, \mu, \alpha, \beta)$ for the problem of finding an (α, β) -down set in T . If (T, μ) and two reals α, β satisfy the conditions in Theorem 4, there is an algorithm solving $DS(T, \mu, \alpha, \beta)$ in linear time. Indeed, according to the proof of Theorem 4, we can reduce $DS(T, \mu, \alpha, \beta)$ to $DS(T', \mu', \alpha', \beta')$ for some rooted tree T' with $|V(T')| < |V(T)|$ in constant time.
- (iii) If $\mu(C) < c$ for all ends C in T , there is an algorithm that finds a (d_1, \dots, d_k) -matching of (H_T, μ) in $O(k|V(T)|)$ time (by using (ii) k times).

2.2 Infinite tree

Let T be a rooted tree with a root r . A **rake** in T is a subset W of $V(T)$ that is the union of two sets W_0 and W_1 , called its handle and its brush respectively, such that W_0 is a saturated chain in T and W_1 is either empty or of the form $w \downarrow_T$ where w is the minimum element in W_1 . Note that if the handle of a rake is an infinite chain then its brush must be an empty set. For each nonempty rake W , let $h(W)$ denote the unique maximum element of W . Let \mathbb{N} stand for the set of positive integers. Let $T^0 := \{r\}$, let $T^i := \bigcup_{v \in T^{i-1}} T^+(v)$ and $T^{<i} := \bigcup_{j=0}^{i-1} T^j$ for all $i \in \mathbb{N}$.

Lemma 1. *Let T be a rooted tree. For each $i \in \mathbb{N}$, let R_i be a rake in T such that $|R_i \cap T^j| \leq 1$ for all $j \in [i]$. Then there exists an increasing map $\phi \in \mathbb{N}^{\mathbb{N}}$ such that $\limsup_{i \rightarrow \infty} R_{\phi(i)}$ is a saturated chain in T .*

Proof. Let $\mathbf{R}_0 := \limsup_{i \rightarrow \infty} R_i$. If \mathbf{R}_0 is a chain in T , then ϕ can be chosen as the identity map. Otherwise, we have $\mathbf{R}_0 \neq \emptyset$ and so $\tau_1 := \min\{i : T^i \cap \mathbf{R}_0 \neq \emptyset\} < \infty$. Choose $v_1 \in T^{\tau_1} \cap \mathbf{R}_0$, let $N_1 = \{i \in \mathbb{N} : v_1 = h(R_i)\}$ and let ϕ_1 be the unique order-preserving bijection from \mathbb{N} to N_1 . Let $\mathbf{R}_1 := \limsup_{i \rightarrow \infty} R_{\phi_1(i)}$. If \mathbf{R}_1 is a chain in T , then ϕ can be chosen to be ϕ_1 . Otherwise, we can find $\tau_2 > \tau_1$ and $v_2 \in \mathbf{R}_1 \cap T^{\tau_2}$, and then let $N_2 = \{i \in N_1 : \{v_2\} = T^{\tau_2} \cap R_i\}$ and let ϕ_2 be the unique increasing bijection from \mathbb{N} to N_2 . Let $\mathbf{R}_2 := \limsup_{i \rightarrow \infty} R_{\phi_2(i)}$. If \mathbf{R}_2 is a chain in T , then ϕ can be chosen to be ϕ_2 . Otherwise, we continue as before and repeat this procedure as far as we can. If we cannot stop at some finite moment

t to get the required map $\phi = \phi_t$, we then must have found an infinite chain v_1, v_2, \dots of vertices of T and an infinite filtration $\mathbb{N} \supseteq N_1 \supseteq N_2 \supseteq \dots$ of infinite subsets of \mathbb{N} . Surely, we can find an increasing map $\phi \in \mathbb{N}^{\mathbb{N}}$ such that $\phi(i) \in N_i$ for all $i \in \mathbb{N}$. It is not difficult to see that $\limsup_{i \rightarrow \infty} R_{\phi(i)}$, which coincides with $\lim_{i \rightarrow \infty} R_{\phi(i)}$ and contains $\{v_i : i \in \mathbb{N}\}$, is an infinite saturated chain in T , finishing the proof. \square

Lemma 2. *Let (T, μ) be a weighted rooted tree on countably many vertices such that μ has the weak Fatou property. For any $\epsilon \geq 0$ and any $n \in \mathbb{N}$, there is an up-set F of T such that $F \subseteq T^{<n+1}$, $|F| < \infty$, and $\mu(F \cup (\bigcup_{u \in F \cap T^n} u \downarrow_T)) \geq \mu(V(T)) - \epsilon$.*

Proof. Let M be the set of minimal elements of T which fall into $T^{<n}$. Note that M corresponds to the set of rays of T which are disjoint from T^n . Since $M \cup T^n$ is countable, its elements can be enumerated as v_1, v_2, \dots . For each $v \in M \cup T^n$, we define the rake R_v to be the union of $v \uparrow_T$ and $v \downarrow_T$. Let $A_\ell := \bigcup_{k=1}^\ell R_{v_k}$ for $\ell \leq |M \cup T^n|$. If $M \cup T^n$ is indeed a finite set, say of size m , we make the convention that $A_\ell := A_m$ for all $\ell > m$. Clearly, $(A_\ell)_{\ell \in \mathbb{N}}$ is an increasing sequence of sets and $\lim_{\ell \rightarrow \infty} A_\ell = V(T)$. Due to the weak Fatou property of μ , there exists a positive integer N such that $\mu(A_N) \geq \mu(V(T)) - \epsilon$. Surely, we can take F to be $A_N \cap T^{<n+1}$, completing the proof. \square

Proof (Proof of Theorem 3). Let d_1, \dots, d_k be k nonnegative reals with $d = \sum_{i=1}^k d_i$. Let us assume that (H_T, μ) does not contain any (d_1, \dots, d_k) -matching and turn to show that we can find a saturated chain C of T such that $\mu(C) \geq c$.

Let r be the root of T . Let $(\epsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive reals such that $\lim_{n \rightarrow \infty} \epsilon_n = 0$. By Lemma 2, for each $n \in \mathbb{N}$ there exists a finite subset F_n of $T^{<n+1}$ such that $F_n = F_n \uparrow_T$, $\mu(F_n \cup (\bigcup_{u \in F_n \cap T^n} u \downarrow_T)) \geq \mu(V(T)) - \epsilon_n$.

Pick $n \in \mathbb{N}$. We consider the finite subdigraph T_n of T induced by F_n . Surely, T_n is a rooted tree rooted at r . For any $A \subseteq F_n$, let \tilde{A} represent the set $A \cup (\bigcup_{v \in A \cap T^n} v \downarrow_T)$. Define a function μ_n from 2^{F_n} to \mathbb{R} by setting $\mu_n(A) = \mu(\tilde{A})$ for all $A \subseteq F_n = V(T_n)$. We now apply Theorem 2 on (T_n, μ_n) and see that, we either have a (d_1, \dots, d_k) -matching in (H_{T_n}, μ_n) or an end C_n of T_n with $\mu_n(C_n) \geq c - \epsilon_n$. Assume, for sake of contradiction, that the former case happens. This means that we have k disjoint down-sets D_1, \dots, D_k of T_n such that $\mu_n(D_i) \geq d_i$ for $i \in [k]$. It follows that $\tilde{D}_1 \downarrow_T, \dots, \tilde{D}_k \downarrow_T$ are k disjoint down-sets of T , such that, thanks to the increasing property of μ , $\mu(\tilde{D}_i \downarrow_T) \geq \mu(\tilde{D}_i) = \mu_n(D_i) \geq d_i$ for all $i \in [k]$. This contradicts our assumption that (H_T, μ) does not contain any (d_1, \dots, d_k) -matching. Therefore, the asserted set C_n must exist and we can set $R_n := \tilde{C}_n$. It is easy to see that R_n is a rake of T satisfying $\mu(R_n) \geq c - \epsilon_n$ and $|R_n \cap T^m| \leq 1$ for all $m \in [n]$.

Finally, we use Lemma 1 to obtain the existence of an increasing map ϕ from \mathbb{N} to itself such that $C := \limsup_{n \rightarrow \infty} R_{\phi(n)}$ is a saturated chain in T . It follows that

$$\begin{aligned} \mu(C) &= \mu(\limsup_{n \rightarrow \infty} R_{\phi(n)}) \\ &\geq \limsup_{n \rightarrow \infty} \mu(R_{\phi(n)}) && \text{(Reverse Fatou)} \\ &= \lim_{n \rightarrow \infty} \mu(R_{\phi(n)}) \\ &\geq \lim_{n \rightarrow \infty} (c - \epsilon_{\phi(n)}) \\ &= c, \end{aligned}$$

which is the end of the proof. \square

To finish off, we mention that an infinite counterpart of Theorem 4 can be established exactly in the same way as we deduce Theorem 3 from Theorem 2.

3 Anticore and base polyhedron

Let X be a countable set. Let \mathcal{M}_X be the set of functions μ from 2^X to \mathbb{R} such that the following hold:

1. $\mu(X) \geq 0$;
2. For every rooted tree T with $V(T) = X$ and every nonnegative reals c and d and positive integer k with $\mu(X) \geq d + (k-1)c$, (H_T, μ) is either (d, k) -fat or $(c, 1)$ -tall or both.

Question 2. How to characterize/understand \mathcal{M}_X ? Is it closed under the taking sum operation?

Here is a very simple observation, which says that to determine \mathcal{M}_X essentially reduces to an understanding of the minimal elements in \mathcal{M}_X .

Observation 1. *Let X be a countable set and μ, μ' be set functions on X with $\mu(X) = \mu'(X)$. If $\mu' \in \mathcal{M}_X$ and $\mu' \leq \mu$, then $\mu \in \mathcal{M}_X$.*

By Theorem 2, for a finite set X , \mathcal{M}_X contains the set of all additive functions on 2^X . Therefore, we are interested in those functions which dominate some additive functions. It turns out that this is already considered a lot by people of various backgrounds.

Let X be a set and let τ be a function from 2^X to \mathbb{R} . The core of τ is the set of additive functions μ on 2^X such that $f(A) \geq \tau(A)$ for all $A \subseteq X$ and $f(X) = \tau(X)$; The **anticore** of τ , is the set of additive functions μ on 2^X such that

$$\mu(A) \leq \tau(A)$$

for all $A \subseteq X$ and

$$\mu(X) = \tau(X).$$

If X is a finite set, the anticore of τ is also known as the **base polyhedron** of τ , thanks to its connection to matroid base polyhedron [16,22]. Since these two concepts are dual to each other and so results about one can easily be translated to results about the other, we will focus on anticore in this section. Note that when we refer to a result on anticores from some literature, we may really mean that that literature reports a counterpart result on cores.

We assume that X is a finite set of size n . It is well-known that the base polyhedron of a set function τ on X is nonempty whenever $\tau(\emptyset) \geq 0$ and τ is submodular [42, Proposition 4.4]. Indeed, it coincides with the convex hull of those marginal worth distributions of τ and has the so-called Shapley value of τ as its center of gravity [52, Theorem 3, Theorem 5]! We recall below a nice and short argument of displaying the nonemptiness of the anticore [52, Theorem 4]. For any permutation $v = (v_1, \dots, v_n)$ of the n elements of X , we can define the marginal worth distribution y of τ to be the additive function $y = y^{\tau, v}$ such that $y(v_i) = \tau(F_i) - \tau(F_{i-1})$, where we use F_i for $\{v_1, \dots, v_i\}$ for any $i \in [n]$ and use F_0 for \emptyset . An inductive argument on $|A|$ easily demonstrates that $y(A) \leq \tau(A)$ for all $A \subseteq X$ and thus y lies in the anticore of τ . The base case follows from $\tau(\emptyset) \geq 0 = y(\emptyset)$. If A is nonempty and i is the maximum integer such that $v_i \in A$, we have

$$\begin{aligned} y(A) &= y(A \setminus \{v_i\}) + y(v_i) \\ &\leq \tau(A \setminus \{v_i\}) + y(v_i) && \text{(By induction assumption)} \\ &= \tau(A \setminus \{v_i\}) + \tau(F_i) - \tau(F_{i-1}) \\ &= \tau(A \cap F_{i-1}) + \tau(A \cup F_{i-1}) - \tau(F_{i-1}) \\ &\leq \tau(A). && \text{(By submodularity of } \tau) \end{aligned}$$

In light of Observation 1, we now see that to prove Theorem 2 it suffices to verify it in the special case that μ is a signed measure, namely an additive function. In the conference version of our paper, we do prepare this signed measure version already [60, Theorem 1]. We choose to establish Theorem 2 in Section 2 via a different approach as we think that Theorem 4 may be of independent interest.

Observation 1 enables us get something more than Theorem 2. Let X be a finite set. A **balancing sequence** γ on 2^X is a sequence of nonnegative reals $\gamma_A, A \in 2^X$, such that $\sum_{i \in A \in 2^X} \gamma_A = 1$ for all $i \in X$. A necessary condition for a map μ from 2^X to \mathbb{R} to have a nonempty anticore is

$$\mu(X) \leq \sum_{A \subseteq X} \gamma_A \mu(A) \tag{9}$$

for all balancing sequences γ on 2^X . Indeed, if f is in the anticore of μ , then

$$\begin{aligned}
\mu(X) &= f(X) \\
&= \sum_{i \in X} f(i) \\
&= \sum_{i \in X} \left(\sum_{A \subseteq X, i \in A} \gamma_A \right) f(i) + \gamma_\emptyset f(\emptyset) \\
&= \sum_{A \subseteq X} \gamma_A f(A) \\
&\leq \sum_{A \subseteq X} \gamma_A \mu(A).
\end{aligned}$$

The well-known Bondareva–Shapley Theorem [30, Theorem 1.1] claims that a set function μ on a finite set X has a nonempty anticore if and only if (9) holds.

Theorem 5. *Let T be a finite rooted tree and let μ be a function on $2^{V(T)}$ for which (9) holds for all balancing sequences γ on 2^X . For any positive integer k and positive reals d and c such that (3) holds, (H_T, μ) is either (d, k) -fat or $(c, 1)$ -tall or both.*

Proof. According to the Bondareva–Shapley Theorem [1, 17], the anticore of μ is nonempty and so we can pick a signed measure f from it. By Theorem 1, (H_T, f) is either (d, k) -fat or $(c, 1)$ -tall or both. Since f is dominated by μ , Observation 1 tells us that (H_T, μ) is either (d, k) -fat or $(c, 1)$ -tall or both. \square

We should indicate here how to view Theorem 2 as a special case of Theorem 5. For any finite set X and any function μ from 2^X to \mathbb{R} , its Lovász extension, denoted $\hat{\mu}$, is the function from \mathbb{R}^X to \mathbb{R} such that $\hat{\mu}(g) = \sum_{i \in [n]} g(v_i)(\mu(S_i) - \mu(S_{i-1}))$, where v_1, \dots, v_n is a permutation of X satisfying $g(v_1) \geq g(v_2) \geq \dots \geq g(v_n)$, $S_0 = \emptyset, S_1 = \{v_1\}, \dots, S_n = \{v_1, \dots, v_n\}$. Clearly, $\hat{\mu}$ is positively homogeneous, namely $\hat{\mu}(cg) = c\hat{\mu}(g)$ for all nonnegative real c . Moreover, if μ is submodular, $\hat{\mu}$ is the support function of the base polyhedron of μ [42, Proposition 4.5] and hence is a convex function on \mathbb{R}^X [42, Theorem 4.16]. Therefore, on the condition that $\mu(\emptyset) \geq 0$ and μ being submodular, it holds for any balancing sequence γ on 2^X that

$$\begin{aligned}
\mu(X) &= \hat{\mu}(1_X) \\
&= \hat{\mu}\left(\sum_{A \in 2^X \setminus \{\emptyset\}} \gamma_A 1_A\right) \\
&\leq \sum_{A \in 2^X \setminus \{\emptyset\}} \gamma_A \hat{\mu}(1_A) \quad (\text{Convexity and positive homogeneity of } \hat{\mu}) \\
&= \sum_{A \in 2^X \setminus \{\emptyset\}} \gamma_A \mu(A) \\
&\leq \sum_{A \in 2^X} \gamma_A \mu(A). \quad (\mu(\emptyset) \geq 0)
\end{aligned}$$

This gives (9) and so we have confirmed that Theorem 2 can be read from Theorem 5.

For a set function on a general set X , the nonemptiness of its anticore can be characterized analogous to the Bondareva–Shapley Theorem [29, p. 230] [46, Theorem 8] [50, p. 2]. More results on anticore of infinite games can be found in [18, Theorem 10][30, Theorem 6.1]. Although an element in the anticore is additive, we may not know for sure that it is inside \mathcal{M}_X and this then makes it difficult to apply Observation 1 and Theorems 1 and 3 to earn more knowledge about \mathcal{M}_X . However, people have studied the concept of the σ -core of a set function, which is the set of σ -additive functions in the core. Note that an additive function which is σ -additive must be F -continuous. It thus looks interesting to see if any results from [10,29,31,43,44,49,51] can be used to combine Observation 1 and Theorems 1 and 3 to generate more members of \mathcal{M}_X for a countably infinite set X . We mention that Lovász extension is a finite version of Choquet integral [11] and that a set function is submodular if and only if its Choquet integral is convex [5, Theorem 1].

4 Path and pseudorandom graph

Ben-Eliezer, Krivelevich and Sudakov [7, Definition 4.1] define a digraph Γ to be d -pseudorandom if for every two disjoint sets $A, B \subseteq V(\Gamma)$ such that $|A|, |B| \geq d$, there is at least one arc of Γ going from A to B . In the same spirit, we call a weighted graph (G, μ) (d_1, \dots, d_k) -**pseudorandom** if for every k disjoint sets $D_1, \dots, D_k \subseteq V(G)$ such that $\mu(D_i) \geq d_i$ for all $i \in [k]$, there is at least one edge of G connecting D_i and D_j for some $\{i, j\} \in \binom{[k]}{2}$. It is known that every d -pseudorandom digraph on n vertices contains a directed path of length $n - 2d + 1$ [7, Lemma 4.4] [58, Proposition 4.1]. Here is a counterpart for pseudorandom weighted graphs.

Theorem 6. *Let $k > 1$ be an integer and let d_1, \dots, d_k, c be $k + 1$ positive reals. Let (G, μ) be a finite connected weighted graph. We assume that μ is submodular, $\mu(\emptyset) \geq 0$ and $\mu(V(G)) \geq \sum_{i=1}^k d_i + (k-1)c$. If (G, μ) is (d_1, \dots, d_k) -pseudorandom, then there exists a path P of G such that $\mu(P) \geq c$.*

Proof. Run the depth-first search (DFS) algorithm on G , we will get a corresponding depth-first search rooted tree T . Applying Theorem 2 on (T, μ) and parameters d_1, \dots, d_k, c , we will either find a path P with $\mu(P) \geq c$ or k disjoint down-sets D_1, \dots, D_k such that $\mu(D_i) \geq d_i$ for all $i \in [k]$. By the property of DFS algorithm [2, Lemma 5.3], there is no edge between disjoint down-sets of T . Since (G, μ) is (d_1, \dots, d_k) -pseudorandom, we see that the only possibility is that the first case happens. \square

Let G be a connected graph. A **spanning rooted tree** of G is a rooted tree T satisfying $V(T) = V(G)$ and $xy \in E(G)$ whenever $x \in T^+(y)$.

Definition 3. *Let G be a connected graph. A BBT **spanning tree** of G is a spanning rooted tree T of G such that*

- 1) If $x \in y \downarrow_T$ and $xy \in E(G)$, then $x \in T^+(y)$;
- 2) If $x \in T^+(x')$, $y \in T^+(y')$ and $xy \in E(G)$, then either $x' \in y' \downarrow_T$ or $y' \in x' \downarrow_T$.

We adopt the name of BBT spanning tree here as this construction is essentially used by Bonamy, Bousquet, and Thomassé in their proof of [9, Theorem 6].

Example 10. Let G be the grid graph with vertex set $\mathbb{Z} \times \mathbb{Z}$ in which two vertices are adjacent if and only if their difference is one of the four elements, $(0, \pm 1)$ and $(\pm 1, 0)$. We display a BBT spanning tree T of G in Fig. 3. Note that T is an infinite caterpillar, its central stalk marked in red is swirling around the root of T , and each vertex outside of the stalk is connected in T to the vertex on the stalk that is adjacent to it in G and is closest to the root along the stalk. The stalk walks to the west for two steps, then to the north for two steps, then to the east for four steps, then to the south for four steps, then to the west for six steps, then to the north for six steps, then to the east for eight steps, and so on.

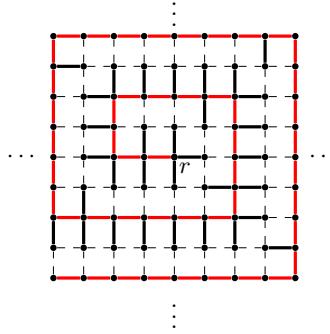


Fig. 3. A BBT spanning tree of the infinite grid graph; See Example 10.

Let G be a graph. For every $r, v \in V(G)$, an induced path from r to v in G is a finite vertex sequence $v_1 = r, v_2, \dots, v_\ell = v$ such that $v_i v_j \in E(G)$ if and only if $i, j \in [\ell]$ and $|i - j| = 1$; We denote the set of induced paths from r to v by $\text{IP}_G(r, v)$, write $\text{IP}_G(r)$ for $\bigcup_{v \in V(G)} \text{IP}_G(r, v)$ and then use the notation IP_G for $\bigcup_{r \in V(G)} \text{IP}_G(r)$. In the sequel, we always assume that $V(G)$ is equipped with a well order $<$, namely a total order in which each nonempty set has a least element. This allows us define a partial order \prec on IP_G such that $(v_1, \dots, v_k) \prec (u_1, \dots, u_s)$ if and only if there exists $t \leq \min(k, s)$ such that $v_i = u_i$ for $i \in [t - 1]$ and $v_t < u_t$. We say that (v_1, \dots, v_k) is an initial segment of (u_1, \dots, u_s) if $k \leq s$ and $u_i = v_i$ for $i \in [k]$. Note that two distinct elements in IP_G are incomparable with respect to \prec if and only if one is an initial segment of the other. If $V(G)$ is a finite set, $(\text{IP}_G(r, v), \prec)$ itself surely still forms a well order. The following result generalizes the idea of [9, Lemma 2].

Lemma 3. *Let G be a connected graph and let $r \in V(G)$. We assume that for every $v \in V(G)$, the total order $(\text{IP}_G(r, v), \prec)$ has a smallest element, which we denote by $P_G(r, v)$. Then G has a BBT spanning tree rooted at r .*

Proof. We first show that if $P_G(r, v) = (v_1, v_2, \dots, v_{k+1})$ for some positive integer k , then $P_G(r, v_k) = (v_1, \dots, v_k)$. If this were not true, let us assume

$$P_G(r, v_k) = (u_1, \dots, u_\ell) \prec (v_1, \dots, v_k). \tag{10}$$

If $v_{k+1} \notin \{u_1, \dots, u_\ell\}$, we see that $(u_1, \dots, u_\ell, v_{k+1}) \prec (v_1, v_2, \dots, v_{k+1})$, violating the definition of $P_G(r, v)$. If $v_{k+1} \in \{u_1, \dots, u_\ell\}$, then there is an $m \in [\ell]$ such that $v_{k+1} = u_m$ and then $(u_1, \dots, u_m) \in \text{IP}_G(r, v_{k+1})$. From $v_{k+1} \notin \{v_1, \dots, v_k\}$ and (10), we conclude that $(u_1, \dots, u_m) \prec (v_1, v_2, \dots, v_{k+1})$, yielding again a contradiction with the definition of $P_G(r, v)$.

Given the fact as illustrated above, we can build a spanning rooted tree T of G as follows. For every $v \in V(G) \setminus \{r\}$, we consider $P_G(r, v)$, say $P_G(r, v) = (v_1, v_2, \dots, v_k)$, and then add (v_k, v_{k-1}) to the arc set of T . It is easy to see that what we obtain is really a rooted spanning tree of G with root r . It remains to check that it is a BBT spanning tree of G . Condition 1) in Definition 3 trivially holds as all arcs of T come from induced paths in G . To check condition 2) in Definition 3, we assume that $P_G(r, x) = (v_1, \dots, v_k)$, $P_G(r, y) = (w_1, \dots, w_s)$, $xy \in E(G)$, $k, s > 1$, and we aim to demonstrate that either $v_{k-1} \in w_{s-1} \downarrow_T$ or $w_{s-1} \in v_{k-1} \downarrow_T$.

Let $q := \max\{i \in \mathbb{N} : v_i = w_i\} \geq 1$. We want to show that $q \geq \min\{k - 1, s - 1\}$. By way of contradiction, assume $q < \min\{k - 1, s - 1\}$. Observe that $\{v_1, \dots, v_k\} \cap \{w_1, \dots, w_s\} = \{v_1, \dots, v_q\}$. Without loss of generality, assume that $v_{q+1} < w_{q+1}$. Let t be the minimum positive integer i such that $v_i w_s \in E(G)$, which is well-defined owing to the assumption of $xy \in E(G)$. Note that $t \geq q + 1$. Then we see that $(v_1, \dots, v_t, w_s) \prec (w_1, \dots, w_{s-1}, w_s)$ in $\text{IP}_G(r, y)$, contradicting the assumption that $P_G(r, y) = (w_1, \dots, w_{s-1}, w_s)$. \square

Example 11. In Fig. 4, we draw a graph G whose vertices are ordered as the usual order for nonnegative integers. Note that $(\text{IP}_G(0, 1), \prec)$ does not have any smallest element. However, it has a BBT spanning tree rooted at 0 as indicated in boldface in Fig. 4.

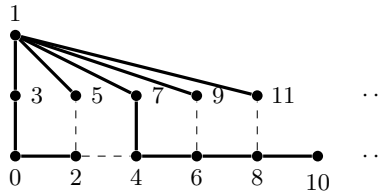


Fig. 4. An infinite graph G and one of its BBT spanning trees.

Bonamy, Bousquet, and Thomassé make use of [9, Lemma 2] and [9, Lemma 3] to derive [9, Theorem 6]. Accordingly, we now employ Lemma 3 and Theorem 2 to establish Theorem 7. Let G be a graph. For any $X \subseteq V(G)$, the **closed neighborhood** of X in G , denoted by $N_G[X]$, is the set of vertices of G which are either in X or adjacent to some elements of X .

Theorem 7. *Let $k > 1$ be an integer and let d_1, \dots, d_k, c be $k + 1$ positive reals. Let (G, μ) be a finite connected weighted graph which is (d_1, \dots, d_k) -pseudorandom. We assume that μ is submodular, $\mu(\emptyset) \geq 0$ and*

$$\max\{\mu(X) : |V(G) \setminus X| \leq 1\} \geq \sum_{i=1}^k d_i + (k-1)c. \quad (11)$$

Then there exist two subsets B and C of $V(G)$ such that $C \subseteq B \subseteq N_G[C]$, $G[C]$ is a path and $\mu(B) \geq c$.

Proof. As G is a finite graph, Lemma 3 allows us choose a BBT spanning tree T of G rooted at any specified vertex. If the maximum value on the left hand side of (11) is achieved at $X = V(T)$, we select r arbitrarily to be the root of T and, for any $X \subseteq V(T)$, let

$$\mathcal{U}_X := \begin{cases} \{r\} \cup \left(\bigcup_{x \in X} T^+(x) \right) & \text{if } r \in X; \\ \bigcup_{x \in X} T^+(x) & \text{if } r \notin X. \end{cases}$$

Otherwise, we assume that the root r of the BBT spanning tree T is a vertex such that $X = V(T) \setminus \{r\}$ is a maximizer of the left hand side of (11), and, for any $X \subseteq V(T)$, let

$$\mathcal{U}_X := \bigcup_{x \in X} T^+(x).$$

Define a submodular function λ on $V(T)$ by setting

$$\lambda(X) := \mu(\mathcal{U}_X).$$

Note that $\lambda(\emptyset) = \mu(\emptyset) \geq 0$ and $\lambda(V(T)) = \max\{\mu(X) : |V(G) \setminus X| \leq 1\} \geq \sum_{i=1}^k d_i + (k-1)c$.

Let us show that it is impossible for (H_T, λ) to have a (d_1, \dots, d_k) -matching. Otherwise, T has $k \geq 2$ disjoint down-sets D_1, \dots, D_k satisfying $\mu(\mathcal{U}_{D_i}) \geq d_i$ for $i \in [k]$. As (G, μ) is (d_1, \dots, d_k) -pseudorandom, there exist $i \neq j$ and an edge $xy \in E(G)$ such that $x \in \mathcal{U}_{D_i}$ and $y \in \mathcal{U}_{D_j}$. We have $x' \in D_i$ and $y' \in D_j$ such that $x \in T^+(x')$ and $y \in T^+(y')$. Since T is a BBT spanning tree of G , we may assume that $y' \in x' \downarrow_T \subseteq D_i$, which is absurd as D_i and D_j cannot include a common element y' !

Hence, by applying Theorem 2 on the weighted rooted tree (T, λ) , we see that T has a saturated chain C with $\lambda(C) \geq c$. Putting $B = \mathcal{U}_C$ then finishes the proof. \square

- Remark 1.* – The step of going from finite to infinite in Section 2.2 is not hard but also not totally trivial; A recent work in the same direction is to get a Gomory-Hu tree of an infinite graph [26].
- A normal spanning tree of a graph G is a spanning rooted tree T of G in which $(x \downarrow_T) \cap (y \downarrow_T) \neq \emptyset$ whenever $xy \in E(G)$. Jung [28] proved that every countable connected graph has a normal spanning tree [14, Theorem 8.2.4]; Surely, when the graph is finite, a normal spanning tree is just a depth-first search tree. Based on any normal spanning tree of a countable connected graph, we can use Theorem 3 to get an infinite counterpart of Theorem 6.
 - In order to emulate the proof of Theorem 7 to get a counterpart for countably infinite graphs, we should go to Theorem 3 and we need to guarantee that every countable connected graph has a BBT spanning tree. We [59] have found that every connected graph has a BBT spanning tree and so such a counterpart does exist.

5 Additive function

In the conference version of this work, we report [60, Corollary 1, Corollary 2] without proof due to the constraint on paper length. It turns out that the statement of those two corollaries are not accurate and we correct them in this final section.

Let P be a finite poset. For any $r, x \in P$, let $\text{CH}_P(r, x)$ denote the set of saturated chains of P from r to x and we write $\text{CH}_P(r)$ for $\bigcup_{x \in P} \text{CH}_P(r, x)$. The elements from $\text{CH}_P(r)$ form a poset in which $A > B$ if and only if $A \subsetneq B$.

Theorem 8. *Let P be a countable poset, let $r \in P$ and let μ be an additive and σ -additive function on 2^P . For every $x \in P$, we assume that $|\text{CH}_P(r, x)|$ is finite and denote it by n_x . For any $k + 1$ nonnegative reals c, d_1, \dots, d_k satisfying $(k - 1)c + \sum_{i=1}^k d_i \leq \mu(r \downarrow_P)$, either there exists a saturated chain C of $r \downarrow_P$ such that $\sum_{u \in C} \frac{\mu(u)}{n_u} \geq c$, or there exist pairwise disjoint down-sets D_1, \dots, D_k of $\text{CH}_P(r)$ such that $\sum_{u \in D_i} \frac{\mu(u)}{n_u} \geq d_i$ for all $i \in [k]$.*

Proof. Construct a rooted tree T on the vertex set $V(T) := \text{CH}_P(r)$ with an arc from A to B if and only if $A \subseteq B$ and $|B \setminus A| = 1$. Note that $\{r\}$ is the root of T . Define an additive function λ on $2^{V(T)}$ such that

$$\lambda(A) = \frac{\mu(x)}{n_x}$$

for all $A \in \text{CH}_P(r, x)$. An application of Theorem 1 on the weighted rooted tree (T, λ) yields the result. \square

For a tree G and $W \subseteq V(G)$, we write $\text{Conv}_G(W)$ for the union of all paths in G connecting vertices of W .

Theorem 9. *Let V be a countable set, let $k \geq 2$ be an integer, and let μ be an additive and σ -additive function on 2^V . Let G be a tree with $V(G) = V$ and*

let $W \subseteq V(G)$. Let d_1, \dots, d_k, c be $k + 1$ positive reals such that $\mu(V(G)) \geq \sum_{i=1}^k d_i + (k - 1)c$. Then there are either k disjoint subsets D_1, \dots, D_k such that $\mu(D_i) \geq d_i$ and $G - D_i$ is a tree containing W for all $i \in [k]$, or there is a path P in $G - \text{Conv}_G(W)$ such that $\max\{\mu(P), \mu(\text{Conv}_G(W) \cup P)\} \geq c$.

Proof. Shrink $\text{Conv}_G(W)$ into a vertex r to get a new tree G' from G . Orient each edge of G' so that every arc will go towards r and we thus get a rooted tree T . Define λ to be the additive function on $2^{V(T)}$ such that

$$\lambda(X) = \begin{cases} \mu(X) & \text{if } r \notin X \subseteq V(T); \\ \mu((X \setminus \{r\}) \cup W) & \text{if } r \in X \subseteq V(T). \end{cases}$$

By virtue of Theorem 1, either (H_T, λ) is (d, k) -fat or T has an end C such that $\lambda(C) \geq c$. This implies our claim, as desired. \square

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