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Top-heavy phenomena for transformations*

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Abstract

Let S be a transformation semigroup acting on a set Ω. The action of S on Ω can be naturally extended to be an action on all subsets of Ω. We say that S is ℓ-homogeneous provided it can send A to B for any two (not necessarily distinct) ℓ-subsets A and B of Ω. On the condition that $k \leq \ell < k + \ell \leq |\Omega|$, we show that every ℓ-homogeneous transformation semigroup acting on Ω must be k-homogeneous. We report other variants of this result for Boolean lattices and projective geometries. In general, any semigroup action on a poset gives rise to an automaton and we associate some sequences of integers with the phase space of this automaton. When the poset is a geometric lattice, we propose to study various possible regularity properties of these sequences, especially the so-called top-heavy property.

Keywords: Grassmannian, homogeneity, incidence operator, strong shape, valuated poset, weak shape.

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1 Introduction

1.1 Transformation and phase space

Let Γ be a *digraph*, namely a pair consisting of its vertex set $\mathrm{V}(\Gamma)$ and arc set $\mathrm{E}(\Gamma) \subseteq \mathrm{V}(\Gamma) \times \mathrm{V}(\Gamma)$. We call Γ *symmetric* if $(u, v) \in \mathrm{E}(\Gamma)$ holds if and only if so does $(v, u) \in \mathrm{E}(\Gamma)$. For any $A \subseteq \mathrm{V}(\Gamma)$, we adopt the notation $\Gamma[A]$ for the subdigraph of Γ induced by A which has vertex set A and arc set $\mathrm{E}(\Gamma) \cap (A \times A)$. The number of weakly connected components and the number of strongly connected components of Γ will be dubbed $\mathrm{wcc}(\Gamma)$ and $\mathrm{scc}(\Gamma)$, respectively.

For a set Ω, all maps from Ω to itself form the set Ω^Ω. For each $g \in \Omega^\Omega$ and $\alpha \in \Omega$, we write αg for the image of α under the map g. The composition of maps provides an

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associative product on the set Ω^Ω and thus turns it into a monoid, namely a semigroup with a multiplicative unit. We call this monoid the *full transformation monoid* on Ω and denote it by $\mathrm{T}(\Omega)$. A subset of $\mathrm{T}(\Omega)$ which is closed under map composition, whether or not it contains the identity map on Ω, is called a *transformation semigroup* acting on Ω. Let S be a transformation semigroup on Ω. We say that S is *transitive on a set* $A \subseteq \Omega$ if for every $\alpha, \beta \in A$ we can find $g \in S$ such that $\alpha g = \beta$; We call S *transitive* if S is transitive on Ω. If the transformation semigroup S is generated by a set $G \subseteq \Omega^\Omega$, namely S consists of products of elements of G of positive length, we call (S, G) a *deterministic automaton* on Ω [60, §1]. The *phase space* of an automaton (S, G) on Ω, denoted by $\Gamma(S, G)$, is the digraph with vertex set Ω and arc set $\{(\alpha, \alpha g) : \alpha \in \Omega, g \in G\}$. When Ω has at least two elements, the claim that S is transitive is equivalent to the claim that $\Gamma(S, G)$ is strongly connected for any generator set G of S. We write $\Gamma(S, S)$ simply as $\Gamma(S)$ and note that each strongly/weakly connected component of $\Gamma(S)$ coincides with a strongly/weakly connected component of $\Gamma(S, G)$ for any generator set G of S. For all work in this paper, we can simply focus on $\Gamma(S)$ instead of considering $\Gamma(S, G)$ for any specific generator set G. We emphasize $\Gamma(S, G)$ from the phase space viewpoint here to highlight the connection between semigroup theory and automaton theory, and to indicate the role played by the choice of G in some problems related to various distance functions on the phase space, say the Černý conjecture. For any set Ω, a subset of $\mathrm{T}(\Omega)$ forms a *permutation group* on Ω whenever it is a transformation semigroup and each element has an inverse in it, namely it is a set of bijective transformations of Ω and is closed under compositions and taking inverses. Permutation groups correspond to reversible deterministic automata.

Let Ω be a set. We follow the common practice to use 2^Ω for the power set of Ω. Note that under the inclusion relation, 2^Ω gives rise to a Boolean lattice P for which $P_i = \binom{\Omega}{i}$ for all nonnegative integer $i \leq |\Omega|$. For each $g \in \mathrm{T}(\Omega)$, let $\overline{g}$ be the element in $\mathrm{T}(2^\Omega)$ that sends each $A \in 2^\Omega$ to $A\overline{g} \doteq \{ag : a \in A\}$. More generally, for each $G \subseteq \mathrm{T}(\Omega)$, $\overline{G}$ refers to the set $\{\overline{g} : g \in G\}$. For any transformation semigroup S on Ω and any generator set G of S, $\overline{S}$, as a semigroup derived from S, is known to be the *powerset transformation semigroup of* S acting on 2^Ω and $(\overline{S}, \overline{G})$ is known to be the *powerset automaton* of (S, G). It may be interesting to iterate the powerset automaton construction and examine the evolution of the phase spaces of the resulting automata.

When discussing transformation semigroups, we may often be more interested in those which preserve some structures, say simplicial maps for simplicial complexes, continuous maps for topological spaces, ordering preserving maps for posets, or adjacency-preserving maps in matrix geometry [47, 59]. Unlike the work on group actions on posets [4, 52] and matroids [19], very little has been done on semigroup actions on these structures [55]. Moving from group actions to semigroup actions is just to consider general deterministic automata instead of reversible ones. We suggest one small problem on "derived transitiveness" of semigroup actions in next section.

1.2 Valuated poset and its shape

For any two sets Ω and Ψ, if they are different or if we do not emphasize that they may be equal, the image of $\omega \in \Omega$ under a map $g \in \Psi^\Omega$ is denoted $g(\omega)$; Note that we often write it as ωg when $\Omega = \Psi$.

A *poset* P consists of a set Ω and a binary relation $<_P$ on it which is transitive and acyclic, namely we require that $\alpha <_P \alpha$ never happens, and that $\alpha <_P \beta$ and $\beta <_P \gamma$ implies $\alpha <_P \gamma$ for all $\alpha, \beta, \gamma \in \Omega$. We often just write P for its ground set Ω and we say the poset P is *finite* if $|P|$ is finite. For each $\alpha \in P$, the *principal ideal* generated by α is the set $\{\beta : \beta <_P \alpha\} \cup \{\alpha\} \subseteq P$, which we denote by $P_\downarrow(\alpha)$; The *principal filter* generated by α is the set $\{\beta : \alpha <_P \beta\} \cup \{\alpha\} \subseteq P$, which we denote by $P_\uparrow(\alpha)$. A map g from a poset P to a poset Q is *order-preserving* if $g(\beta) \in Q_\downarrow(g(\alpha))$ holds whenever $\beta \in P_\downarrow(\alpha)$. We use $\mathrm{End}(P)$ to denote the set of all order-preserving maps from P to itself.

Let $\mathbb{Z}_{\geq 0}$ be the set of nonnegative integers which carries a natural poset structure such that $a < b$ in $\mathbb{Z}_{\geq 0}$ if and only if $b - a$ is a positive integer. A *valuation* on a poset P is an order-preserving map r_P from P to the poset $\mathbb{Z}_{\geq 0}$; We call $\mathrm{r}_P(x)$ the rank of x in the valuated poset. When we say P is a *valuated poset*, we are considering the poset P together with a valuation r_P, though the valuation may be only implicitly indicated. The *rank* of a valuated poset P, denoted by $\mathrm{r}(P)$, is the maximum value of $\mathrm{r}_P(\alpha)$ for $\alpha \in P$ if it exists and is ∞ otherwise. For a poset P, the symbols like $<_P$ and r_P will often be abbreviated to $<$ and r when no confusion can arise. Let P be a valuated poset. For any $k \in \mathbb{Z}_{\geq 0}$, we write P_k for the set $\{\alpha \in P : \mathrm{r}(\alpha) = k\}$. We call the sequence $|P_0|, |P_1|, \ldots$ the *shape* of the valuated poset and refer to it by $\mathsf{S}(P)$. If $\mathrm{r}(P) < \infty$, $\mathsf{S}(P)$ is a sequence of $\mathrm{r}(P) + 1$ nonnegative integers.

Let P be a valuated poset and let S be a subsemigroup of $\mathrm{End}(P)$. The *weak shape of P under the action of S* is the sequence

$$\mathrm{wcc}(\Gamma(S)[P_0]), \mathrm{wcc}(\Gamma(S)[P_1]), \ldots$$

which we denote by $\mathsf{WS}(S, P)$; While the *strong shape of P under the action of S* is the sequence

$$\mathrm{scc}(\Gamma(S)[P_0]), \mathrm{scc}(\Gamma(S)[P_1]), \ldots$$

which we denote by $\mathsf{SS}(S, P)$. Note that

$$\mathsf{S}(P) = \mathsf{WS}(S, P) = \mathsf{SS}(S, P)$$

when the semigroup S consists of the identity transformation from $\mathrm{End}(P)$.

The main purpose of this note is to propose the study of the possible regularity in the strong/weak shape of a semigroup acting on a valuated poset.

1.3 Geometric lattice and top-heavy property

A matroid M consists of a ground set $\mathscr{E}_M$ and a rank function r_M from $2^{\mathscr{E}_M}$ to the set of nonnegative integers plus infinity such that the rank axioms are satisfied [13, §1.5]. The flats of a matroid M, ordered by inclusion, form a very pretty structure, called the *matroid lattice* of M and denoted by $\mathrm{F}(M)$. For each nonnegative integer t, let $\mathrm{F}_t(M)$ be the set of all rank-t flats of the matroid M. A *geometric lattice* is an atomic and semimodular lattice which does not have any infinite chain [57, p. 305]. We mention that a geometry lattice is cryptomorphic to a natural object called combinatorial geometry [57, Theorem 23.1] and that finite geometric lattice is nothing but finite matroid lattice [34, p. 163, Birkhoff's Theorem]. A geometric/matroid lattice has a natural valuated poset structure, where the valuation is given by its rank function. For example, for a matroid M, all elements in $\mathrm{F}_t(M)$ have rank t. In a geometric lattice, the elements of rank 1, 2 and 3 are viewed as points, lines and planes, respectively, thus giving geometric intuitions to many results about geometric lattices.

For a linear space V and each nonnegative integer k, we use $\mathrm{Gr}(k, V)$ for the set of all k-dimensional linear subspaces of V and we call $\cup_{k=0}^{\infty} \mathrm{Gr}(k, V)$ the Grassmannian of V, which is denoted by $\mathrm{Gr}(V)$. For each finite dimension linear space V, $\mathrm{Gr}(V)$ is surely a geometric lattice where elements from $\mathrm{Gr}(k,V)$ have rank k.

Example 1.1. Let n and k be two positive integers such that $k<n$. Fix a non-degenerate inner product on $\mathbb{Q}^n$, say $\langle,\rangle$. For each $g\in \mathrm{GL}_n(\mathbb{Q})$, let $g^\top$ stand for the adjoint of g, namely the element such that $\langle ug,v\rangle=\langle u,vg^\top\rangle$ for all $u,v\in\mathbb{Q}^n$, and we write $g_\#$ for $(g^{-1})^\top$. Let $S\le \mathrm{GL}_n(\mathbb{Q})$ be a matrix group acting on $\mathbb{Q}^n$. If $\overline{S}$ is transitive on the set of all dimension-k subspaces and if $g_\#\in S$ for all $g\in S$, then $\overline{S}$ is transitive on the set of dimension-$(n-k)$ subspaces. To see this, fix a pair of subspaces (U,U') which are orthogonal complements to each other with respect to $\langle,\rangle$ and $(\dim U,\dim U')=(k,n-k)$. For each $g\in S$, we can see that $U\overline{g}$ and $U'\overline{g_\#}$ are orthogonal complements to each other with respect to the given inner product $\langle,\rangle$. Considering the set of pairs $\{(U\overline{g},U'\overline{g_\#}):\ g\in S\}$, we see that the transitivity on $\mathrm{Gr}(k,\mathbb{Q}^n)$ implies transitivity on $\mathrm{Gr}(n-k,\mathbb{Q}^n)$.

Motivated by Example 1.1, here is a very simple question on the very simple geometric lattice $\mathrm{Gr}(\mathbb{Q}^3)$. Surprisingly, we could not find a discussion of it in the literature.

Question 1.2. [1] If S is a general matrix group acting on $\mathbb{Q}^3$, can we still draw the conclusion that $\overline{S}$ is transitive on $\mathrm{Gr}(1,\mathbb{Q}^3)$ from the assumption of its transitivity on $\mathrm{Gr}(2,\mathbb{Q}^3)$? What about only assuming that S is a matrix semigroup?

Some seemingly weird properties of sequences turn out to be ubiquitous when we are examining some interesting structures or processes [6, 10, 11, 27, 51, 54]. We review some of them below. Let $c_0,c_1,\dots,$ be a sequence of $n+1$ real numbers, where n can be finite or infinite. We call it *t-top-heavy* if $c_k\le t$ whenever there exists an integer ℓ such that $k\le\ell\le k+\ell\le n$ and $c_\ell\le t$; We call it *top-heavy* if it is t-top-heavy for all $t\in\mathbb{R}$, namely $c_k\le c_\ell$ holds for all k,ℓ such that $k\le\ell\le k+\ell\le n$; We call it *unimodal* if you cannot find three distinct integers i,j,k such that $0\le i<j<k\le n$ and $c_i-c_j>0>c_j-c_k$; We call it *log-concave* if $c_i^2\ge c_{i-1}c_{i+1}$ for all $i=1,\dots,n-1$. When n is finite, we call the sequence *real-rooted* provided the polynomial $c_0+c_1x+\cdots+c_nx^n$ in the unknown x only has real roots and we call it *ultra-log-concave* provided $\frac{c_0}{\binom{n}{0}},\dots,\frac{c_n}{\binom{n}{n}}$ forms a log-concave sequence. Note that Question 1.2 is about the possible 1-top-heavy property of the strong shape of $\mathrm{Gr}(\mathbb{Q}^3)$ under a matrix semigroup action.

In 1970s, two log-concavity conjectures [54, Conjecture 3] appeared in combinatorics community which claims that the sequences of Whitney numbers of both the first kind and the second kind of a finite matroid are log-concave. The first conjecture is verified by Adiprasito, Huh and Katz [1]. Mason [38] has made variants and stronger versions of the second conjecture; But even the original conjecture is still open. Dowling and Wilson [22] conjectured that the sequence of Whitney numbers of the second kind of a finite matroid is top-heavy. When restricted to finite realizable matroids, this top-heavy conjecture is proved by Huh and Wang [26].

The second log-concavity conjecture as described above, which is about the Whitney numbers of the second kind [45], simply says that the shape of every geometric lattice is log-concave. The above-mentioned top-heavy conjecture says that the shape of every finite geometric lattice is top-heavy. On the condition that these two conjectures are both true, we

[1] Jiarui Fei told us that, by a simple dimension argument, on the condition that the action of a group S on $\mathbb{Q}^n$ is algebraic, it is transitive on $\mathrm{Gr}(k,\mathbb{Q}^n)$ if and only if it is transitive on $\mathrm{Gr}(n-k,\mathbb{Q}^n)$.

know that the shape of a finite geometric lattice is both log-concave (and hence unimodal) and top-heavy. Can we draw this conclusion for the strong/weak shape of some semigoup actions on some geometric lattices?

Boolean lattices, partition lattices and projective/affine geometries are some most well-known geometric lattices. It is easy to see that their shapes are all ultra-log-concave (and hence real-rooted) and top-heavy [35]. The main result of this paper, Theorems 2.1 and 2.12, is to deduce the top-heavy property for the strong/weak shape of some semigroups acting on Boolean lattices and projective/affine geometries. The semigroups considered by us are those derived from "simple" transformations. We also report a bit our effort of tackling the same problem for partition lattices and the Vámos matroid.

In Section 2, we will present our main results as well as pertinent problems, examples, and remarks. The first three subsections are devoted to Boolean lattices, partition lattices and projective/affine geometries. The last subsection is a simple discussion in the context of matroids. In order to present our proofs of the main results, we develop some technical tools in Section 3. In the sequel, we provide in Sections 4 to 7 all the proofs missing from Sections 2.1 to 2.4. We indicate some further research problems in Section 8, thus concluding the paper.

2 A top-heavy promenade

2.1 Boolean lattice and homogeneity

For any set Ω, the set $\mathrm{B}_\Omega \doteq \cup_{k=0}^{\infty}\binom{\Omega}{k}$ forms a poset under the inclusion relationship, which is often known as the *Boolean semiring over* Ω – the set 2^Ω gives rise to the Boolean algebra over Ω. When we view B_Ω as a valuated poset, unless stated otherwise, the valuation will be $\mathrm{r}(A) = |A|$ for all $A \in \mathrm{B}_\Omega$. If Ω is a finite set, B_Ω coincides with 2^Ω and is referred to as a Boolean lattice.

Let A and Ω be two sets with $A \subseteq \Omega$. For any $g \in \Omega^\Omega$, write $g|_A$ for the restriction of g on A. Let S be a transformation semigroup on Ω. For any positive integer $k \le |\Omega|$, we name S *k-homogeneous* if the transformation semigroup $\overline{S}$ is transitive on $\binom{\Omega}{k}$, that is, $\mathrm{scc}(\Gamma(\overline{S})[\binom{\Omega}{k})] = 1$. The *stabiliser permutation group* of (S, A) is the permutation group $S_A \doteq \{g|_A :\ g \in S, A\overline{g} = A\}$ acting on A. The *relative transformation semigroup* of (S, A) is the transformation semigroup $\tilde{S}_A \doteq \{g|_A :\ g \in S, A\overline{g} \subseteq A\}$ acting on A. Note that $\tilde{S}_A$ may not be transitive on A even when S is transitive on A.

Theorem 2.1. *Let Ω be a set. Let S be a transformation semigroup on Ω and let Γ be the phase space of $\overline{S}$.*

(1) $\mathsf{SS}(\overline{S}, \mathrm{B}_\Omega)$ *is* 1*-top-heavy.*

(2) *If $|\Omega| < \infty$, then both* $\mathsf{WS}(\overline{S}, 2^\Omega)$ *and* $\mathsf{SS}(\overline{S}, 2^\Omega)$ *are top-heavy.*

(3) *Let k and ℓ be two integers such that $k \le \ell \le k + \ell \le |\Omega| + 1$. Let $A \in \binom{\Omega}{k}$ and $B \in \binom{\Omega}{\ell}$. If $|\Omega| < \infty$ and S is ℓ-homogeneous, then* $\mathrm{scc}(\Gamma(S_A)) = \mathrm{wcc}(\Gamma(S_A)) \le \mathrm{wcc}(\Gamma(S_B)) = \mathrm{scc}(\Gamma(S_B))$.

Question 2.2. Take a finite set Ω and two integers k and ℓ such that $k \le \ell < k+\ell \le |\Omega|+1$. Let S be an ℓ-homogeneous transformation semigroup acting on Ω. For any $A \in \binom{\Omega}{k}$ and $B \in \binom{\Omega}{\ell}$, does it always hold that $\mathrm{wcc}(\Gamma(\tilde{S}_A)) \le \mathrm{wcc}(\Gamma(\tilde{S}_B))$?

When restricting to permutation groups, the results in Theorem 2.1 are all known more than 40 years ago: Claim (1) for an infinite set Ω was discovered by Brown [12, Corollary 1]; Claims (1) and (2) for a finite set Ω were derived by Livingstone and Wagner [36, Theorem 1]; Claim (3), as well as a positive answer to Question 2.2 for permutation groups, was proved by Cameron [15, Proposition 2.3] under the mild restriction of $k+\ell \leq |\Omega|$. Let G be a group acting on a finite set Ω. By Theorem 2.1 (2), or more precisely Livingstone-Wagner Theorem [36, Theorem 1], we know that the strong/weak shape of 2^Ω under the action of $\overline{G}$ is a symmetric unimodal distribution. This means that, for any two integers k and ℓ such that $k \leq \ell < k+\ell \leq |\Omega|$, the number of $\overline{G}$-orbits on $\binom{\Omega}{\ell}$ is equal to the sum of a nonnegative integer c plus the number of $\overline{G}$-orbits on $\binom{\Omega}{k}$. As an improvement of this fact, Siemons [49, Corollary 4.3] found a natural linear space whose dimension equals this integer c and he [49, Theorem 4.2] even obtained an algorithm to reconstruct the $\overline{G}$-orbits on $\binom{\Omega}{k}$ from the information on the $\overline{G}$-orbits on $\binom{\Omega}{\ell}$ without reference to the group G.

Question 2.3. Let Ω be a finite set, and let k and ℓ be two integers such that $k \leq \ell < k+\ell \leq |\Omega|$. Let S be a transformation semigroup on Ω and let Γ be the phase space of $\overline{S}$.

(1) Is there a counterpart of [49, Corollary 4.3] which explains the nonnegativeness of the integer $\mathrm{wcc}(\Gamma[\binom{\Omega}{\ell}]) - \mathrm{wcc}(\Gamma[\binom{\Omega}{k}])$?

(2) If S is $(\ell+1)$-homogeneous, is there a counterpart of [49, Corollary 4.3] which explains the nonnegativeness of the integer $\mathrm{scc}(\Gamma(S_B)) - \mathrm{scc}(\Gamma(S_A))$ for any $A \in \binom{\Omega}{k}$ and $B \in \binom{\Omega}{\ell+1}$?

(3) Is there any algorithm to determine the weakly connected components of $\Gamma[\binom{\Omega}{k}]$ from the weakly connected components of $\Gamma[\binom{\Omega}{\ell}]$ without reference to the transformation semigroup S?

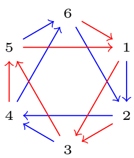
Example 2.4. Let Ω be a set carrying a linear order $\prec$. A map $g \in \Omega^\Omega$ is order-preserving with respect to $\prec$ provided αg is not bigger than βg in $\prec$ whenever α is not bigger than β in $\prec$. Let S be the monoid consisting of all order-preserving maps on Ω with respect to the given linear order $\prec$. It is easy to see that S is ℓ-homogeneous for all $\ell \leq |\Omega|$. Note that the only permutation contained in S is the identity map in case that Ω is a finite set. This suggests that you may not be able to read Theorem 2.1 or answer Question 2.3 directly from those known facts on permutation groups.

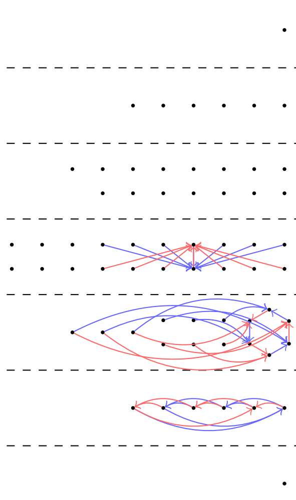
Example 2.5. Let $\Omega = \{1, \ldots, 6\}$. Let r and b be two maps in $\mathrm{T}(\Omega)$ such that

$$r(1) = r(2) = 3, \quad r(3) = r(4) = 5, \quad r(5) = r(6) = 1;$$
$$b(6) = b(1) = 2, \quad b(2) = b(3) = 4, \quad b(4) = b(5) = 6.$$

Let $S = \langle r, b\rangle$. On the left of Fig. 1, we depict the phase space $\Gamma(S, \{r, b\})$; On the right of Fig. 1, we display both the strong shape and the weak shape of 2^Ω under the action of $\overline{S}$. Both weak shape and strong shape are unimodal and top-heavy. But neither of them is log-concave. Note that the peak of the weak shape does not happen at the middle rank 3.

Example 2.6. Let Ω be a set of size $n \geq 3$ and let S be a transformation semigroup acting on Ω. If $\mathsf{SS}(\overline{S}, 2^\Omega)$ is not a sequence of all ones and has at least two ones at the beginning





Vertex set	wcc	scc
$\binom{\Omega}{6}$	1	1
$\binom{\Omega}{5}$	6	6
$\binom{\Omega}{4}$	15	15
$\binom{\Omega}{3}$	7	19
$\binom{\Omega}{2}$	1	10
$\binom{\Omega}{1}$	1	1
$\binom{\Omega}{0}$	1	1

Figure 1: $\Gamma(S, \{r, b\})$ and $\Gamma(\overline{S}, \{\overline{r}, \overline{b}\})[\binom{\Omega}{k}]$, $k \in \{0, 1, \dots, 6\}$. See Example 2.5.

of it, then it cannot be log-concave. For example, if S is the alternating group of order $n \geq 4$, then $\mathsf{SS}(\overline{S}, 2^\Omega)$ is not log-concave. If the semigroup S is 2-homogeneous but not 3-homogoneous, we also obtain that $\mathsf{SS}(\overline{S}, 2^\Omega)$ is not log-concave.

Example 2.7. Let n and k be two integers such that $1 \leq k \leq n$. Let Ω be a set of size n and take $X \in \binom{\Omega}{k}$. Let S be the set $\{f \in \mathrm{T}(\Omega) : \ f|_X = \mathrm{Id}\,|_X, \Omega\overline{f} = X\}$. Note that S is a transformation semigroup on 2^Ω satisfying

$$\mathrm{wcc}(\Gamma(\overline{S})[\binom{\Omega}{i}] = \begin{cases} 1, & \text{if } 0 \leq i \leq k; \\ \binom{n}{i}, & \text{if } k+1 \leq i \leq n. \end{cases}$$

This shows that the sequence $\mathsf{WS}(\overline{S}, 2^\Omega)$ is unimodal and top-heavy and that it is not log-concave when $n \geq 2$. Note that $\mathsf{SS}(\overline{S}, 2^\Omega)$ is a sequence of all ones.

Question 2.8. Let S be a transformation semigroup acting on an n-element set Ω. When can we conclude that the strong/weak shape of 2^Ω under the action of $\overline{S}$ is unimodal?

Neumann [41] asked whether every λ-homogeneous permutation group is θ-homogeneous for all cardinals $\lambda > \theta \geq \aleph_0$. Assuming Martin's Axiom, Shelah and Thomas [48] gave a negative answer to it. Hajnal [25] supplied an example to show that 2^θ-homogeneity does not imply θ-homogeneity. For each statement in Theorem 2.1 and Question 2.2 and 2.3, it is interesting to see whether or not it holds in the case that Ω is an infinite set. We are also wondering if the rich theory on oligomorphic permutation groups [16] should have a counterpart for transformation semigroups.

For any two positive integers k and ℓ, we say that a transformation semigroup S acting on Ω is (k, ℓ)-*homogeneous* provided for every $A \in \binom{\Omega}{k}$ and $B \in \binom{\Omega}{\ell}$ we can find $g \in S$ such that $A\overline{g} \cap B \in \{A\overline{g}, B\}$. Note that Araújo and Cameron [3] have studied (k, ℓ)-homogeneous permutation groups. Beyond the homogeneity property discussed so far, there has been an active study of those permutation groups which are transitive on the set of all ordered or unordered partitions of a set of a given shape [2, 21, 37, 40]. There are many ways to define corresponding properties for transformation semigroups. A systematic study of the extension of the results on permutation groups to transformation semigroups should be fruitful.

2.2 Partition lattice

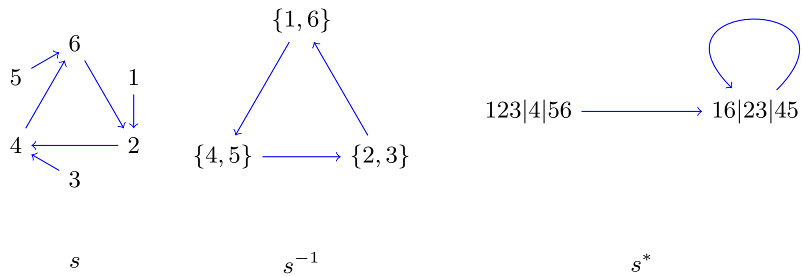


Figure 2: A map, its inverse and the derived action on partitions.

Let Ω be a set. For any map $s \in \Omega^\Omega$, we define its *kernel map*, denoted by s^{-1}, to be the map from 2^Ω to 2^Ω that sends $X \in 2^\Omega$ to $Xs^{-1} = \{y \in \Omega : \; ys \in X\} \in 2^\Omega$. To illustrate the definition, we depict the phase space of a map s on the left of Fig. 2 and part of the phase space of s^{-1} in the middle of Fig. 2. A *partition* of Ω is a set of nonempty disjoint subsets of Ω whose union is Ω. We call these elements of a partition its *blocks*. Write $\mathrm{P}(\Omega)$ for the set of all partitions of Ω. When $|\Omega| < \infty$, the set $\mathrm{P}(\Omega)$ together with the refinement relation forms a geometric lattice, which we call the *partition lattice* of Ω. Note that the rank of a partition in this geometric lattice is $|\Omega|$ minus the number of its blocks. Let $\mathrm{P}_k(\Omega)$ be the set of rank-k partitions of Ω, namely, those partitions of Ω of size $|\Omega| - k$. Each transformation $s \in \Omega^\Omega$ induces a transformation s^* of 2^Ω such that $\Pi s^* = \{\pi s^{-1} : \; \pi \in \Pi\} \setminus \{\emptyset\}$ for all $\Pi \in \mathrm{P}(\Omega)$. We demonstrate part of the phase space of s^* on the right of Fig. 2 for the map s as shown on the left there. Let S be a transformation semigroup on Ω. We have a derived transformation semigroup $S^* := \{s^* : \; s \in S\}$ on $\mathrm{P}(\Omega)$, which we call the *kernel space* of S. We say that S is *k-kernel homogeneous* if for all $\Pi, \Pi' \in \mathrm{P}_k(\Omega)$ there exists $s \in S$ such that $\Pi s^* = \Pi'$, which surely implies $\mathrm{scc}(\Gamma(S^*)[\mathrm{P}_k(\Omega)]) \leq 1$.

Example 2.9. On the left of Fig. 3, we depict the so-called Černý automaton $\mathcal{C}_4 = \Gamma(S, G)$, where $G = \{a, b\}$ consists of two transformations on a four-element set Ω. On the right of Fig. 3, we depict the automaton $\Gamma(S^*, G^*)$ where S^* is acting on $\mathrm{P}(\Omega)$. Observe that $\mathsf{WS}(S^*, \mathrm{P}(\Omega)) = (1, 1, 1, 1)$ and $\mathsf{SS}(S^*, \mathrm{P}(\Omega)) = (1, 2, 2, 1)$ are both unimodal and top-heavy.

For any finite set Ω, $A \in 2^\Omega$, $\pi \in \mathrm{P}(\Omega)$ and $s \in \Omega^\Omega$, it holds

$$\mathrm{r}(A) \geq \mathrm{r}(A\overline{s}), \; \mathrm{r}(\pi) \leq \mathrm{r}(\pi s^*).$$

This difference between Boolean lattice and partition lattice somehow hints at our difficulty of turning the following conjecture into a result like Theorem 2.1.

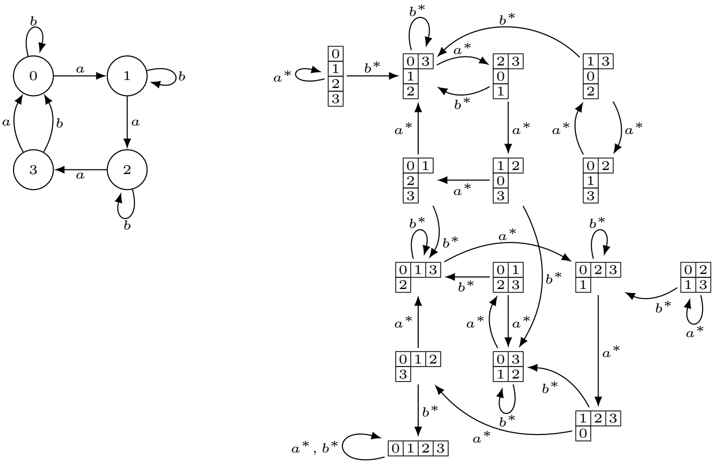


Figure 3: Černý automaton $\mathcal{C}_4$ and its kernel space. See Example 2.9.

Conjecture 2.10. *Let Ω be a finite set and let S be a semigroup acting on Ω. Then both* $\mathsf{WS}(S^*, \mathrm{P}(\Omega))$ *and* $\mathsf{SS}(S^*, \mathrm{P}(\Omega))$ *are top-heavy.*

For each set Ω and each positive integer $k \leq |\Omega|$, we use $\mathrm{P}(\Omega, k)$ for the set of partitions of Ω into k blocks.

Question 2.11. (1) Take two positive integers k and ℓ with $k < \ell$. Let Ω be an infinite set and let S be a semigroup S acting on Ω. If S^* is transitive on $\mathrm{P}(\Omega, k)$, is it true that S^* is transitive on $\mathrm{P}(\Omega, \ell)$?

(2) The shapes of all Dowling lattices, which include all partition lattices, are real-rooted [8]. What about the top-heavy property of the (strong/weak) shapes of Dowling lattices?

2.3 Subspace lattice

Let Ω be a finite set of size n, let F be a finite field and let k be a nonnegative integer such that $k \leq n$. We mention that $\mathrm{Gr}(k, F^{\Omega})$ is a q-analogue of $\binom{\Omega}{k}$ and their relationship is like the one between Johnson graphs and Grassmann graphs [42]. For each prime power q, we write $\mathbb{F}_q$ for the q-element finite field and write $\mathrm{Mat}_n(\mathbb{F}_q)$ for the multiplicative semigroup of all n by n matrices over $\mathbb{F}_q$. The set of all linear subspaces of $\mathbb{F}_q^n$ is denoted by $\mathcal{P}_{q,n} \doteq \mathrm{Gr}(\mathbb{F}_q^n)$ and the set of all dimension-k linear subspaces of $\mathbb{F}_q^n$ is denoted by $\mathcal{P}_{q,n}^k \doteq \mathrm{Gr}(k, \mathbb{F}_q^n)$. By an affine subspace of a linear space V, we mean either the empty set or a translate of a linear subspace of V. We regard the empty set as a dimension-(-1) affine subspace. The set of all affine subspaces of $\mathbb{F}_q^n$ is denoted by $\mathcal{A}_{q,n}$ and the set of all dimension-k affine subspaces of $\mathbb{F}_q^n$ is denoted by $\mathcal{A}_{q,n}^{k+1}$. Note that $\mathcal{P}_{q,n}$ and $\mathcal{A}_{q,n}$ are known as projection geometry and affine geometry over the field $\mathbb{F}_q$. As a geometric lattice, the rank of an element in $\mathcal{P}_{q,n}^k$ and the rank of an element in $\mathcal{A}_{q,n}^k$ are both k.

We are ready to display Theorem 2.12, a q-analogue of Theorem 2.1. If the semigroup $S \leq \mathrm{Mat}_n(\mathbb{F}_q)$ is a subgroup of the general linear group $\mathrm{GL}_n(\mathbb{F}_q)$, Theorem 2.12 was already reported by Stanley in 1982 [52, Corollary 9.9].

Theorem 2.12. *Let n be a positive integer and let q be a prime power. Let $S \leq \mathrm{Mat}_n(\mathbb{F}_q)$ be a linear transformation semigroup acting on $\mathbb{F}_q^n$. For each $g \in S$, write $g^{\mathcal{P}}$ for $\overline{g}|_{\mathcal{P}_{q,n}}$ and write $g^{\mathcal{A}}$ for $\overline{g}|_{\mathcal{A}_{q,n}}$. Let $S^{\mathcal{P}}$ be the transformation semigroup $\{g^{\mathcal{P}} : g \in S\}$ acting on $\mathcal{P}_{q,n}$ and let $S^{\mathcal{A}}$ be the transformation semigroup $\{g^{\mathcal{A}} : g \in S\}$ acting on $\mathcal{A}_{q,n}$. Then $\mathsf{SS}(S^{\mathcal{P}}, \mathcal{P}_{q,n}$, $\mathsf{WS}(S^{\mathcal{P}}, \mathcal{P}_{q,n})$, $\mathsf{SS}(S^{\mathcal{A}}, \mathcal{A}_{q,n})$ and $\mathsf{WS}(S^{\mathcal{A}}, \mathcal{A}_{q,n})$ are all top-heavy.*

Remark 2.13. Kantor [30, Theorem 2] determined all the ordered-basis-transitive finite geometric lattices of rank at least three: Roughly speaking, they are Boolean lattices, projective (affine) geometries, and four sporadic designs. Kantor's classification theorem along with Theorems 2.1 and 2.12 may be a basis for getting homogeneity results about ordered-basis-transitive matroids.

Question 2.14. A general projective geometry is defined to be a modular combinatorial geometry that is connected in the sense that the point set cannot be expressed as the union of two proper flats [57, p. 313]. Can we establish a counterpart of Theorem 2.12 for general projective geometries?

In mathematics we encounter quite some nice duality phenomena, say Chow's Theorem [42, Corollary 3.1] and many duality concepts for matroids [13]. Let us report the following duality result of Stanley [52, Corollary 9.9].

Theorem 2.15 (Stanley)**.** *Let F be a finite field and let k and n be two positive integers with $k < n$. For any subgroup G of $\mathrm{GL}(n, F)$, the number of orbits of the action of G on $\mathrm{Gr}(k, F^n)$ must be the same with the number of orbits of G acting on $\mathrm{Gr}(n-k, F^n)$.*

By analysing the proofs of Theorem 2.15 in Section 6.2, we intend to understand the challenge of extending some results on group actions to that on semigroup actions.

2.4 A glimpse of matroid

In previous subsections, we discuss those poset endomorphisms which are derived from either set transformations or linear transformations. Since finite geometric lattice just encodes information of finite matroids, it is natural to ask why not directly consider matroids and morphisms among matroids, namely those transformations which preserve "independence structure".

Let M_1 and M_2 be two matroids and let f be a map from $\mathscr{E}_{M_1}$ to $\mathscr{E}_{M_2}$. We call f a *weak map from M_1 to M_2* provided

$$\mathrm{r}_{M_1}(A) \geq \mathrm{r}_{M_2}(A\overline{f})$$

holds for all $A \subseteq \mathscr{E}_{M_1}$, and we call f a *strong map from M_1 to M_2* provided the preimage of any flat in M_2 is a flat of M_1 [31, 33, 50]. It is known that all strong maps must be weak maps.

Let M be a matroid on the ground set $\mathscr{E}_M = \Omega$. Let $\mathrm{T}_M(\Omega)$ ($\mathrm{T}^*_M(\Omega)$) be the monoid consisting of all elements of $\mathrm{T}(\Omega)$ which are weak (strong) maps from M to itself. If we know that S is a subsemigroup of $\mathrm{T}_M(\Omega)$ ($\mathrm{T}^*_M(\Omega)$) acting on Ω, we can define a digraph $\Gamma_{M,t}(S)$ on $\mathrm{F}_t(M)$ as follows: for any $X, Y \in \mathrm{F}_t(M)$, there is an arc from X to Y if and only if there is $g \in S$ such that the minimum flat containing $X\overline{g}$ in M is Y. What is the relationship between the connectivity of $\Gamma_{M,t}(S)$ and $\Gamma_{M,r}(S)$ for different t and r? We can ask the same question by imposing the extra condition that every element $f \in S$ is a

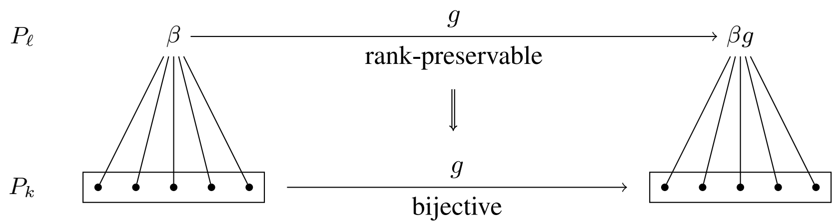


Figure 4: An (ℓ, k)-hereditary endomorphism.

bijection of Ω. If the matroid is a very special uniform matroid, namely a matroid in which all sets are independent, one can see that what is discussed in Section 1.3 becomes a very special case of this general setting.

Example 2.16. Let M be the Vámos matroid [23, p. 252] and let S be a subsemigroup of $\mathrm{T}^*_M(\mathscr{E}_M)$. It holds $\mathrm{wcc}(\Gamma_{M,1}(S)) \leq \mathrm{wcc}(\Gamma_{M,2}(S)) \leq \mathrm{wcc}(\Gamma_{M,3}(S))$ and $\mathrm{scc}(\Gamma_{M,1}(S)) \leq \mathrm{scc}(\Gamma_{M,2}(S)) \leq \mathrm{scc}(\Gamma_{M,3}(S))$.

Remark 2.17. Compared with the Fundamental Theorem of Projective (Affine) Geometry [17, 43], we think that weak/strong maps and bijective weak/strong maps for matroids are natural extensions of linear transformations and invertible linear transformations for linear spaces. We also mention the well-adopted viewpoint that full permutation group and the full transformation semigroup can be interpreted as the general linear group and the linear transformation semigroup over the field with one element.

3 Valuated poset and incidence operator

3.1 Our approach

To prepare for a proof of our main results listed in Section 2, we will introduce a key property and then present a key lemma for our work. The key property is the so-called hereditary endomorphisms. The key lemma is Lemma 3.2, which gives us some information of the strong/weak shapes of a poset under some semigroup action, provided the semigroup consists of hereditary endomorphisms and that some linear map associated with the poset is injective.

Let P be a valuated poset. For any nonnegative integers $k \leq \ell$, we call the poset P (k, ℓ)-*finite* provided $P_k \neq \emptyset$, $P_\ell \neq \emptyset$ and the set $P_\ell \cap P_\uparrow(\alpha)$ is finite for every $\alpha \in P_k$; We call P (ℓ, k)-*finite* provided $P_k \neq \emptyset$, $P_\ell \neq \emptyset$ and the set $P_\downarrow(\beta) \cap P_k$ is finite for every $\beta \in P_\ell$; We call $g \in \mathrm{End}(P)$ a (k, ℓ)-*hereditary endomorphism* if for all $\alpha \in P_k$ which satisfies $\mathrm{r}_P(g(\alpha)) = \mathrm{r}_P(\alpha) = k$ it happens that g induces a bijection from the set $P_\ell \cap P_\uparrow(\alpha)$ to $P_\ell \cap P_\uparrow(\alpha g)$; We call $g \in \mathrm{End}(P)$ an (ℓ, k)-*hereditary endomorphism* if for each $\beta \in P_\ell$, $\mathrm{r}_P(\beta g) = \mathrm{r}_P(\beta) = \ell$ ensures that g induces a bijection from the set $P_k \cap P_\downarrow(\beta)$ to $P_k \cap P_\downarrow(\beta g)$. See Fig. 4 for an illustration. For any $k, \ell \in \mathbb{Z}_{\geq 0}$, we designate by $\mathrm{hEnd}_{k,\ell}(P)$ the set of all (k, ℓ)-hereditary endomorphisms of the valuated poset P.

Let S be a transformation semigroup on a valuated poset P and let G be a generating set of S. For any two nonnegative integers k and ℓ with $k \leq \ell \leq \mathrm{r}(P)$, we set $\Pi_{S,G}(k, \ell)$ to be the digraph with vertex set P_k and arc set

$$\{(\alpha, \alpha') \in P_k \times P_k : \exists g \in G, \beta \in P_\ell \text{ s.t. } \beta g \in P_\ell, \alpha' = \alpha g, \alpha \in P_\downarrow(\beta)\};$$

We set $\Pi_{S,G}(\ell, k)$ to be the digraph with vertex set P_ℓ and arc set

$$\{(\alpha, \alpha') \in P_\ell \times P_\ell : \exists g \in G, \beta \in P_k \text{ s.t. } \beta g \in P_k, \alpha' = \alpha g, \alpha \in P_\uparrow(\beta)\}.$$

We use the shorthand $\Pi_S(k, \ell)$ for $\Pi_{S,S}(k, \ell)$.

Lemma 3.1. *Let P be a valuated poset. Take two nonnegative integers k and ℓ such that $k, \ell \leq \mathrm{r}(P)$ and that P is (ℓ, k)-finite. Let S be a sub-semigroup of $\mathrm{hEnd}_{\ell,k}(P)$, let G be a generator set of S, and let $\Gamma \doteq \Gamma(S, G)$.*

(1) *Let $\beta \in P_\ell$ and let $\alpha \in P_k$ be an element comparable with β. Assume that $g, h \in S$ are two elements such that $\beta g \in P_\ell$, $\beta gh = \beta$. Then there exists $f \in S$ such that $\beta g f \in P_\ell$, $\alpha g f = \alpha$.*

(2) *If every weakly connected component of $\Gamma[P_\ell]$ is strongly connected, then so is $\Pi_{S,G}(k, \ell)$.*

Proof. Claim (2) is immediate from (1) and so our task is just to prove (1). Without loss of generality, we assume that $k < \ell$. Since $\beta(gh) = \beta$ and $gh \in S \leq \mathrm{hEnd}_{\ell,k}(P)$, it follows that gh induces a permutation on $P_k \cap P_\downarrow(\beta)$. But from the assumption that P is (ℓ, k)-finite, we see that $P_k \cap P_\downarrow(\beta)$ is a finite set, which contains α. This means that there exists a positive integer r such that $\alpha(gh)^r = \alpha$. Accordingly, for $f = (hg)^{r-1}h \in S$ it holds $(\beta g)f = (\beta g)(hg)^{r-1}h = \beta(gh)^r = \beta \in P_\ell$ and $(\alpha g)f = (\alpha g)(hg)^{r-1}h = \alpha(gh)^r = \alpha$, finishing the proof. □

For any set Ω, $\mathbb{Q}^\Omega$ refers to the linear space of all rational functions on Ω with finite supports. If P is a (k, ℓ)-finite valuated poset, the *incidence operator* $\zeta_P^{k,\ell} : \mathbb{Q}^{P_k} \to \mathbb{Q}^{P_\ell}$ is the linear operator such that for all $f \in \mathbb{Q}^{P_k}$ and $\beta \in P_\ell$, we have

$$(\zeta_P^{k,\ell}(f))(\beta) = \begin{cases} \sum_{\alpha \in P_k \cap P_\downarrow(\beta)} f(\alpha), & \text{if } k \leq \ell; \\ \sum_{\alpha \in P_k \cap P_\uparrow(\beta)} f(\alpha), & \text{if } k > \ell. \end{cases} \tag{3.1}$$

Lemma 3.2. *Let P be a valuated poset and let k and ℓ be two nonnegative integers such that P_k and P_ℓ are both nonempty finite sets. Let S be a sub-semigroup of $\mathrm{hEnd}_{\ell,k}(P)$ and let Γ stand for $\Gamma(S)$. Assume that $\zeta_P^{k,\ell}$ is an injective linear map from $\mathbb{Q}^{P_k}$ to $\mathbb{Q}^{P_\ell}$.*

(1) $\mathrm{wcc}(\Gamma[P_k]) \leq \mathrm{wcc}(\Pi_S(k, \ell)) \leq \mathrm{wcc}(\Gamma[P_\ell])$.

(2) $\mathrm{scc}(\Gamma[P_k]) \leq \mathrm{scc}(\Pi_S(k, \ell)) \leq \mathrm{scc}(\Gamma[P_\ell])$.

Proof. (1) The first inequality is a consequence of the fact that $\mathrm{E}(\Pi_S(k, \ell)) \subseteq \mathrm{E}(\Gamma[P_k])$.

Let $W \subseteq \mathbb{Q}^{P_\ell}$ be the subspace of all functions which are constant on each weakly connected component of $\Gamma[P_\ell]$; Let $V \subseteq \mathbb{Q}^{P_k}$ be the subspace of all functions which are constant on each weakly connected component of $\Pi_S(k, \ell)$. Note that $\dim(V) = \mathrm{wcc}(\Pi_S(k, \ell))$ and $\dim(W) = \mathrm{wcc}(\Gamma[P_\ell])$ and so it suffices to demonstrate $\dim(V) \leq \dim(W)$.

By symmetry, we only deal with the case of $k \leq \ell$. For every $f \in V$ and every arc $(\beta, \beta g)$ of $\Gamma[P_\ell]$, we have

$$\begin{aligned}
(\zeta_P^{k,\ell}(f))(\beta g) &= \sum_{\alpha' \in P_k \cap P_\downarrow(\beta g)} f(\alpha') \\
&= \sum_{\alpha \in P_k \cap P_\downarrow(\beta)} f(\alpha g) \qquad (g \in \mathrm{hEnd}_{\ell,k}(P)) \\
&= \sum_{\alpha \in P_k \cap P_\downarrow(\beta)} f(\alpha) \qquad (f \in V) \\
&= (\zeta_P^{k,\ell}(f))(\beta).
\end{aligned} \tag{3.2}$$

This says that $\zeta_P^{k,\ell}(f) \in W$ for all $f \in V$. Hence, by the injectivity of $\zeta_P^{k,\ell}$, $\dim(V) \leq \dim(W)$, as wanted.

(2) The first inequality is a consequence of the fact that $\mathrm{E}(\Pi_S(k,\ell)) \subseteq \mathrm{E}(\Gamma[P_k])$.

Let $W' \subseteq \mathbb{Q}^{P_\ell}$ be the subspace of all functions which are constant on each strongly connected component of $\Gamma[P_\ell]$; Let $V' \subseteq \mathbb{Q}^{P_k}$ be the subspace of all functions which are constant on each strongly connected component of $\Pi_S(k,\ell)$. Note that $\dim(V') = \mathrm{scc}(\Pi_S(k,\ell))$ and $\dim(W') = \mathrm{scc}(\Gamma[P_\ell])$ and so it suffices to demonstrate $\dim(V') \leq \dim(W')$. Take $f \in V'$. As $\zeta_P^{k,\ell}$ is injective, we aim to show that $\zeta_P^{k,\ell}(f) \in W'$.

By symmetry, we only deal with the case of $k \leq \ell$. Assume that β and βg are from the same strongly connected component of $\Gamma[P_\ell]$, where $g \in S$. By Lemma 3.1, for every $\alpha \in P_k \cap P_\downarrow(\beta)$, α and αg fall into the same strongly connected component of $\Gamma[P_k]$ and so, as $f \in V'$,

$$f(\alpha) = f(\alpha g). \tag{3.3}$$

This allows us to write

$$\begin{aligned}
(\zeta_P^{k,\ell}(f))(\beta g) &= \sum_{\alpha' \in P_k \cap P_\downarrow(\beta g)} f(\alpha') \\
&= \sum_{\alpha \in P_k \cap P_\downarrow(\beta)} f(\alpha g) \qquad (g \in \mathrm{hEnd}_{\ell,k}(P)) \\
&= \sum_{\alpha \in P_k \cap P_\downarrow(\beta)} f(\alpha) \qquad (\text{Eq. } (3.3)) \\
&= (\zeta_P^{k,\ell}(f))(\beta),
\end{aligned}$$

proving that $\zeta_P^{k,\ell}(V') \subseteq W'$, as desired. □

In order to apply Lemma 3.2, we may need to have some results to guarantee the injectivity of an incidence operator. In this regard, a good understanding of the incidence algebra of a poset may be useful [34, 61]. We mention that Guiduli [5, Theorem 9.4] has established an injectivity result for the so-called rank-regular semi-lattices. It may also be quite useful if the following conjecture [32, Conjecture 1.1] can be verified.

Conjecture 3.3 (Kung)**.** *Let P be a finite geometric lattice. Let k and ℓ be two positive integers such that $k \leq \ell \leq \frac{\mathrm{r}(P)}{2}$. Then $\ker(\zeta_P^{k,\ell}) = \{0\}$.*

3.2 Incidence operator as an intertwiner

For $f \in \Psi^\Omega$, we sometimes need to talk about $f(\omega)$ for $\omega \notin \Omega$. Following the practice of those mathematics with natural multivalued operations [7, 14, 58], we create a universal "don't care" symbol $\star \notin \Psi$ and will set $f(\omega) = \star$. We often regard $\star$ as all possible values in Ψ and so, whenever we have some addition operation $+$ on Ψ, we extend it to $\Psi \cup \{\star\}$ by setting $\star + \psi = \star$ for all $\psi \in \Psi \cup \{\star\}$.

Let P be a valuated poset. Let k, ℓ be two nonnegative integers no greater than $\mathrm{r}(P)$. Let $g \in P^P$. For $f \in \mathbb{Q}^{P_k}$, we write $fg^{\dagger,k}$ for the element in $(\{\star\} \cup \mathbb{Q})^{P_k}$, where $\star$ stands for "don't care" and can be thought of as the whole set $\mathbb{Q}$, such that the following holds for all $\beta \in P_k$:

$$fg^{\dagger,k}(\beta) = \begin{cases} f(\beta g), & \text{if } \beta g \in P_k; \\ \star, & \text{if } \beta g \notin P_k. \end{cases}$$

Denote by $\mathrm{Fix}\, g^{\dagger,k}$ the set of $f \in \mathbb{Q}^{P_k}$ for which

$$fg^{\dagger,k}(\beta) \in \{f(\beta), \star\}$$

holds for all $\beta \in P_k$. If $g \in \mathrm{hEnd}_{\ell,k}(P)$, we say that it is a *good* (ℓ, k)*-hereditary endomorphism of* P provided that for any $\beta \in P_\ell$ with $\beta g \notin P_\ell$ it holds $\alpha g \notin P_k$ for some $\alpha \in P_k$ which is comparable to β in P. Assuming that g is a good (ℓ, k)-hereditary endomorphism of P, for any $\beta \in P_\ell$ and $f \in \mathbb{Q}^{P_k}$ we will have

$$\begin{aligned} (\zeta_P^{k,\ell}(f) g^{\dagger,\ell})(\beta) &= (\zeta_P^{k,\ell}(f))(\beta g) \\ &= \sum_{\alpha' \in P_k \cap (P_\downarrow(\beta g) \cup P_\uparrow(\beta g))} f(\alpha') \\ &= \sum_{\alpha \in P_k \cap (P_\downarrow(\beta) \cup P_\uparrow(\beta))} f(\alpha g) \\ &= (\zeta_P^{k,\ell}(fg^{\dagger,k}))(\beta) \end{aligned}$$

in case that $\beta g \in P_\ell$, and that

$$\begin{aligned} (\zeta_P^{k,\ell}(f) g^{\dagger,\ell})(\beta) &= (\zeta_P^{k,\ell}(f))(\beta g) \\ &= \star \\ &= (\zeta_P^{k,\ell}(fg^{\dagger,k}))(\beta) \end{aligned}$$

in case that $\beta g \notin P_\ell$. This observation can be summarized by the commutative diagram in Fig. 5, which implies that $\mathrm{Fix}\, g^{\dagger,k}$ is mapped by $\zeta_P^{k,\ell}$ to $\mathrm{Fix}\, g^{\dagger,\ell}$ for all good (ℓ, k)-hereditary endomorphisms g of P.

Example 3.4. (1) Let Ω be a set of size n. Assume that $2 \le k < \ell \le n$. Here is an easy observation used often in the study of synchronizing automata: For any $g \in \Omega^\Omega$ and any $A \in \binom{\Omega}{\ell}$, we have $|A\overline{g}| = \ell$ if and only if $|B\overline{g}| = k$ for all $B \in \binom{A}{k}$. This conclusion is surely not valid any more when $k \le 1$. Note that $\overline{g}$ is a good (ℓ, k)-hereditary endomorphism of the Boolean lattice 2^Ω for each $g \in \Omega^\Omega$.

(2) Take integers n, k, ℓ such that $2 \le k < \ell \le n$ and let q be a prime power. Let P be either $\mathcal{P}_{q,n}$ or $\mathcal{A}_{q,n}$. Similar to the above claim on Boolean lattice, $\overline{M}$ is a good (ℓ, k)-hereditary endomorphism of P for each $M \in \mathrm{Mat}_n(\mathbb{F}_q)$.

$$\begin{array}{ccc} f & \xrightarrow{\zeta_P^{k,\ell}} & \zeta_P^{k,\ell}(f) \\ {\scriptstyle g^{\dagger,k}}\big\downarrow & & \big\downarrow{\scriptstyle g^{\dagger,\ell}} \\ fg^{\dagger,k} & \xrightarrow[\zeta_P^{k,\ell}]{} & \zeta_P^{k,\ell}(fg^{\dagger,k}) \end{array}$$

Figure 5: The incidence operator intertwines with every good hereditary endomorphism.

4 Boolean semiring

Let Ω be a set. For the valuated poset $P = \mathrm{B}_\Omega$ and $0 \leq k < \ell \leq |\Omega|$, we write the incidence operator $\zeta_P^{k,\ell}$ defined in Eq. (3.1) as $\zeta_\Omega^{k,\ell}$. That is,

$$(\zeta_\Omega^{k,\ell}(f))(B) = \sum_{A \in \binom{B}{k}} f(A),$$

for all $f \in \mathbb{Q}^{\binom{\Omega}{k}}$ and $B \in \binom{\Omega}{\ell}$.

Following a common approach in establishing homogeneity of permutation groups [15, 39] [20, pp. 20-22], we will make use of the ensuing classical result about incidence matrices. For a simple proof of it, we refer the reader to [18, Corollary] and [49, Theorem 2.4]. Note that it is a positive answer to Conjecture 3.3 for Boolean lattices.

Lemma 4.1. *Let Ω be a finite set. Let k and ℓ be two integers such that $k \leq \ell \leq k+\ell \leq |\Omega|$. Then* $\ker \zeta_\Omega^{k,\ell} = \{0\}$.

Let Ω be a set and S be a transformation semigroup on Ω. Let $\Omega^\sharp \doteq \{(\omega, C) : \omega \in C \in 2^\Omega\}$ and, for each $g \in S$, let $g^\sharp$ be the transformation on $\Omega^\sharp$ which sends (ω, C) to $(\omega g, C\overline{g})$ for all $(\omega, C) \in \Omega^\sharp$. Let $S^\sharp$ stand for the transformation semigroup on $\Omega^\sharp$ consisting of all elements $g^\sharp$ for $g \in S$. For all positive integers ℓ, we use the following notation:

$$\Omega_\ell^\sharp \doteq \{(\omega, C) : \omega \in C \in \binom{\Omega}{\ell}\}$$

and

$$\Gamma_\ell^\sharp(S) \doteq \Gamma(S^\sharp)[\Omega_\ell^\sharp].$$

Here is a result analogous to Lemma 3.1.

Lemma 4.2. *Let m be a positive integer and let S be an m-homogeneous transformation semigroup acting on a set Ω. Then the digraph $\Gamma_m^\sharp(S)$ is symmetric. Especially,* $\mathrm{wcc}(\Gamma_m^\sharp(S)) = \mathrm{scc}(\Gamma_m^\sharp(S))$.

Proof. Take $(\omega, C) \in \Omega_m^\sharp$ and $g \in S$ such that $|C\overline{g}| = m$. Our task is to show the existence of $h \in S$ such that $(\omega g, C\overline{g})h^\sharp = (\omega, C)$. As S is m-homogeneous, we can find $f \in S$ such that $C\overline{gf} = (C\overline{g})\overline{f} = C$. Hence, the fact that $|C| = m < \infty$ allows us to obtain a positive integer r for which $(gf)^r|_C$ is the identity map on C. This means that we can choose h to be $f(gf)^{r-1}$. □

Lemma 4.3. *Let Ω be a set, let m be an integer satisfying $|\Omega| \geq m > 1$, and let S be a transformation semigroup on Ω. For every $X \in \binom{\Omega}{m}$, it holds*

$$\mathrm{scc}(\Gamma(S_X)) = \mathrm{wcc}(\Gamma(S_X)) \leq \mathrm{wcc}(\Gamma^\sharp_m(S)) \leq \mathrm{scc}(\Gamma^\sharp_m(S)). \tag{4.1}$$

Moreover, if S is m-homogeneous, then

$$\mathrm{scc}(\Gamma(S_X)) = \mathrm{wcc}(\Gamma(S_X)) = \mathrm{wcc}(\Gamma^\sharp_m(S)) = \mathrm{scc}(\Gamma^\sharp_m(S)). \tag{4.2}$$

Proof. It is trivial to see that $\mathrm{wcc}(\Gamma(S_X)) = \mathrm{scc}(\Gamma(S_X))$ and $\mathrm{wcc}(\Gamma^\sharp_m(S)) \leq \mathrm{scc}(\Gamma^\sharp_m(S))$. Let us call each strongly/weakly connected component of $\Gamma(S_X)$ a component. To prove Eq. (4.1), let us find an injective map ψ from the set of components of $\Gamma(S_X)$ to the set of weakly connected components of $\Gamma^\sharp_m(S)$.

For each $\gamma \in X$, let the weakly connected component of $\Gamma^\sharp_m(S)$ containing (γ, X) be $\psi'(\gamma)$. Take γ_1, γ_2 from the same component of $\Gamma(S_X)$. We may assume that $\gamma_1 g = \gamma_2$ and $X\overline{g} = X$ for some $g \in S$. As $(\gamma_1, X)g^\sharp = (\gamma_1 g, X\overline{g}) = (\gamma_2, X)$, we see that $\psi'(\gamma_1) = \psi'(\gamma_2)$. For each component C of $\Gamma(S_X)$, we can now choose any $\gamma \in C$ and get a well-defined map ψ by setting $\psi(C) = \psi'(\gamma)$. For every weakly connected component $C^\sharp$ of $\Gamma^\sharp_m(S)$, let $\phi(C^\sharp)$ be the set $\{\gamma \in X : (\gamma, X) \in C^\sharp\}$. It is routine to check that $\phi\psi(C) = C$ for every component C of $\Gamma(S_X)$, proving that ψ is injective, as desired.

Assume now S is m-homogeneous. It follows from Lemma 4.2 that $\mathrm{wcc}(\Gamma^\sharp_m(S)) = \mathrm{scc}(\Gamma^\sharp_m(S))$. We thus call each strongly/weakly connected component of $\Gamma^\sharp_m(S)$ simply a component. Since S is m-homogeneous, for every component $C^\sharp$ of $\Gamma^\sharp_m(S)$, we have $\phi(C^\sharp) \neq \emptyset$. This verifies that ϕ and ψ are inverses of each other. We thus get Eq. (4.2) and so finish the proof. □

For any positive integers s, t and m, let $\mathrm{R}(s, t, m)$ denote the minimum integer $N \geq s$ such that for any set Υ with $|\Upsilon| \geq N$ and any partition of $\binom{\Upsilon}{s}$ into t equivalence classes one can always find one equivalence class which contains $\binom{A}{s}$ for some $A \in \binom{\Upsilon}{m}$. The existence of this number is guaranteed by Ramsey's Theorem [24, Chap. 2] [44].

Proof of Theorem 2.1. (1) If Ω is a finite set, the result follows from Lemma 3.2 (2) and Lemma 4.1. We now assume that Ω is an infinite set. Surely, it suffices to show that S is k-homogeneous under the assumption that it is $(k+1)$-homogeneous for each nonnegative integer k. The proof below essentially follows the proof presented by Bercov and Hobby for [9, Corollary 1] and also the proof of Roy for [46, Theorem].

Fix an element $Y \in \binom{\Omega}{k+1}$. Since S is $(k+1)$-homogeneous, for every $X \in \binom{\Omega}{k}$ we can find $g_X \in S$ such that $X\overline{g_X} \in \binom{Y}{k}$ and $(X, X\overline{g_X}) \in \mathrm{E}(\Pi_S(k, k+1))$. We define a partition $\mathscr{P}$ of $\binom{\Omega}{k}$ into equivalence classes such that Z_1 and Z_2 are equivalent if and only if $Z_1\overline{g_{Z_1}} = Z_2\overline{g_{Z_2}}$. Thanks to Ramsey's Theorem, we know the existence of the finite number $\mathrm{R}(k, k+1, k+1)$. This means that there exist $W \in \binom{Y}{k}$ and $Y' \in \binom{\Omega}{k+1}$ such that $W'\overline{g_{W'}} = W$ for all $W' \in \binom{Y'}{k}$. Take any $X \in \binom{\Omega}{k}$. By virtue of the fact that S is $(k+1)$-homogeneous, we can find $h \in S$ such that $X\overline{h} \in \binom{Y'}{k}, X\overline{hg_{X\overline{h}}} = W$ and $(X, W) \in \mathrm{E}(\Pi_S(k, k+1))$. So far, what we see is that all elements of $\binom{\Omega}{k}$ can reach the vertex W in the spanning subdigraph $\Pi_S(k, k+1)$ of $\Gamma[\binom{\Omega}{k}]$ in one step. Applying Lemma 3.1 (2) now then yields (1). Instead of utilizing Lemma 3.1, another way to see

(1) is to further show the existence of $U \in \binom{\Omega}{k}$ such that U can reach all elements of $\binom{\Omega}{k}$ in $\Gamma[\binom{\Omega}{k}]$ in one step. This can be done similar to the above process of getting the existence of W. Since S is $(k+1)$-homogeneous, for every $X \in \binom{\Omega}{k}$ we can find $h_X \in S$ and $Y_X \in \binom{Y}{k}$ such that $Y_X\overline{h_X} = X$. We define a partition $\mathscr{P}$ of $\binom{\Omega}{k}$ into equivalence classes such that X and Z are equivalent if and only if $Y_X = Y_Z$. We can now continue with Ramsey's Theorem as above but we shall leave it to interested readers to fill in the details.

(2) This is direct from Lemmas 3.2 and 4.1.

(3) Since S is ℓ-homogenous, it follows from Lemma 4.3 that

$$\mathrm{wcc}(\Gamma(S_A)) = \mathrm{scc}(\Gamma(S_A)) \le \mathrm{wcc}(\Gamma^\sharp_k(S))$$

and

$$\mathrm{wcc}(\Gamma(S_B)) = \mathrm{scc}(\Gamma(S_B)) = \mathrm{wcc}(\Gamma^\sharp_\ell(S)).$$

It then remains to prove $\mathrm{wcc}(\Gamma^\sharp_\ell(S)) \ge \mathrm{wcc}(\Gamma^\sharp_k(S))$.

We regard $\Omega^\sharp$ as a valuated poset by putting $\mathrm{r}((\alpha, X)) = |X|$ and requiring $(\alpha, X) < (\beta, Y)$ if and only if $\alpha = \beta \in \Omega$ and $X \subsetneq Y \subseteq \Omega$. Note that $S^\sharp \subseteq \mathrm{hEnd}_{\ell,k}(\Omega^\sharp)$. In view of Lemma 3.2 (1), it is sufficient to show that $\zeta^{k,\ell}_{\Omega^\sharp}$ is injective.

For each nonnegative integer m and each $\alpha \in \Omega$, let $\Omega^\sharp_{m,\alpha} \doteq \{(\alpha, A) : (\alpha, A) \in \Omega^\sharp_m\}$. Corresponding to the partition $\Omega^\sharp_k = \cup_{\alpha\in\Omega}\Omega^\sharp_{k,\alpha}$ and $\Omega^\sharp_\ell = \cup_{\beta\in\Omega}\Omega^\sharp_{\ell,\beta}$, the $\Omega^\sharp_k \times \Omega^\sharp_\ell$ matrix $\zeta^{k,\ell}_{\Omega^\sharp}$ is viewed as a partitioned matrix with blocks $\zeta_{\alpha,\beta}$, which are the submatrices with row index set $\Omega^\sharp_{k,\alpha}$ and column index set $\Omega^\sharp_{\ell,\beta}$, where $\alpha, \beta \in \Omega$. Observe that

$$\zeta_{\alpha,\beta} = \begin{cases} \zeta^{k-1,\ell-1}_{\Omega\setminus\{\alpha\}}, & \text{if } \alpha = \beta; \\ 0, & \text{otherwise.} \end{cases}$$

Since $(k-1) + (\ell-1) \le |\Omega| - 1$, it follows from Lemma 4.1 that $\zeta_{\alpha,\alpha} = \zeta^{k-1,\ell-1}_{\Omega\setminus\{\alpha\}}$ is of full row rank for all $\alpha \in \Omega$. This implies that $\zeta^{k,\ell}_{\Omega^\sharp}$ is an injective linear map, as desired. □

Remark 4.4. Let Ω be a set, which is not necessarily finite. Let k and ℓ be two integers with $k \le \ell \le k + \ell \le |\Omega|$. For all $f \in \mathbb{Q}^{\binom{\Omega}{\ell}}$ and $A \in \binom{\Omega}{k}$, we put

$$(\zeta^{\ell,k}_\Omega(f))(A) = \sum_{A\subseteq B} f(B).$$

Making use of Lemma 4.1, it is easy to see that the linear transformation $\zeta^{\ell,k}_\Omega : \mathbb{Q}^{\binom{\Omega}{\ell}} \to \mathbb{Q}^{\binom{\Omega}{k}}$ is always a surjective map. Unfortunately, we do not see if this observation is helpful to get a counterpart of Theorem 2.1 (2) or Theorem 2.1 (3) when Ω is an infinite set.

5 A graded Möbius algebra

Möbius algebra is a semigroup algebra which plays an important role in combinatorics [34, §3.6]. Huh and Wang [26] introduced a graded Möbius algebra for geometric lattices. Let L be a finite geometric lattice with rank function (valuation) r. Define a $\mathbb{Q}$-algebra $M(L, \mathbb{Q})$, called the *graded Möbius algebra* of L [26], to be the linear space with L as a $\mathbb{Q}$-basis together with a multiplication given by

$$xy = \begin{cases} x \vee y, & \text{if } \mathrm{r}(x) + \mathrm{r}(y) = \mathrm{r}(x \vee y), \\ 0, & \text{if } \mathrm{r}(x) + \mathrm{r}(y) > \mathrm{r}(x \vee y), \end{cases}$$

and extended by linearity and distributivity. For any non-negative integers $k \le \ell$, it is easy to see that the linear map $\xi_L^{k,\ell}$ as specified below is well-defined:

$$\begin{array}{cccc} \xi_L^{k,\ell}: & \mathbb{Q}^{L_k} & \to & \mathbb{Q}^{L_\ell} \\ & \phi & \mapsto & (\sum_{x \in L_1} x)^{\ell-k}\phi. \end{array}$$

We call a finite geometric lattice a *realizable lattice* if it is the matroid lattice of a finite realizable matroid. Here is the main result of Huh and Wang [26, Theorem 6] in their work on solving the realzable case of the top-heavy conjecture of Dowling-Wilson.

Theorem 5.1 (Huh and Wang)**.** *Let L be a finite realizable geometric lattice with rank r. For any integers k and ℓ such that $k \le \ell \le k + \ell \le r$, the linear map $\xi_L^{k,\ell}$ is injective.*

Remark 5.2. (1) The partition lattice $\mathrm{P}(\Omega)$ is isomorphic with the flat lattice of the graphic matroid of the complete graph on Ω. Note that a graphic matroid is regular, namely it is representable over over all possible fields. This means that finite partition lattices are realizable.

(2) Assume that L is a either a Boolean lattice, or a subspace lattice or a partition lattice. It is easy to see that $\xi_L^{k,\ell} = C_{L,k,\ell}\zeta_L^{k,\ell}$ for some positive integer $C_{L,k,\ell}$ which is deermined by L, k and ℓ. Especially, $\xi_L^{k,k+1} = \zeta_L^{k,k+1}$. An important message here is that $\zeta_L^{k,\ell}$ is injective if and only if so is $\xi_L^{k,\ell}$.

Kung verified Conjecture 3.3 for partition lattices of finite sets [32, Theorem 1.3]. We can improve his result a little bit now.

Lemma 5.3. *Let Ω be a finite set. Let k and ℓ be two integers such that $k \le \ell \le k+\ell \le |\Omega|$. Then* $\ker(\zeta_{\mathrm{P}(\Omega)}^{k,\ell}) = \{0\}$.

Proof. Theorem 5.1 and Remark 5.2. □

Let Ω be a finite set and let k and ℓ be two integers such that $k \le \ell \le k + \ell \le |\Omega|$. By virtue of Lemma 5.3, $\ker(\zeta_{\mathrm{P}(\Omega)}^{k,\ell}) = \{0\}$. So, to prove Conjecture 2.10 via Lemma 3.2, we want to have $s^* \in \mathrm{hEnd}_{\ell,k}(\mathrm{P}(\Omega))$ for all $s \in \Omega^\Omega$. It is a pity that what we can have instead is $s^* \in \mathrm{hEnd}_{k,\ell}(\mathrm{P}(\Omega))$ for all $s \in \Omega^\Omega$.

6 Finite linear space

6.1 Top-heavy shape

Let k, ℓ, n be three integers such that $0 \le k \le \ell \le n$. Let q be a prime power. As q-analogues of the set incidence operator specified in Eq. (3.1), we define two linear transformations $M_{q,n}^{k,\ell}: \mathbb{Q}^{\mathcal{P}_{q,n}^k} \to \mathbb{Q}^{\mathcal{P}_{q,n}^\ell}$ and $N_{q,n}^{k,\ell}: \mathbb{Q}^{\mathcal{A}_{q,n}^k} \to \mathbb{Q}^{\mathcal{A}_{q,n}^\ell}$ as follows:

$$(M_{q,n}^{k,\ell}(f))(Y) \doteq \sum_{X \le Y, X \in \mathcal{P}_{q,n}^k} f(X),$$

and

$$(N_{q,n}^{k,\ell}(f'))(Y') \doteq \sum_{X' \leq Y', X' \in \mathcal{A}_{q,n}^k} f(X'),$$

for all $f \in \mathbb{Q}^{\mathcal{P}_{q,n}^k}$, $Y \in \mathcal{P}_{q,n}^\ell$ and $f' \in \mathbb{Q}^{\mathcal{A}_{q,n}^k}$, $Y' \in \mathcal{A}_{q,n}^\ell$.

Kantor [28, Theorem] obtained a q-analogue of Lemma 4.1. Moreover, he [29, Theorem 1] deduced a q-analogue of the aforementioned result of Livingstone and Wagner [36, Theorem 1]. We will need his q-analogue of Lemma 4.1 for proving our Theorem 2.12. Note that Lemma 6.1 says that Conjecture 3.3 holds for affine/projective geometries.

Lemma 6.1 (Kantor). *Let k, ℓ and n be three nonnegative integers such that $k \leq \ell \leq k+\ell \leq n$ and let q be any prime power. Then both $M_{q,n}^{k,\ell}$ and $N_{q,n}^{k,\ell}$ are injective.*

Proof of Theorem 2.12. Note that $S^{\mathcal{P}} \subseteq \mathrm{hEnd}_{k,\ell}(\mathcal{P}_{q,n})$ and $S^{\mathcal{A}} \subseteq \mathrm{hEnd}_{k,\ell}(\mathcal{A}_{q,n})$. The results thus follow readily from Lemmas 3.2 and 6.1. □

6.2 Duality: A result of Stanley

First Proof of Theorem 2.15. Let F be a field and Ω be a set. For each linear subspace $U \leq F^\Omega$, let $U^\perp$ be the subspace of F^Ω given by

$$U^\perp \doteq \{f \in F^\Omega : \sum_{\omega \in \Omega} f(\omega)g(\omega) = 0 \text{ for all } g \in U\}.$$

Take a matrix $A \in F^{\Omega\times\Omega}$ and record its transpose by $A^\top$. For any $f \in F^\Omega$, which can be thought of as a row vector indexed by Ω, the image of f under the action of A, written as fA, can be thought of as the product of the row vector f and the matrix A. The matrix A induces a transformation $\widehat{A}$ on $\mathrm{Gr}(F^\Omega)$ such that $U \in \mathrm{Gr}(F^\Omega)$ is sent to $U\widehat{A} \doteq \{fA : f \in U\}$. It is easy to see that for any $U, W \in \mathrm{Gr}(V)$ we have the implication

$$U\widehat{A} = W \Longrightarrow W^\perp \widehat{A^\top} \leq U^\perp; \tag{6.1}$$

Especially, when $A \in \mathrm{GL}_n(F)$ it holds

$$U\widehat{A} = W \Longleftrightarrow W^\perp \widehat{A^\top} = U^\perp. \tag{6.2}$$

According to Taussky and Zassenhaus [56], we can find $P \in \mathrm{GL}_n(F)$ such that $A^\top = PAP^{-1}$. This means that Eqs. (6.1) and (6.2) become

$$U\widehat{A} = W \Longrightarrow (W^\perp \widehat{P})\widehat{A} \leq U^\perp \widehat{P}$$

and

$$U\widehat{A} = W \Longleftrightarrow (W^\perp \widehat{P})\widehat{A} = U^\perp \widehat{P}, \tag{6.3}$$

respectively. It is well-known that q-binomial coefficients (Gaussian coefficients) occur in pairs, namely in any n-dimensional linear space over a finite field, the number of k-dimensional subspaces is equal to the number of $(n-k)$-dimensional subspaces [23, Proposition 5.31] [53, §3]. In general, as a consequence of Eq. (6.3), for any $A \in \mathrm{GL}_n(F)$, the number of k-dimension subspaces of F^n fixed by $\widehat{A}$ equals to the number of $(n-k)$-dimension subspaces of F^n fixed by $\widehat{A}$. If F is a finite field and G is a subgroup of $\mathrm{GL}_n(F)$, in view of the Orbit Counting Lemma (also known as Burnside's Lemma), the above discussion leads to a proof of Theorem 2.15. □

$$\begin{array}{ccc} f & \xrightarrow{M_{q,n}^{k,\ell}} & M_{q,n}^{k,\ell}(f) \\ \downarrow g^{\ddagger,k} & & \downarrow g^{\ddagger,\ell} \\ fg^{\ddagger,k} & \xrightarrow[M_{q,n}^{k,\ell}]{} & M_{q,n}^{k,\ell}(fg^{\ddagger,k}) \end{array}$$

Figure 6: The incidence operator intertwines with every linear isomorphism g.

Second Proof of Theorem 2.15. Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ and let k be a positive integer fulfilling $k \leq \frac{n}{2}$. The group G can be seen as a permutation group acting on both $\mathrm{Gr}(n-k, \mathbb{F}_q^n) = \mathcal{P}_{q,n}^k$ and $\mathrm{Gr}(n-k, \mathbb{F}_q^n) = \mathcal{P}_{q,n}^{n-k}$; We use W_k and W_{n-k} for the two permutation modules accordingly. From Lemma 6.1, we see that $M_{q,n}^{k,n-k}$ is an $\mathbb{F}_q$-linear isomorphism from $\mathcal{P}_{q,n}^k$ to $\mathcal{P}_{q,n}^{n-k}$. From Fig. 5 and Example 3.4, we have the commutative diagram in Fig. 6 for $2 \leq k \leq \frac{n}{2}$; Assuming that g comes from the group G, clearly our deduction of Fig. 5 shows that Fig. 6 is also valid for $k = 1$. This then shows that W_k and W_{n-k} are isomorphic permutation modules for G. In particular, the number of orbits of G on $\mathcal{P}_{q,n}^k$ and the number of its orbits on $\mathcal{P}_{q,n}^{n-k}$ must be equal. □

The above two proofs apply to a set of invertible linear operators over finite linear spaces. If we have a single linear operator $A \in \mathrm{Mat}_n(F)$, by considering its action on the linear space obtained by "collapsing" the kernel of A to zero, we can somehow still say something similarly as above. When we have a subsemigroup S of the full linear transformation monoid acting on a finite linear space, different elements of S may have different kernels and that makes it nontrivial to glean global information about the semigroup action. In general, if we have a transformation g on a set Ω, we get a partition of Ω in which two elements α and β fall into the same part provided $\alpha g = \beta g$, and we call this partition the kernel of g. For a permutation group, all elements of it have the same kernel. For a transformation semigroup, the existence of different kernels may make some arguments for permutation group invalid. We will study the kernels of elements from a transformation semigroup in [62].

7 Vámos matroid

Proof of Example 2.16. A simple calculation shows that $\ker(\zeta_{\mathrm{F}(M)}^{k,\ell}) = \{0\}$ for $(k,\ell) \in \{(1,2),(1,3),(2,3)\}$. By Lemma 3.2, we will be done if we can show that $f' \in \mathrm{hEnd}_{\ell,k}(\mathrm{F}(M))$ for $(k,\ell) \in \{(1,2),(1,3),(2,3)\}$.

If we know that f is a bijection or that $|\mathscr{E}_M \overline{f}| \leq 2$, we can easily check that $f' \in \mathrm{hEnd}_{\ell,k}(\mathrm{F}(M))$, as wanted. We intend to find a contradiction under the hypothesis that neither of them holds.

By assumption, we can take three distinct elements x, y, z in $\mathscr{E}_M \overline{f}$ such that $|xf^{-1}| \geq 2$. Let A be the minimum flat containing $\{x,y,z\}$ and let $B = Af^{-1}$. Observe that $|A| \in \{3,4\}$. Since f is a strong map, B is a flat containing at least four elements and so $|B| \in \{4,8\}$.

CASE 1. $|B| = 8$.

Take any $X \in \binom{A}{2}$. Note that X must be a flat and thus so is Xf^{-1}. Since $|\mathscr{E}_M \overline{f}| \geq 3$, we deduce that the flat Xf^{-1} is not equal to $\mathscr{E}_M$ and so $|Xf^{-1}| \leq 4$. Considering that $|A| \in \{3, 4\}$, we find that $|A| = 4$ and each element in A has two perimages under f. Note that every element in $\binom{A}{2}$ is a flat. It follows that $\{Xf^{-1} : X \in \binom{A}{2}\}$ is a set of six distinct flats and each of them contains four elements, which cannot happen for the Vámos matroid M.

CASE 2. $|B| = 4$.

Thanks to the assumption of $|B| = 4$, we see that $C = \{x, y\}$ is a flat in M satisfying $|Cf^{-1}| = 3$. Note that no three-elements subset of any four-elements flat in M can be a flat. This means that Cf^{-1} is not a flat, violating the assumption that f is a strong map. □

8 Concluding remarks

Lemmas 4.1, 5.3 and 6.1 suggest the following strengthening of Kung's Conjecture (Conjecture 3.3).

Conjecture 8.1. *Let P be a geometric lattice. Let k and ℓ be two integers such that $k \leq \ell \leq k + \ell \leq \mathrm{r}(P)$. Then $\zeta_P^{k,\ell}$ is an injective map.*

Remark 8.2. Let M be a matroid of rank r. Let S be a subsemigroup of $T^*_M(\mathscr{E}_M)$. For every $f \in S$, let $f' : \mathrm{F}(M) \to \mathrm{F}(M)$ be the map sending a flat $X \in \mathrm{F}(M)$ to the minimum flat containing $X\overline{f}$ in M. Assume that $f' \in \mathrm{hEnd}_{\ell,k}(\mathrm{F}(M))$ for every $f \in S$. In light of Lemma 3.2, if Conjecture 8.1 is valid for the lattice $\mathrm{F}(M)$, we will be able to conclude that both the sequence $(\mathrm{wcc}(\Gamma_{M,0}(S)), \ldots, \mathrm{wcc}(\Gamma_{M,r}(S)))$ and the sequence $(\mathrm{scc}(\Gamma_{M,0}(S)), \ldots, \mathrm{scc}(\Gamma_{M,r}(S)))$ are top-heavy.

We mention that the corresponding result of Huh and Wang on the top-heavy conjecture, [26, Theorem 2], not only proves that, for a realizable finite matroid the number of its k-flats is less than the number of its ℓ-flats for some suitable k and ℓ, but also establishes a matching from its set of k-flats into its set of ℓ-flats such that each k-flat is contained in the ℓ-flat matched to it. A natural next step is to see if our work here on the top-heavy property can be extended in this direction as well.

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