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Note

# Minimum light number of lit-only $\sigma$-game on a tree

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**Abstract**

Let $T$ be a tree with $\ell$ leaves. Each vertex of $T$ is assigned a state either lit or off. An assignment of states to all the vertices of $T$ is called a configuration. The lit-only $\sigma$-game allows the player to pick a lit vertex and change the states of all its neighbours. We prove that for any initial configuration one can make a sequence of allowable moves to arrive at a configuration in which the number of lit vertices is no greater than $\lceil \frac{\ell}{2} \rceil$. We also give examples to show that the bound $\lceil \frac{\ell}{2} \rceil$ cannot be relaxed to $\lfloor \frac{\ell}{2} \rfloor$.


*Keywords:* Binary field; Orbit; Leaves; Hamming weight

## 1. The maximum-orbit-weight problem

Let $S$ be a finite set, $w$ be a real-valued weight function on $S$, $G$ be a permutation group acting on $S$. Under the action of $G$, $S$ is divided into disjoint orbits. Two elements $x, y \in S$ are said to be *equivalent*, denoted by $x \sim y$, if they lie in the same orbit $\bar{x} = \bar{y}$. We define the *weight* of the orbit $\bar{x}$ to be $w(\bar{x}) = \min_{x \sim y} w(y)$. The *maximum-orbit-weight problem* is to estimate $\max_{x \in S} w(\bar{x})$.

**Example 1.** Let $S = \mathbb{F}^n$ be the linear space of $n$-tuples over a finite field $\mathbb{F}$, the weight function $w$ be the Hamming weight $w(x) = \#\{i \mid x_i \neq 0\}$, $G$ be a subspace of $S$, the group action be vector addition of elements of $G$. Then each orbit has size equal to the cardinality of $G$ and the maximum orbit weight coincides with the covering radius of the linear code $G$.

The $\sigma$-game, introduced by Sutner [25], and many of its variations or generalizations have been the research topics of many papers [1–32]. We are interested in the maximum-orbit-weight problem for some special group action that has a close connection with the $\sigma$-game on a graph.

Let $T$ be a graph with vertex set $V(T) = \{v_1, \ldots, v_n\}$. Each vector $x = (x_1, \ldots, x_n) \in \mathbb{F}_2^n$ denotes an assignment of 0's and 1's to the vertices of $T$, namely $v_i$ receives value $x_i$ for each $i$. $x$ is called a *configuration* of $T$ and can be

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represented as follows: draw the graph $T$ on the plane and represent $v_i$ by a bullet if $x_i = 1$; or by a circle if $x_i = 0$; or by a circle with question mark inside if the value of $x_i$ is unclear or irrelevant. A *move* at $v \in V(T)$ of the *σ-game* played on $T$ toggles the states of all the vertices which are adjacent to $v$ in $T$. Let $\alpha_i$ be the $i$-th row of the adjacency matrix of $T$. It is easy to see that the move at $v$ transforms the configuration of $T$ from $x$ to $x + \alpha_i$. Therefore, the set of all moves of the $\sigma$-game on $T$ forms an abelian group $A$, which is isomorphic to the linear subspace of $\mathbb{F}_2^n$ spanned by $\{\alpha_1, \ldots, \alpha_n\}$.

For any configuration $x$, we say that $v_i$ is *lit* in $x$ if $x_i = 1$, and $v_i$ is *off* in $x$ if $x_i = 0$. The *light number* of $x$ is the total number of its lit vertices, i.e., the light number is the Hamming weight $w(x)$. For any given configuration $x$, the *minimum light number problem* is to play the $\sigma$-game aiming to transform $x$ into a configuration $y$ with as few light number as possible. Note that such $y$ belongs to $\overline{x} = x + A$ and $w(y) = w(\overline{x})$. Thus, we are led to the maximum-orbit-weight problem if we concern with the upper bound $L(T) = \max_{x \in \mathbb{F}_2^n} w(\bar{x})$. As mentioned in Example 1, $L(T)$ is the covering radius of the linear code $A$.

For the $\sigma$-game, the group $A$ is abelian and the order of the moves does not matter. For this reason, the maximum-orbit-weight problem is relatively easy to solve. In [30], we consider the $\sigma$-game played on a graph that is obtained form a tree with $\ell \geq 2$ leaves by attaching some loops. We show there that any given configuration can be transformed to a configuration with light number no greater than $\lfloor \frac{\ell}{2} \rfloor$ and all lit vertices are distributed among the leaves of the original tree.

This note is devoted to the maximum-orbit-weight problem for a class of nonabelian groups. If we forbid any move at an off vertex in the $\sigma$-game, we come to one of its variations, the so-called *lit-only σ-game* [4]. In other words, a move of the lit-only $\sigma$-game at $v_i$ transforms the configuration of $T$ from $x$ to $x + x_i\alpha_i = x(I + e_i^\top \alpha_i) = xA_i$, where $I$ is the identity matrix and $e_i$ is the $i$-th unit row vector. Notice that the set of all moves of the lit-only $\sigma$-game corresponds to the subgroup of $SL(n, \mathbb{F}_2)$ generated by $\{A_1, \ldots, A_n\}$.

From now on, we use the light number as the weight function $w$ and concern with the maximum-orbit-weight problem for the lit-only $\sigma$-game on a graph $T$. Unless stated otherwise, all moves mentioned below refer to moves in the lit-only $\sigma$-game.

**Problem 2.** Estimate $L'(T) = \max_{x \in \mathbb{F}_2^n} w(\bar{x})$ for a simple graph $T$ with $n$ vertices.

The main result of this paper, as described below, is our tiny first step in tackling Problem 2.

**Theorem 3.** *$L'(T) \leq \lceil \frac{\ell}{2} \rceil$ for any tree $T$ with $\ell$ leaves.*

The direct motivation for Theorem 3 comes from the work of Chuah and Hu on Lie algebras [9,10], in which they addressed Problem 2 for configurations of some special trees, namely Vogan diagrams. During the course of his work on generalizing the results of Chuah and Hu, Chang [6] made implicitly the conjecture that half of the number of leaves should be an upper bound of the maximum orbit weight for the lit-only $\sigma$-game on a tree. We remark that after informing Chang of our work in proving Theorem 3, he told us that one of his students also obtained similar results. Though we still have no details of their work, we do hope that our work here and subsequent work in preparation will be of independent interest.

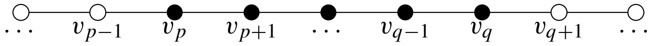
## 2. Preliminaries

This section is devoted to some definitions and lemmas.

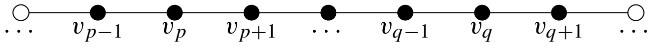
**Definition 4.**

- $\deg(v)$ is the number of vertices adjacent to a vertex $v$.
- A vertex $v$ is called a *leaf* if $\deg(v) = 1$; a *stem* if $\deg(v) = 2$; and a *fork* if $\deg(v) \geq 3$.
- A graph is called a *tree* if it is connected and has no cycle.
- A tree is called a *path* if each vertex is either a leaf or a stem. We write $(v_1, v_2, \ldots, v_n)$ to represent the path with $n$ vertices $v_1, \ldots, v_n$ and $n - 1$ edges connecting $v_i$ and $v_{i+1}$ for each $i = 1, \ldots, n - 1$.
- A tree $T$ is called *star-like* if it has exactly one fork. This fork is called the *root* of $T$.
- The smallest connected subgraph $T'$ that contains all forks of a tree $T$ is called the *inner tree* of $T$.
- Let $T$ be a graph. For any $S \subseteq V(T)$, we use $T - S$ to denote the subgraph of $T$ induced by $V(T) \setminus S$. For subgraphs $T_1, \ldots, T_p$ of $T$, we write $\cup_{i=1}^{p} T_i$ for the subgraph of $T$ induced by $\cup_{i=1}^{p} V(T_i)$.

- In a $\sigma$-game that transforms one configuration to another, if there are $t$ moves at a vertex $v$, we say that $v$ is *used t* times.
- Consider a configuration of the path $(v_1, v_2, \ldots, v_n)$. Suppose the vertices $v_p, v_{p+1}, \ldots, v_q$ are lit and all other vertices are off initially, $1 \le p \le q \le n$.

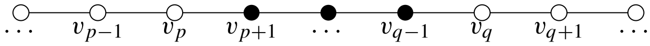



(i) When $q - p$ is even, execution of successive moves at $v_p, v_{p+2}, \ldots, v_q$ changes the states of $v_{p-1}$ and $v_{q+1}$ to lit and leave all other states unchanged. Note that $v_{p-1}/v_{q+1}$ may be a dummy vertex.



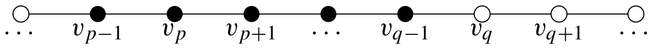

This sequence of moves is called an *inflate*.

(ii) When $q - p$ is even and $q - p \ge 2$, successive moves at $v_{p+1}, v_{p+3}, \ldots, v_{q-1}$ transform the states of $v_p$ and $v_q$ to off but leave all other states unchanged.



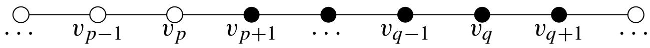

This sequence of moves is called a *deflate*.

(iii) When $q - p$ is odd, by a series of moves at $v_p, v_{p+2}, \ldots, v_{q-1}$, we find that the state of $v_{p-1}$ becomes lit and that of $v_q$ becomes off and all other states are unaffected. Note that $v_{p-1}$ is dummy when $p = 1$.




This sequence of moves is called a *left-shift*.

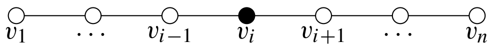
(iv) When $q - p$ is odd, making moves consecutively at $v_q, v_{q-2}, \ldots, v_{p+1}$ brings the game into the configuration that the lit vertices are $v_{p+1}, v_{p+2}, \ldots, v_{q+1}$. Note that $v_{q+1}$ is dummy when $q = n$.




This sequence of moves is called a *right-shift*.

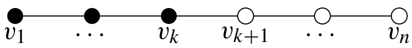
We begin with studying paths and star-like trees.

Let $T$ be the path $(v_1, v_2, \ldots, v_n)$. For each $i = 1, \ldots, n$, let $e_i$ be the configuration of $T$ where $v_i$ is the only lit vertex (see the figure below).

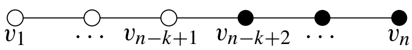
$e_i$ :



**Lemma 5.** *Consider the lit-only $\sigma$-game on the path $T = (v_1, v_2, \ldots, v_n)$. Any nonzero configuration $x$ of the path $T$ can be transformed without using $v_1$ to some $e_i$, $1 \le i \le n$. In particular, $w(\bar{x}) = 1$.*

**Proof.** For any configuration of $T$, if $v_i, v_{i+1}, \ldots, v_{j-1}$ is off and $v_j$ is lit, after sequential moves at $v_j, v_{j-1}, \ldots, v_{i+1}$, the vertex $v_i$ is turned lit. In this way, beginning from the very left, each vertex can be turned lit without using $v_1$ until all vertices on its right are off. That is, for any nonzero configuration $x$ we can find $1 \le k \le n$ such that $x$ is equivalent to $e_1 + \cdots + e_k$.

$e_1 + \cdots + e_k$ :


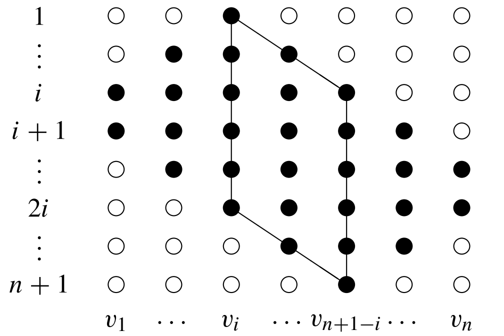

When $k$ is odd, $e_1 + \cdots + e_k$ can be deflated to $e_{\frac{k+1}{2}}$ without using $v_1$. When $k$ is even, $n - k + 1$ right-shifts transform $e_1 + \cdots + e_k$ without using $v_1$ into $e_{n-k+2} + \cdots + e_n$,




which can then be deflated to $e_{n+1-\frac{k}{2}}$. □

**Lemma 6.** *Let $T$ be the path $(v_1, v_2, \ldots, v_n)$ and $i \le \frac{n+1}{2}$. The configuration $e_i$ can be transformed to $e_{n+1-i}$ with each $v_j$ used $i$ times, $j = i, i+1, \ldots, n+1-i$. Correspondingly, $e_{n+1-i}$ can be transformed to $e_i$ with each $v_j$ used $i$ times, $j = i, i+1, \ldots, n+1-i$.*

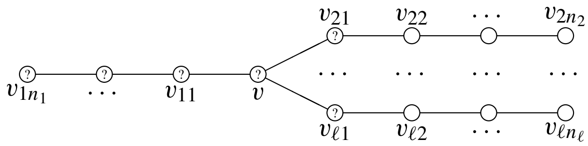
**Proof.** It follows from an observation of the chart below that records the changes of configuration.




□

**Lemma 7.** $w(\bar{x}) \leq \lceil \frac{\ell}{2} \rceil$ *for any nonzero configuration* $x$ *of a star-like tree* $T$ *with* $\ell$ *leaves.*

**Proof.** Let $v$ be the root of $T$. Note that $\ell = \deg(v) \geq 3$ and we can write $T = \cup_{i=1}^{\ell} P_i$, where $P_i$ is an induced path of $T$ with $v$ as one endpoint, say $P_i = (v, v_{i1}, \ldots, v_{in_i})$, $i = 1, \ldots, \ell$. Without loss of generality, assume $n_1 = \max(n_1, \ldots, n_\ell)$. For each $i = 2, \ldots, \ell$ in that order, if the configuration of the path $P_i \cup P_1$ is nonzero, we may apply Lemmas 5 and 6 to get a configuration such that the only lit vertex of $P_i \cup P_1$ lies in $P_1$. Note that, for any $2 \leq i < j \leq n$, the moves on $P_j \cup P_1$ does not affect the states of vertices on $\{v_{i2}, \ldots, v_{in_i}\}$. Therefore, after this sequence of moves, each lit vertex of $T$ lies in $V(P_1) \cup \{v_{21}, \ldots, v_{\ell-1,1}\}$ and there is exactly one lit vertex on $P_1$.

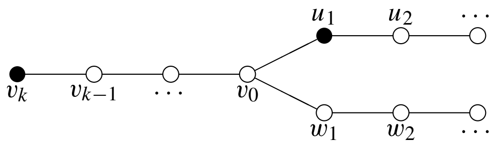



By possible moves on $P_1$, we can set $v$ lit, and then a possible move at $v$ will change the number of lit vertices in $\{v_{21}, \ldots, v_{\ell 1}\}$ to be at most $\lfloor \frac{\ell-1}{2} \rfloor$. Finally, by Lemma 5, the number of lit vertices in $P_1$ can be reduced to 1 by moves inside $\{v_{11}, \ldots, v_{1n_1}\}$. Now, we see that $w(\bar{x}) \leq \lfloor \frac{\ell-1}{2} \rfloor + 1 = \lceil \frac{\ell}{2} \rceil$. □

In order to establish Theorem 3, three more lemmas are prepared below. They are tools to treat the local configurations of a game. For the first one, one vertex is forbidden to be used so that outside states is unaffected by local moves while for the the second and third ones the state of some fixed vertex can be changed taking advantage of outside moves.

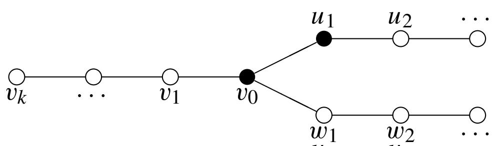
**Lemma 8.** *Let* $k \geq 0$ *and let* $T$ *be a tree with the following configuration*

$X$ :

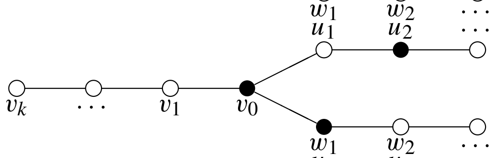



*where* $u_2$ *and/or* $w_2$ *may be dummy.* $X$ *can be transformed to one of the following configurations by moves restricted inside* $\{v_{k-1}, \ldots, v_0, u_1, w_1\}$ *only.*

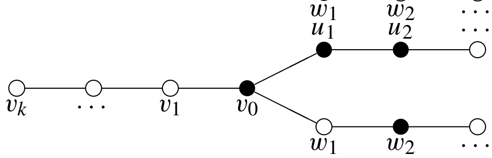
(i) $k \equiv 0 \bmod 4$ :



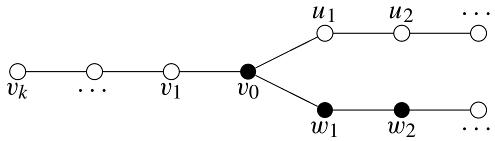

(ii) $k \equiv 1 \bmod 4$ :




(iii) $k \equiv 2 \bmod 4$ :

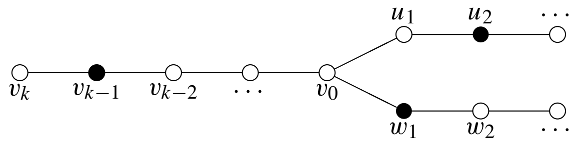


(iv) $k \equiv 3 \bmod 4$ :



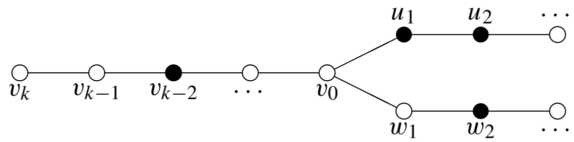

**Proof.** Successive moves at $u_1, v_0, \ldots, v_{k-1}$ transform the configuration $X$ to

$X'$ :



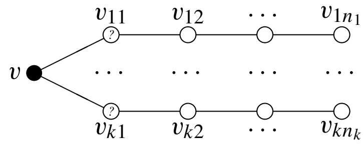

Then, sequencial moves at $w_1, v_0, \ldots, v_{k-2}$ turn the configuration $X'$ into

$X''$ :




Repeating the successive moves at $u_1/w_1, v_0, \ldots, v_{i-1}$ where $v_i$ is lit, we eventually get to the desired configuration. $\square$

**Lemma 9.** *Let $T$ be a tree drawn as below, $k = \deg(v) \geq 1$. Suppose $v$ is always reset to lit after each move. Then any configuration of $T$ can be transformed to*




*where at most $\lfloor \frac{k}{2} \rfloor$ vertices of $\{v_{11}, \ldots, v_{k1}\}$ are lit.*

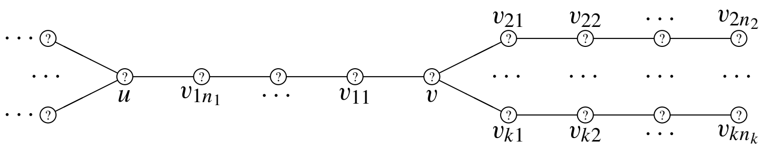
**Proof.** Since $v$ is always reset to lit, any configuration of $(v, v_{i1}, \ldots, v_{in_i})$ can be transformed to $(1, ?, 0, \ldots, 0)$. Finally, if necessary, a move at $v$ reduces the number of lit vertices in $\{v_{11}, \ldots, v_{k1}\}$ to be at most $\lfloor \frac{k}{2} \rfloor$. $\square$

**Lemma 10.** *Let $T$ be the path $(v_1, v_2, \ldots, v_n)$ and let $1 \leq p \leq n$ be a number such that $n - p$ is even. Suppose $v_p$ is always reset to lit after each move. Then any configuration of $T$ can be transformed to $e_p$.*

**Proof.** By Lemma 9, any configuration of $T$ can be transformed to either $e_p$ or $e_p + e_{p+1}$. To conclude the proof, it is enough to show that $e_p \sim e_p + e_{p+1}$. For each $i = p+1, p+2, \ldots, n$, we define $X_i = \begin{cases} e_i + e_p, \text{ when } i - p \text{ is odd;} \\ e_i + e_p + e_{p-1}, \text{ when } i - p \text{ is even.} \end{cases}$ Successive moves at $v_i, v_{i-1}, \ldots, v_p$ transform the configuration $X_i$ to $X_{i+1}$. Hence, we have $e_p + e_{p+1} = X_{p+1} \sim \cdots \sim X_n$. Finally, the sequence of moves at $v_n, v_{n-1}, \ldots, v_p$ gives $X_n \sim e_p$. This ends the proof. $\square$

## 3. Proof of Theorem 3

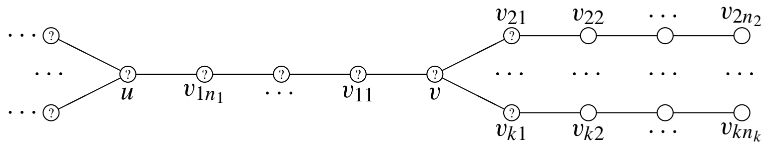
**Proof.** We prove Theorem 3 by induction on the number of vertices of $T$. Let $T'$ be the inner tree of $T$. By virtue of Lemmas 5 and 7, we need only deal with the case that $T'$ has at least two vertices. Let $v$ be a leaf of $T'$. Then, we have $\deg(v) = k \geq 3$, $\ell \geq k + 1$, and that $T$ can be drawn as



where $\deg(u) \geq 3$, $(v, v_{i1}, v_{i2}, \ldots, v_{in_i})$ is an induced path of $T$ with $n_i$ edges, $n_1 \geq 0$, $n_i \geq 1$, $\forall i = 2, \ldots, k$. When $n_1 = 0$, we identify $v_{1n_1}$ with $v$.

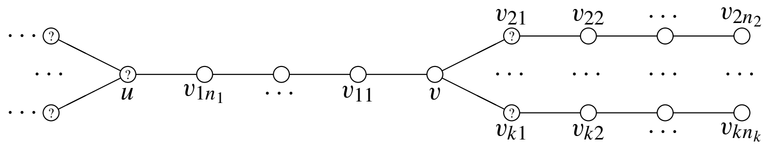
Suppose that the given configuration $x$ is equivalent to one for which all vertices on the left side of $v$ are off. Since $T_1 = \bigcup_{i=2}^{k}(v, v_{i1}, \ldots, v_{i,n_i})$ is a tree with $k-1$ leaves, by Lemmas 5 and 7 we know that further moves inside $V(T_1)$ result in a configuration of $T$ where no more than $\lceil \frac{k-1}{2} \rceil$ lit vertices can be found in $V(T_1)$ and at most one vertex of $V(T) \setminus V(T_1)$ is switched from off to lit. This shows that $w(\bar{x}) \leq \lceil \frac{k-1}{2} \rceil + 1 \leq \lceil \frac{\ell}{2} \rceil$ and hence we are done.

Consequently, we are allowed to assume that we are working with a configuration $x$ for any of whose equivalent configurations there always exists a lit vertex on the left side of $v$. This means that $v$ can always be set to lit by moves at some vertices on its left during the play. By Lemma 9, the configuration can be transformed to




where at most $\lfloor \frac{k-1}{2} \rfloor$ elements of $\{v_{21}, \ldots, v_{k1}\}$ are lit.

For each vertex $v, v_{11}, \ldots, v_{1n_1}$ in that order, if there exists a lit vertex on its left, then it can be set to off by moves at some sequence of vertices on its left; otherwise, we simply have $w(\bar{x}) \leq \lfloor \frac{k-1}{2} \rfloor + 1 \leq \lfloor \frac{\ell}{2} \rfloor$. Therefore, to prove the theorem, it suffices to consider the case that the configuration $x$ can be transformed to




where at most $\lfloor \frac{k-1}{2} \rfloor$ vertices from $\{v_{21}, \ldots, v_{k1}\}$ are lit. To deal with this configuration, we distinguish three cases.

CASE 1: all vertices $v_{21}, \ldots, v_{k1}$ are off. Let $T_2 = T - \bigcup_{i=1}^{k}\{v, v_{i1}, \ldots, v_{in_i}\}$, which has $\ell - k + 1$ leaves. By induction, we can make a sequence of moves on $T_2$ so that the resulting configuration has at most $\lceil \frac{\ell-k+1}{2} \rceil$ lit vertices in $V(T_2)$ and the only possible vertex outside $T_2$ which may become lit again is $v_{1n_1}$. This leads to $w(\bar{x}) \leq \lceil \frac{\ell-k+1}{2} \rceil + 1 \leq \lceil \frac{\ell}{2} \rceil$.

CASE 2: $k \geq 4$ and there exist one lit vertex $v_{i1}$ and one off vertex $v_{j1}$, $2 \leq i, j \leq k$. Let $T_3 = T - \{v_{i1}, \ldots, v_{in_i}, v_{j1}, \ldots, v_{jn_j}\}$. $T_3$ has $\ell - 2$ leaves. By induction, $x$ is equivalent to a configuration that has at most $\lceil \frac{\ell-2}{2} \rceil$ lit vertices in $V(T_3)$ via a series of moves restricted in $V(T_3)$. Note that any move at a vertex from $V(T_3)$ leaves unchanged the number of lit vertices in $\{v_{i1}, \ldots, v_{in_i}, v_{j1}, \ldots, v_{jn_j}\}$. Consequently, $w(\bar{x}) \leq \lceil \frac{\ell-2}{2} \rceil + 1 = \lceil \frac{\ell}{2} \rceil$.

CASE 3: $k = 3$ and one of $v_{21}$ and $v_{31}$ is lit, the other is off.

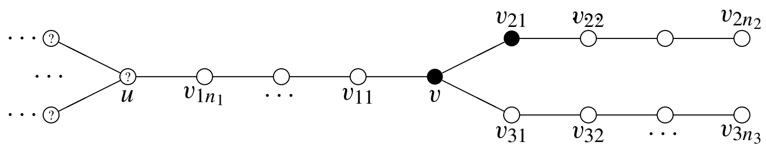
SUBCASE 3.1: $n_2$ or $n_3$ is even. Since we have assumed that there always exists a lit vertex on the left side of $v$, we can apply Lemma 10 to the path $(v_{2n_2}, \ldots, v_{21}, v, v_{31}, \ldots, v_{3n_3})$, followed by those transformations described in the paragraph immediately before CASE 1, and then go back to CASE 1.

SUBCASE 3.2: both $n_2$ and $n_3$ are odd. Without loss of generality, we assume that $n_2 \geq n_3$. Furthermore, because there always exists a lit vertex on the left side of $v$, we can utilize certain series of moves on the left of $v$ to make $v$ lit, and then make a move at $v$ to swap the states of $v_{21}$ and $v_{31}$ whenever needed. So we can assume that $v_{21}$ is lit when $n_1$ is even; and $v_{31}$ is lit when $n_1$ is odd. Proceeding with those transformations described in the paragraph immediately before CASE 1 and then using the induction hypothesis, we may assume that there are at most $\lceil \frac{\ell-2}{2} \rceil$ lit vertices in $V(T) \setminus \bigcup_{i=1}^{3}\{v, v_{i1}, \ldots, v_{in_i}\}$ and assume again that all vertices on the right of $u$ are off with the only possible exceptions of $v_{21}$, $v_{31}$ and $v_{1n_1}$.

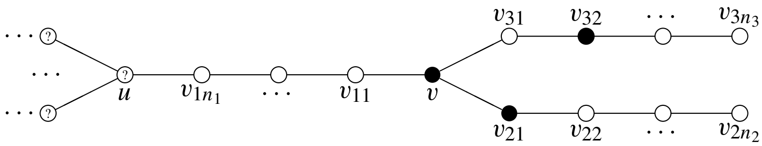
SUBCASE 3.2.1: $v_{1n_1}$ is off. We have $w(\bar{x}) \leq \lceil \frac{\ell-2}{2} \rceil + 1 = \lceil \frac{\ell}{2} \rceil$.

SUBCASE 3.2.2: $v_{1n_1}$ is lit. By Lemma 8, the configuration can be transformed by moves inside $\{v_{21}, v_{31}, v, v_{11}, \ldots, v_{1,n_1-1}\}$ to one of the following cases:

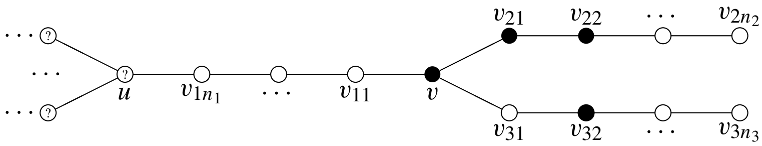
(i) $n_1 \equiv 0 \bmod 4$,



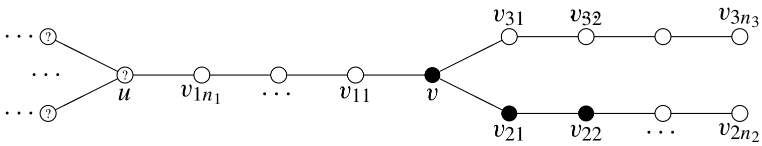
(ii) $n_1 \equiv 1 \bmod 4$,




(iii) $n_1 \equiv 2 \bmod 4$,



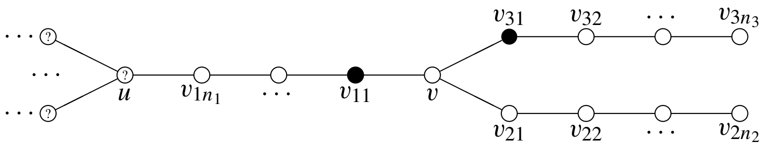

(iv) $n_1 \equiv 3 \bmod 4$,




where $v_{22}$ or $v_{32}$ is dummy if $n_2 = 1$ or $n_3 = 1$, respectively. Note that the apparent interchange of the branches for $\{v_{21}, \ldots, v_{2n_2}\}$ and $\{v_{31}, \ldots, v_{3n_3}\}$ when $n_1$ is odd is because of our assumption that $v_{31}$ is initially lit if and only if $n_1$ is odd.

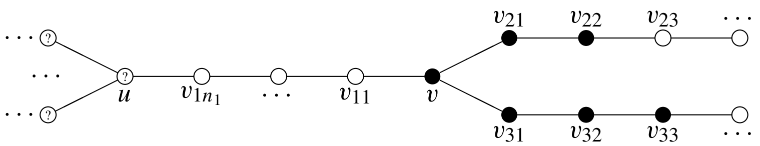
For case (i), successive moves at $v_{21}, v_{22}, \ldots, v_{2n_2}$ reduce the number of lit vertices in $\{v, v_{21}, \ldots, v_{2n_2}\}$ to be 1 and leave all other states unchanged.

For case (ii), successive moves at $v$ and $v_{31}$ transform the configuration to




As $n_2 \geq n_3$, Lemma 6 guarantees that there is a sequence of moves on the path $P = (v_{3n_3}, \ldots, v_{31}, v, v_{21}, \ldots, v_{2n_2})$ which uses $v$ $n_3$ times and transforms the configuration of $P$ to have only one lit state at $v_{2,n_2-n_3+1}$. Recall that $n_3$ is odd and so the final state of $v_{11}$ is off.

For case (iii), if $n_3 = 1$ then a move at $v_{21}$ reduces the number of lit vertices in $\{v, v_{21}, \ldots, v_{2n_2}\}$ to 1. Otherwise, we have $n_2 \geq n_3 \geq 3$, and a move at $v_{32}$ transforms the configuration to



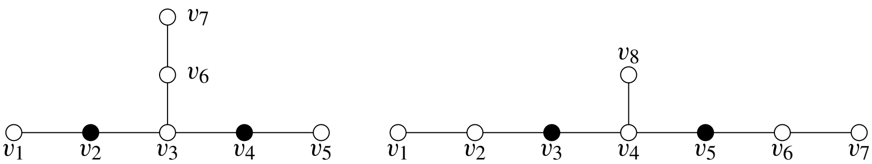

Clearly, three successive right-shifts to the path $(v_{3n_3}, \ldots, v_{31}, v, v_{21}, \ldots, v_{2n_2})$ will set the states of $v_{31}, v_{32}, v_{33}$ all off. The fact that $n_2 \geq 2$ guarantees that $v$ is used exactly twice in this process and so the state of $v_{11}$ remains off. By Lemma 5, the number of lit vertices in $\{v, v_{21}, \ldots, v_{2n_2}\}$ can be further reduced to 1 by moves restricted to $\{v_{21}, \ldots, v_{2n_2}\}$.

For case (iv), a move at $v_{21}$ reduces the number of lit vertices in $\{v, v_{21}, \ldots, v_{2n_2}\}$ to 1.

For all these cases (i), (ii), (iii) and (iv), we have $w(\bar{x}) \leq \lceil \frac{\ell-2}{2} \rceil + 1 = \lceil \frac{\ell}{2} \rceil$.

Since we have verified $w(\bar{x}) \leq \lceil \frac{\ell}{2} \rceil$ for all initial configurations $x$, the result follows from the principle of induction. $\square$

## 4. Concluding remarks




We wrote a computer program which enumerates the whole orbit of any input configuration. It reports that $w(\bar{x}) = 2$ for the two configurations $x$ depicted above. Therefore, the bound $\lceil \frac{\ell}{2} \rceil$ in Theorem 3 cannot be relaxed to $\lfloor \frac{\ell}{2} \rfloor$. Many other examples have been found for which $w(\bar{x}) = \frac{\ell+1}{2}$. It would be interesting to determine all those extremal configurations.

In Section 1, we mention that the lit-only $\sigma$-game corresponds to a group action and list a set of generators of that group. We wonder whether a better understanding of the structure of that group would provide some tools to estimate $w(\bar{x})$. When the game is played on the line graph of a tree, Wu has found that this group is $S_n$, where $n$ is the number of vertices of the tree, and further deduced some related results [31].

For a given tree, we can add loops to some vertices of it and thus make each of them be adjacent to itself. We make here the conjecture that our Theorem 3 for trees still holds for the wider class of graphs thus obtained. Note that for this general situation we not necessarily have a group action as the process of playing the game may be irreversible. We refer to [13,14] and the references therein for an interesting discussion on the lit-only $\sigma$-game on a graph where all vertices have a loop.

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We thank Gerard Chang, whose plenary talk on Vogan diagrams in the Third Pacific Rim Conference on Mathematics aroused our interest in lit-only $\sigma$-game. We are also grateful to the editor and referees. Without their advisory suggestion and kind help, this paper could not appear here in the current form.

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