A LINEAR-TIME ALGORITHM FOR THE 1-FIXED-ENDPOINT PATH COVER PROBLEM ON INTERVAL GRAPHS

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Abstract. Let G be an interval graph and take one of its vertices x. Can we find in linear time a minimum number of vertex disjoint paths of G which cover the vertex set of G and have x as one of their endpoints? This paper provides a positive answer to this problem. In the course of developing such an algorithm, we explore the possibility of getting insight on the path structure of interval graphs via greedy graph searches.

Key words. forward degree sequence, Hamiltonian path, interval representation, normal vertex ordering, 2-fixed-endpoint Hamiltonian path problem.

AMS subject classifications. 05C38, 05C62, 05C85, 68R10, 68W40

1. Introduction.

Interval graphs form a class of very simple and natural mathematical structures. Do we really understand the structure of a set of intervals on the real line? In the field of number theory, many innocent looking problems on a set of integers, namely a set of degenerate intervals, turn out to be headache triggers. Finding minimum path covers with endpoint constraints is a basic algorithmic task in theory and in applications. Then, why not test our understanding of interval graphs by seeing to which extent we can solve relevant minimum path cover problems on them? This paper belongs to a series of our papers which use several sweeps of graph searches on an interval graph to explore its path property; those properties developed in this note will lead to a linear time algorithm for solving the 1-fixed-endpoint path cover problem on interval graphs.

The remainder of this opening section first prepares a body of notation and terminology, which the readers might like to look up only when checking some technical details later, and then outlines the background and main contribution of this paper.

1.1. Notation.

For any two integers i and j with $i \leq j$, we write $[i, j]$ for the set of integers k such that $i \leq k \leq j$. For a map σ defined on $[i, j]$, $\sigma[i, j]$ stands for $\{\sigma(k) : k \in [i, j]\}$. For any integer j, it is convenient to write $[j]$ for $[1, j]$ and hence $\sigma[j]$ for $\sigma[1, j]$. For an n-element set S and a bijective map $\sigma \in S^{[n]}$, we often view σ as an ordering of the set S and we thus write $\sigma(i)$ as σ_i in such a situation and put $\sigma_{[i,j]}$ for the ordering/sequence $\sigma_i, \ldots, \sigma_j$ for any two integers i and j satisfying $1 \leq i \leq j \leq n$.

In this paper, all graphs are assumed to be finite, simple, undirected and loopless. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Take $v \in V(G)$. The *open neighborhood* of v in G, denoted by $N_G(v)$, is the set of vertices adjacent to v in G. The *closed neighborhood* of v in G, denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. Let S be any subset of $V(G)$. We write $N_G(S)$ for $(\cup_{v\in S} N_G(v)) \setminus S$ and write $N_G[S]$ for $N_G(S) \cup S = \cup_{v\in S} N_G[v]$. We follow the convention that $G[S]$ denotes the subgraph of G induced by S. For simplicity, we often write $G[V(G) - S]$ as $G - S$ and write $G - S$ as $G - v$ when S is a singleton set $\{v\}$. For any two disjoint vertex subsets A and B

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of G, we reserve the notation $E_G(A, B)$ for the set of edges of G between A and B. A *path* P in G of length $n-1$ is a sequence of distinct vertices $x_1, \ldots, x_n$ such that $x_i x_{i+1} \in E(G)$ for all $i \in [n-1]$, where x_1 and x_n are the *endpoints* of this path. We use the notation $(x_1, \ldots, x_n)$ for this path P, call it an x_1, x_n-path and write $V(P)$ for the set $\{x_i : i \in [n]\}$. A *cycle* C in G is a cyclic sequence of distinct vertices $x_1, \ldots, x_n$ such that $x_i x_{i+1} \in E(G)$ for all $i \in [n]$, where the subscripts are computed modulo n. We adopt the notation $[x_1, \ldots, x_n]$ for this cycle C and record the set $\{x_i : i \in [n]\}$ by $V(C)$. Two vertices x_i and x_j are *consecutive* in the cycle $[x_1, \ldots, x_n]$ provided $j - i \equiv 1 \pmod{n}$. We call a path P of G a *Hamiltonian path*, or an HP for short, of G if $V(P) = V(G)$. Similarly, a cycle C of G satisfying $V(C) = V(G)$ is called a *Hamiltonian cycle*, or an HC for short, of G. The graph G is *traceable/Hamiltonian* provided it possesses an HP/HC. If $P = (x_1, \ldots, x_s)$ and $Q = (y_1, \ldots, y_t)$ are two paths of G such that $V(P) \cap V(Q) = \emptyset$ and $x_s y_1 \in E(G)$, then we denote the path $(x_1, \ldots, x_s, y_1, \ldots, y_t)$ by $P + Q$.

Let G be a graph and take $S \subseteq V(G)$. An *S-fixed-endpoint path cover* of G is a set $\mathcal{P}$ of vertex-disjoint paths that covers $V(G)$ in which no vertex from S can appear as an inner vertex of some path in $\mathcal{P}$. The *size* of a path cover is the number of paths in it. A *minimum S-fixed-endpoint path cover* of G is an S-fixed-endpoint path cover of G which is of smallest size. We write $\mathrm{PC}(G, S)$ for the problem of finding a minimum S-fixed-endpoint path cover of G. We write $\mathrm{HP}(G, S)$ for the problem of finding a Hamiltonian path of G with S being a subset of the endpoints. We make the convention that $\mathrm{PC}(G, x)$, $\mathrm{HP}(G)$, $\mathrm{HP}(G, x, y)$ refers to $\mathrm{PC}(G, \{x\})$, $\mathrm{HP}(G, \emptyset)$ and $\mathrm{HP}(G, \{x, y\})$, respectively, and so on. For any positive integer k, the *k-fixed-endpoint path cover problem*, or the $k\,$PC problem, on G is to solve $\mathrm{PC}(G, S)$ for any given $S \in \binom{V(G)}{k}$. For $k \in \{0, 1, 2\}$, the *k-fixed-endpoint Hamiltonian path* problem, or the $k\,$HP problem, for G is to solve $\mathrm{HP}(G, S)$ for any given $S \in \binom{V(G)}{k}$. Note that an answer to the $k\,$HP problem naturally follows from an answer to the $k\,$PC problem. Let $\mathrm{pc}(G)$ denote the smallest size of any path cover of G and, for any $x \in V(G)$, let $\mathrm{pc}(G, x)$ denote the smallest size of any $\{x\}$-fixed-endpoint path cover of G. It is straightforward to see that

$$\mathrm{pc}(G) \leq \mathrm{pc}(G, x) \leq \mathrm{pc}(G) + 1. \tag{1.1}$$

We warn the reader that, when we talk about a path cover, we really mean a sequence of paths in which each path should be read as a vertex sequence and so has a natural direction (and so it makes sense to talk about its first vertex and last vertex). This convention is important in understanding several algorithms discussed in the paper.

Let G be a graph with n vertices and $\pi \in V(G)^{[n]}$ a vertex ordering of G. For each $i \in [n]$, the *forward degree* of π_i with respect to the graph G and the vertex ordering π is $\mathrm{d}_{G,\pi}(i) = |N_G(\pi_i) \cap \{\pi_{i+1}, \ldots, \pi_n\}|$. The *forward degree sequence* of G with respect to π is $\mathrm{d}_{G,\pi} = (\mathrm{d}_{G,\pi}(1), \ldots, \mathrm{d}_{G,\pi}(n))$. We call π a *k-thick ordering* provided $n \geq k+1$ and $\mathrm{d}_{G,\pi}(i) \geq \min\{k, n-i\}$ holds for every $i \in [n]$. The ordering π is called a *k-thick path* or just a *k-thick HP* if it is both a path of G and k-thick [15]. Assume that

$$\{i \in [n] : \pi_i \pi_{i+1} \notin E(G)\} = \{\mathbb{t}_1, \ldots, \mathbb{t}_r\}, \tag{1.2}$$

where $r \geq 1$ and $1 \leq \mathbb{t}_1 < \cdots < \mathbb{t}_r = n$. Let $\mathbb{t}_0 = 0$, $\mathbb{s}_i = \mathbb{t}_{i-1} + 1$ for $i \in [r]$. Then we see that

$$P_1 = \pi_{[\mathbb{s}_1, \mathbb{t}_1]}, \ldots, P_r = \pi_{[\mathbb{s}_r, \mathbb{t}_r]}, \tag{1.3}$$

form a path cover of G of size r, which we call the *path cover corresponding to* π.

Let V be a set. An *interval assignment* for V is a map $\mathcal{I}$ which sends each $x \in V$ to a nonempty interval $\mathcal{I}(x) = [\ell_{\mathcal{I}}(x), r_{\mathcal{I}}(x)]$ on the real line. For any interval assignment $\mathcal{I}$ for V, its *adjoint*, denoted by $\overleftarrow{\mathcal{I}}$, is the map which sends $x \in V$ to $\overleftarrow{\mathcal{I}}(x) = [-r_{\mathcal{I}}(x), -\ell_{\mathcal{I}}(x)]$. The *intersection graph* of an interval assignment $\mathcal{I}$ on V, also called the intersection graph of the set of intervals $\{\mathcal{I}(v) :\ v \in V\}$, denoted by $\mathrm{G}_{\mathcal{I}}$, has vertex set V and edge set $\{uv \in \binom{V}{2} :\ \mathcal{I}(u) \cap \mathcal{I}(v) \neq \emptyset\}$. An *interval representation* of a graph G is an interval assignment $\mathcal{I}$ for $V(G)$ such that, for every two different vertices x and y in G, $xy \in E(G)$ if and only if $\mathcal{I}(x) \cap \mathcal{I}(y) \neq \emptyset$. Obviously, $\overleftarrow{\mathcal{I}}$ is an interval representation of a graph G if and only if so is $\mathcal{I}$. An *interval graph* is a graph possessing an interval representation, namely a graph G which is isomorphic to $\mathrm{G}_{\mathcal{I}}$ for some interval assignment $\mathcal{I}$. With no loss of generality, when talking about an interval assignment $\mathcal{I}$ for a set V in this paper, we always assume that the endpoints of those intervals $\mathcal{I}(x), x \in V$, are all distinct, namely the size of the set $\{\ell_{\mathcal{I}}(x), r_{\mathcal{I}}(x) :\ x \in V\}$ is $2|V|$. Pick an interval assignment $\mathcal{I}$ for a finite set V. Let $\mathrm{rv}(\mathcal{I})$ be the element $b \in V$ such that $\ell_{\mathcal{I}}(b) = \max\{\ell_{\mathcal{I}}(x) :\ x \in V\}$ and call it the *rightmost element* for $\mathcal{I}$; we let $\mathrm{lv}(\mathcal{I})$ be the element $a \in V$ such that $r_{\mathcal{I}}(a) = \min\{r_{\mathcal{I}}(x) :\ x \in V\}$ and call it the *leftmost element* for $\mathcal{I}$. For every $x, y \in V$, we declare $\mathcal{I}(x) < \mathcal{I}(y)$ whenever $r_{\mathcal{I}}(x) < \ell_{\mathcal{I}}(y)$. For any two subsets M and N of V, we say that M is *to the left* of N with respect to $\mathcal{I}$, denoted by $M <_{\mathcal{I}} N$, if $\mathcal{I}(x) < \mathcal{I}(y)$ for each $x \in M$ and each $y \in N$. In particular, if M (or N) contains only one element x, then we write $x <_{\mathcal{I}} N$ (or $M <_{\mathcal{I}} x$) instead of $M <_{\mathcal{I}} N$. We often directly call any (maximal) clique of $\mathrm{G}_{\mathcal{I}}$ as a (maximal) clique of $\mathcal{I}$. The Helly property of intervals on the real line claims that $\cap_{v \in C}\mathcal{I}(v)$ is a nonempty interval for every clique C of $\mathcal{I}$. For any two different maximal cliques C_1 and C_2 of $\mathcal{I}$, $\cap_{v \in C_1}\mathcal{I}(v)$ and $\cap_{v \in C_2}\mathcal{I}(v)$ must be disjoint nonempty intervals and we say that C_1 is to the left of C_2 provided $\cap_{v \in C_1}\mathcal{I}(v)$ is to the left of $\cap_{v \in C_2}\mathcal{I}(v)$. This then allows us to talk about the leftmost and rightmost maximal cliques of $\mathcal{I}$, the two rightmost maximal cliques of $\mathcal{I}$, namely the rightmost and the second rightmost maximal cliques of $\mathcal{I}$, and so on. We use $\mathrm{lc}(\mathcal{I})$ and $\mathrm{rc}(\mathcal{I})$ for the *leftmost clique* for $\mathcal{I}$ and the *rightmost clique* for $\mathcal{I}$, respectively. Clearly, we have $\mathrm{lc}(\mathcal{I}) = \{v \in V :\ \ell_{\mathcal{I}}(v) \le r_{\mathcal{I}}(\mathrm{lv}(\mathcal{I}))\}$ and $\mathrm{rc}(\mathcal{I}) = \{v \in V :\ r_{\mathcal{I}}(v) \ge \ell_{\mathcal{I}}(\mathrm{rv}(\mathcal{I}))\}$.

1.2. The problem. For interval graphs, the Hamiltonian path problem, namely the 0 HP problem [1, Theorem 7][9, §5] and in general, the path cover problem, namely the 0 PC problem [9, Corollary 2] possess very simple linear time algorithms. The first such simple algorithm is already reported by Keil [12] in 1985 and similar simple algorithms are also given in [1, 9]. When restricted to proper interval graphs, much more general k PC problems have been shown to be linear time solvable [4, 17]. For a graph G and $x \in V(G)$, we can attach a new pendant vertex to x and obtain a graph H from G. It is clear that $\mathrm{pc}(G, x) = \mathrm{pc}(H)$. Thus, the 0 PC problem and the 1 PC problem are equivalent on graph classes that are closed under attaching pendant vertices. Note that interval graphs are not closed under this graph modification operation. In 1993, Damaschke [9] asks whether or not there exist polynomial time algorithms for solving the 1 HP problem and the 2 HP problem on interval graphs. In 2010, Asdre and Nikolopoulos [3] propose a $O(|V(G)|^3)$ time algorithm for solving the 1 PC problem (and hence the 1 HP problem) on any interval graph G. The main objective of this paper is to establish a $O(|V(G)| + |E(G)|)$ time algorithm for solving the 1 PC problem on any interval graph G (Theorem 6.4). The basic idea of our algorithm is very different from the one of Asdre and Nikolopoulos [3]. Our algorithm builds on two simple graph search algorithms to be introduced in § 2. We believe that we already have a polynomial time algorithm for solving the general 2 HP problem on interval graphs, which will use the linear time algorithm for solving the interval graph 1 HP problem here as a subprogram and will heavily rely on the mathematical observations developed in § 4 of this paper.

Due to Eqs. (1.2) and (1.3), one can imagine that a close examination of some graph search algorithms will help us discover the rich structure of paths in graphs and design some easily implementable, search-based algorithms for the fixed-endpoint path cover problem. Many natural graph search algorithms have been proposed for various purposes and a huge theory around them is still growing quickly. Much of this graph search theory is motivated by its first success in dealing with some important graph classes, especially interval graphs [1, 3, 6, 7, 8, 9, 10, 12, 14, 16]. To solve the 0 PC problem for a graph with a given interval representation, it is natural to first of all try to sweep through the graph from left to right to generate a good vertex ordering. If there is a fixed-endpoint constraint, it becomes a bit more nontrivial to design a search strategy for solving the path cover problem.

The basic building blocks of our algorithm are two very simple ones to be described in § 2. It is amazing that they can generate inductively very structured vertex sequences at different scale and from different parts of a graph, allowing us to assemble the pieces together to form a required path cover or HP in all possible situations. In a typical scenario when running our algorithm, we have a good vertex ordering π of some subgraph. Our algorithm can easily move forward when π is a 2-thick HP and when π is not a path. In the remaining case, we cut the subgraph into the left part and the right part according to the information squeezed from the forward degree sequence of π. A careful examination of the Normal Ordering Algorithm (c.f. § 2) shows that, in a number of steps proportional to the size of a neighborhood around that cutting point, we can obtain from π good orderings on the left/right part and so we can repeat the same procedure on the smaller parts. Moreover, by discovering the nice landscape of those good vertex orderings of several subgraphs, we can guarantee that our problem for the whole graph can be reduced to corresponding problems on the left/right graphs. Since each such reduction costs linear time, this explains in a high level how we can get a linear time algorithm.

The rest of this paper is organized as follows. In § 2, we introduce two quite simple graph search algorithms, one visiting the vertex set cyclically and the other linearly. Before delving into long strings of reasonings about the path structures of interval graphs, § 3 outlines more details of our strategy for solving the 1 PC problem on interval graphs and makes use of a small example to illustrate the behaviors of our basic graph search algorithms. We link the discussions in § 3 to many technical results presented later in the paper, hoping to help the readers better understand our main algorithm and its analysis. § 4 is devoted to some simple properties of some good vertex orderings of interval graphs, which also lays the foundation for our future work in developing a polynomial time algorithm for solving the 2 HP problem on interval graphs. We demonstrate in § 5 our algorithm for solving the 1 HP problem on interval graphs and then use it in § 6 as a subprogram to get the algorithm for solving the 1 PC problem on interval graphs. To solve the 1 HP problem, we indeed already develop an algorithm in § 5 to solve some special 2 HP problem on interval graphs. Both § 5 and § 6 present very rich properties of paths in interval graphs. We close the paper with some remarks in § 7.

2. Two simple algorithms. We will need the following simple algorithm to search for paths/cycles in interval graphs, which we now state for general graphs.

Procedure HC(G, π);
{ Input: a graph G with n vertices and a 2-thick HP π of G; Output: an HC ρ of G in which π_1, π_2 are consecutive and π_{n-1}, π_n are also consecutive; }
begin
Let π_2, π_j be two different neighbors of π_1 in G.
If $j = n$, return $\rho = [\pi_1, \ldots, \pi_n]$.
Else, do Procedure HC($G[\pi[j-1, n]], \pi_{[j-1,n]}$) to get an HC of $G[\pi[j-1, n]]$, say $\rho' = [\pi_{j-1}, \pi_j, x_1, \ldots, x_h]$.
Output $\rho = [\pi_1, \pi_2, \ldots, \pi_{j-1}, x_h, x_{h-1}, \ldots, x_1, \pi_j]$.
end;

LEMMA 2.1. *The algorithm* HC(G, π) *is correct.*

Proof. We shall proceed by induction on $|V(G)| = n$. If $n \leq 2$, then G cannot have any 2-thick HP and so there is nothing to prove. Suppose $n \geq 3$. Since π is 2-thick, $\mathrm{d}_{G,\pi}(1) \geq 2$. Assume that π_2 and π_j are two neighbors of π_1 where $j \geq 3$.

If $j = n$, the cycle $\rho = [\pi_1, \ldots, \pi_n]$ clearly fulfils the requirement.

Assume now $j < n$. Because $\pi_{[j-1,n]}$ is a 2-thick path, the induction hypothesis shows that the output ρ' of HC($G[\pi[j-1, n]], \pi_{[j-1,n]}$) is an HC of $G[\pi[j-1, n]]$ in which π_{j-1}, π_j are consecutive and π_{n-1}, π_n are consecutive. We can thus check that the output ρ of HC(G, π) has the claimed properties, as desired. □

LEMMA 2.2. *The algorithm* HC(G, π) *can be implemented in linear time.*

Proof. We shall proceed by induction on $|V(G)|$. Finding two neighbors of π_1 in G costs $O(|N_G(\pi_1)|)$ time. By induction hypothesis, Procedure HC($G[\pi[j-1, n]], \pi_{[j-1,n]}$) takes $O(|\pi[j-1, n]| + |E(G[\pi[j-1, n]])|)$ time. So, we need $O(|N_G(\pi_1)|) + O(|\pi[j-1, n]| + |E(G[\pi[j-1, n]])|) \leq O(|V(G)| + |E(G)|)$ time to complete the algorithm. □

The design and analysis of our algorithm for solving the interval graph 1 PC problem rely heavily on a careful analysis of the following greedy (first-fit) algorithm proposed in various forms in [1, 9, 12]. We will refer to this algorithm as the *Normal Ordering Algorithm* hereafter, as it visits/orders the vertex set of a given interval graph in a "normal" way.

Procedure NO($G, \mathcal{I}, a$);
{ Input: an interval graph G, an interval representation $\mathcal{I}$ of G and a vertex $a \in V(G)$; Output: an ordering π of $V(G)$; }
begin
Every vertex of G is unvisited. Let $\pi_1 = a$ and visit it.
For $i = 1$ to $|V(G)| - 1$ do
rule 1: If there is some unvisited vertex of $N_G(\pi_i)$, then visit $\pi_{i+1} = x$ where $x \in N_G(\pi_i)$ such that $r_{\mathcal{I}}(x) = \min\{r_{\mathcal{I}}(y) : y \in N_G(\pi_i) \setminus \pi[i]\}$;
rule 2: Else visit $\pi_{i+1} = x$ where x is the vertex of $V(G) \setminus \pi[i]$ such that $r_{\mathcal{I}}(x) = \min\{r_{\mathcal{I}}(y) : y \in V(G) \setminus \pi[i]\}$.
end;

LEMMA 2.3. *[1, Theorem 7] The algorithm Procedure* NO($G, \mathcal{I}, a$) *has a linear time implementation.*

Let G be an interval graph and let $\mathcal{I}$ be an interval representation of G. The *normal vertex ordering of* G *with respect to* $\mathcal{I}$ is the output of the algorithm NO($G, \mathcal{I}, \mathrm{lv}(\mathcal{I})$).

A normal vertex ordering of G which also turns out to be an HP of G is referred to as a *normal HP* of G. The following fundamental lemma has been used in many algorithms on interval graphs [1, 9, 10, 12].

LEMMA 2.4. *[1, Theorem 7] Let G be an interval graph. Then $\mathrm{pc}(G) = r$ if and only if each path cover corresponding to any normal vertex ordering of G has size r.*

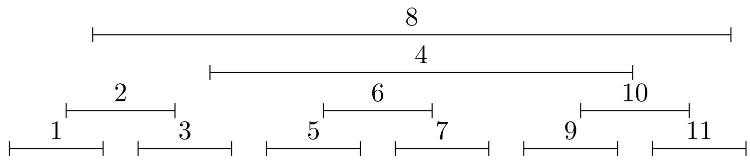

FIG. 2.1. *An interval assignment $\mathcal{I}$ for* [11].

EXAMPLE 2.5. *Let $\mathcal{I}$ be the interval assignment for* [11] *as described in Fig. 2.1 and let $G = \mathrm{G}_{\mathcal{I}}$. It is easy to see that*

$$1 = \mathrm{lv}(\mathcal{I}) \text{ and } 2 = \mathrm{lv}(\mathcal{I}_{[2,8]}).$$

The normal vertex ordering of G with respect to $\mathcal{I}$ is

$$\pi = 1, 2, \dots, 11;$$

the normal vertex ordering of G with respect to $\overleftarrow{\mathcal{I}}$ is

$$\tilde{\pi} = 11, 10, 9, 4, 7, 6, 5, 8, 3, 2, 1;$$

and the normal vertex ordering of $G[[2,8]]$ with respect to $\mathcal{I}_{[2,8]}$ is

$$\tau = 2, 3, \dots, 8.$$

We can compute their forward degree sequences:

$$\begin{cases} \mathrm{d}_{G,\pi} = (2,2,2,6,2,2,1,3,1,1,0), \\ \mathrm{d}_{G,\tilde{\pi}} = (2,3,2,5,2,2,1,3,1,1,0), \\ \mathrm{d}_{G[[2,8]],\tau} = (2,2,4,2,2,1,0). \end{cases} \tag{2.1}$$

This tells us that π is a 1-thick normal vertex ordering of G and τ is a 2-thick normal vertex ordering of $G[[2,8]]$. The output of Procedure $\mathrm{HC}(G[[2,8]], \tau)$ *is $\rho = [2, 3, \dots, 8]$. Obviously, ρ is an HC of $G[[2,8]]$, as predicted by Lemma 2.1. Let us list some (but not all) HPs of G which starts at $5 = \pi_5$: $P_1 = (5,6,7,8,1,2,3,4,9,10,11)$, $P_2 = (5,6,7,4,3,2,1,8,9,10,11)$, $P_3 = (5,6,7,4,9,10,11,8,1,2,3)$ and $P_4 = (5,6,7,4,9,10,11,8,3,2,1)$.*

3. Tour guide. This section is in two parts. The first part gives some hints on why a linear time algorithm to the 1 PC problem on interval graphs can be possible. The second part introduces some facts about the concrete interval assignment in Fig. 2.1 for which the reader might like to come back from time to time when reading various lemmas in the remaining parts of the paper.

3.1. A road map. We try to motivate our algorithm by describing several useful facts about paths in interval graphs in a not very accurate way. We cite some results to be proved in later sections here so that the reader may better tell the direction of our subsequent proof.

Take a graph G and pick its two vertices x and y. We write $1\,\mathrm{HP}(G,x) = 0$ to mean that G has no HP starting from x, and write $1\,\mathrm{HP}(G,x) = 1$ otherwise; we write $2\,\mathrm{HP}(G,x,y) = 0$ to mean that G has no HP with x and y as endpoints, and write $2\,\mathrm{HP}(G,x,y) = 1$ else.

As seen from the proof of Theorem 6.4, to solve the 1 PC problem on interval graphs in linear time, our main contribution is reflected in Theorem 6.2 and Lemma 6.3. For the discussions in the remainder of this subsection, let us fix an interval graph G on n vertices and $x \in V(G)$, let $\mathcal{I}$ be an interval representation of G, and let π and $\tilde{\pi}$ be the normal vertex orderings of G with respect to $\mathcal{I}$ and $\overleftarrow{\mathcal{I}}$, respectively.

The ensuing facts summarize what is really proved by Theorem 6.2. Let $\pi_{[\mathbb{s}_1,\mathbb{t}_1]}$, $\ldots$, $\pi_{[\mathbb{s}_r,\mathbb{t}_r]}$ be the path cover of G corresponding to π. If $r = 1$, then $\mathrm{pc}(G,x) = 2 - 1\,\mathrm{HP}(G,x)$. We assume now $r > 1$. Let $R = \pi[\mathbb{s}_r,\mathbb{t}_r]$. If $x \in R$, then $\mathrm{pc}(G,x) = r + 1 - 1\,\mathrm{HP}(G[R],x)$. Suppose $x \notin R$. Let $\tilde{\pi}'$ be the normal vertex ordering of $G[R]$ with respect to $\overleftarrow{\mathcal{I}_R}$. If $N_G(\tilde{\pi}'_{|R|}) \setminus R = \emptyset$, then $\mathrm{pc}(G,x) = \mathrm{pc}(G-R,x) + 1$; otherwise, by letting $N_G(\tilde{\pi}'_{|R|}) \setminus R = \{\pi_{\mathbb{j}_1},\ldots,\pi_{\mathbb{j}_h}\}$, $\ell_{\mathcal{I}}(\pi_{\mathbb{j}_s}) = \max\{\ell_{\mathcal{I}}(\pi_{\mathbb{j}_k}) :\ k \in [h]\}$ and $G' = G[\pi[\mathbb{s}_r - 1] \setminus \{\pi_{\mathbb{j}_s}\}]$, $\mathrm{pc}(G',x) = \mathrm{pc}(G,x)$ holds. Theorem 6.2 reduces our task to solving $1\,\mathrm{HP}(G,x)$, which is dealt with in Theorem 5.9. We briefly indicate the route to Theorem 5.9 in the sequel.

Lemma 2.4 shows that G is traceable if and only if every normal vertex ordering of G is an HP of G. This means that $1\,\mathrm{HP}(G,x) = 0$ if π is not 1-thick. By Lemma 2.1, $1\,\mathrm{HP}(G,x) = 1$ if π is 2-thick. Thus, to determine $1\,\mathrm{HP}(G,x)$, we can focus on the case that π is 1-thick but not 2-thick. By Lemma 2.4 again, $\tilde{\pi}$ is an HP of G. Take $\iota \in [n-2]$ such that $\mathrm{d}_{G,\pi}(\iota) = 1$. Let $R = \pi[\iota+2,n]$. Let $(a,b) = (\mathrm{lv}(\mathcal{I}),\mathrm{rv}(\mathcal{I}))$. Note that $(a,b) = (\pi_1,\tilde{\pi}_1)$.

Claim A (c.f. Theorem 4.4): If $1\,\mathrm{HP}(G,x) = 1$, then $2\,\mathrm{HP}(G,x,a) = 1$ or $2\,\mathrm{HP}(G,x,b) = 1$.

Claim B (c.f. Lemma 4.7 (v)): The set R appears consecutively in every HP of G.

As $R \neq V(G)$, Claim B implies

$$2\,\mathrm{HP}(G,u,v) = 0 \tag{3.1}$$

for all $u,v \in R$. Note that Lemma 5.4 (ix) is basically a strengthening of Eq. (3.1).

By Lemma 4.2 (i), $b \in R$ holds. If $x \in R$, combining Eq. (3.1) and Claim A gives $1\,\mathrm{HP}(G,x) = 2\,\mathrm{HP}(G,x,a)$; If $x \in N_G(R)$, Lemma 5.4 (vii) gives $1\,\mathrm{HP}(G,x) = 0$. Suppose $x \notin N_G[R]$. Lemma 5.5 (v) implies $2\,\mathrm{HP}(G,x,a) \le 2\,\mathrm{HP}(G,x,b)$ and thus it follows from Claim A that $1\,\mathrm{HP}(G,x) = 2\,\mathrm{HP}(G,x,b)$. By symmetry between $(\overleftarrow{\mathcal{I}},\tilde{\pi},a)$ and $(\mathcal{I},\pi,b)$, it suffices to suppose $x \notin N_G[R]$ and show how to determine $2\,\mathrm{HP}(G,x,b)$.

Claim C (c.f. Theorem 4.6 (b)): Take $\mathcal{R} \subseteq V(G)$ such that $\ell_{\mathcal{I}}(v) > \ell_{\mathcal{I}}(u)$ holds for all $v \in \mathcal{R}$ and $u \notin \mathcal{R}$. For each $z \in \mathcal{R}$ satisfying $1\,\mathrm{HP}(G[\mathcal{R}],z) = 1$, $N_G(\tilde{\pi}_{|\mathcal{R}|}) \supseteq N_G(z) \setminus \mathcal{R}$ holds.

Let $L = V(G) \setminus R$. Remember that $b = \tilde{\pi}_1 \in R = \pi[\iota+2,n]$. Since $\tilde{\pi}$ is an HP of G, we obtain $2\,\mathrm{HP}(G,x,b) \ge 2\,\mathrm{HP}(G[L \cup \{\tilde{\pi}_{|R|}\}],x,\tilde{\pi}_{|R|})$. If G has an HP θ leading from x to b, Claim B says that $\theta[n-|R|+1,n]=R$. Let $z=\theta_{n-|R|+1}\in R$. By Lemma 4.2 (i), we can take $\mathcal{R}=R$ in Claim C and so find that one can replace z by $\tilde{\pi}_{|\mathcal{R}|}$ in $\theta_{[|L|+1]}$ to get an HP of $G[L\cup\{\tilde{\pi}_{|R|}\}]$ going from x to $\tilde{\pi}_{|R|}$. This verifies $2\,\mathrm{HP}(G,x,b)=2\,\mathrm{HP}(G[L\cup\{\tilde{\pi}_{|R|}\}],x,\tilde{\pi}_{|R|})$. Note that this reduction may not really decrease the size of the problem as it may happen $G=G[L\cup\{\tilde{\pi}_{|R|}\}]$.

Take an element of $N_G(\tilde{\pi}_{|R|})\setminus R$ (c.f. Eq. (5.10)) with maximum $\ell_{\mathcal{I}}$ value, say w. Note that this w corresponds to $\pi_{\mathbb{j}_s}$ in Eq. (5.11). By some nontrivial reasoning, we can deduce $2\,\mathrm{HP}(G,x,b)=2\,\mathrm{HP}(G[L\cup\{\tilde{\pi}_{|R|}\}],x,\tilde{\pi}_{|R|})=2\,\mathrm{HP}(G[L],x,w)$ in Lemma 5.5 (v). So far, $G[L]$ has fewer number of vertices than G. But the new trouble is that w may not be the vertex $\mathrm{rv}(\mathcal{I}_L)$. To proceed, again, we need to look at the thickness of the normal vertex orderings of $G[L]$, we need to cut $G[L]$ into the left part and the right part and we need to apply the two simple algorithms introduced in § 2. Basically, what our algorithm does now to shrink the problem size properly is reflected in Lemma 5.5 (iv.b) and (iv.c) while Lemma 5.5 (iv.a) helps to give linear time implementation of this reduction.

3.2. Example 2.5 revisited. This subsection aims to enable the readers have a chance to see some technical claims in § 4 and § 5 in a small example. We return to Example 2.5 and list below a bunch of facts about the interval assignment $\mathcal{I}$ shown in Fig. 2.1.

We follow the notation in Example 2.5. By Eq. (2.1), $7=\min\{i\in[10]:\ \mathrm{d}_{G,\pi}(i)=1\}$. Let $\iota=7$, $R=\{9,10,11\}$ and $L=[8]$. Let $(\mathbb{j}_1,\mathbb{j}_2)=(4,8)$ and so $N_G(\tilde{\pi}_{|R|})\cap L=N_G(\tilde{\pi}_3)\cap L=N_G(9)\cap L=\{4,8\}=\{\mathbb{j}_1,\mathbb{j}_2\}$. Note that $\ell_{\mathcal{I}}(\mathbb{j}_1)=\max\{\ell_{\mathcal{I}}(\mathbb{j}_t):\ t\in[2]\}$. Each item below carries the label of a result appeared later and the technical statement of that result may be easier to swallow after having a look at the corresponding fact for Fig. 2.1.

Lemma 4.1 (ii). $\pi_{10}\pi_{11}\in E(G)$ and $\{\pi_{10},\pi_{11}\}\subseteq\{\pi_8,\pi_{10},\pi_{11}\}=\mathrm{rc}(\mathcal{I})$.

Lemma 4.1 (iii). Let $H=G[\pi[4,9]]$. Then $\pi_4\in\mathrm{lc}(\mathcal{I}_{[4,9]})$ and $(4,5,\ldots,9)$ is the output of $\mathrm{NO}(H,\mathcal{I}_{[4,9]},4)$.

Lemma 4.2 (i). It holds $\mathcal{I}(7)<\mathcal{I}(y)$ and $\ell_{\mathcal{I}}(z)<\ell_{\mathcal{I}}(y)$ for every $y\in R$ and every $z\in L$. In particular, this shows the existence of $v\in R$ such that $N_G(R)=N_G(v)\cap L$ – indeed, we can take $v=9$ in this case. For any $i=\pi_i\in N_G(R)=N_G(9)=\{4,8\}$, it holds $i>1$ and $\pi[i-1,8]\subseteq N_G[\pi_i]$.

Theorem 4.4. P_4 is an HP of G connecting 5 and $1=\mathrm{lv}(\mathcal{I})$; both P_1 and P_2 are HPs of G between 5 and $11=\mathrm{rv}(\mathcal{I})$.

Lemma 4.5. The normal vertex ordering of $H=G[\pi[4,9]]$ with respect to $\mathcal{I}_{[4,9]}$ is $\sigma=5,6,7,4,9,8$. Let $X=\{7,8\}$. The graph $H-X=G[\pi[4,9]]-X$ is traceable and $\min\{r_{\mathcal{I}}(\sigma_{7-j}):\ j\in[2]\}=r_{\mathcal{I}}(9)\geq r_{\mathcal{I}}(7)=\min\{r_{\mathcal{I}}(v):\ v\in X\}$.

Lemma 4.7 (i). $\pi_\iota=\pi_7=7<_{\mathcal{I}}\{9,10,11\}=R$.

Lemma 4.7 (ii). $\ell_{\mathcal{I}}(x)<r_{\mathcal{I}}(7)<r_{\mathcal{I}}(x)$ holds for all $x\in N_G(R)=\{4,8\}$.

Lemma 4.7 (iv). $N_G(R)=\{4,8\}\subseteq\{4,7,8\}=\mathrm{rc}(\mathcal{I}_{[8]})=\mathrm{rc}(\mathcal{I}_L)$.

Lemma 4.7 (v). R appears consecutively in P_1,P_2,P_3 and P_4.

Lemma 4.8 (i). $\tilde{\pi}_{[11-1-7]}=\tilde{\pi}_{[3]}=(11,10,9)$ is the normal vertex ordering of $G[R]$ with respect to $\overleftarrow{\mathcal{I}_R}$.

Lemma 4.8 (ii). The pairs $(y,z)\in L\times R$ such that y and z appear consecutively in one of P_1, P_2, P_3 or P_4 are $(4,9),(8,9)$ and $(8,11)$. Note that both 4 and 8, as possible y, fall into $N_G(9)=N_G(\tilde{\pi}_3)$.

Lemma 5.3 (i). $r_{\mathcal{I}}(7)<\ell_{\mathcal{I}}(9)<r_{\mathcal{I}}(\mathbb{j}_i)$ holds for all $i\in[2]$.

Lemma 5.3 (ii). Recall that $N_G(9)=N_G(R)=\{4,8\}=\{\mathbb{j}_1,\mathbb{j}_2\}$. It holds $[3,8]\subseteq N_G[4]$ and $[7,8]\subseteq N_G[8]$.

Lemma 5.3 (iii). It holds $N_G[\mathbb{j}_1] \cap L = N_G[4] \cap L = [3,8] \subseteq N_G[8] = N_G[\mathbb{j}_2]$, $[\mathbb{j}_1 - 1, 8] = [3,8] = N_G[4] \cap N_G[8] \cap L$ and $r_{\mathcal{I}}(\mathbb{j}_1) < r_{\mathcal{I}}(\mathbb{j}_2)$.

Lemma 5.3 (iv). It holds $\mathbb{j}_1 = 4 > 1$ and $\mathbb{j}_2 = 8 > 5 = \mathbb{j}_1 + 1$.

Lemma 5.3 (v). $N_G(\mathbb{j}_1 - 1) \cap [\mathbb{j}_1, 11] = N_G(3) \cap [4,11] = \{4,8\} = \{\mathbb{j}_1, \mathbb{j}_2\}$; $N_G(\mathbb{j}_2 - 1) \cap [\mathbb{j}_2, 11] = N_G(7) \cap [8,11] = \{8\} = \{\mathbb{j}_2\}$.

Lemma 5.5 (v). Let $\varsigma = (\mathbb{j}_1, \tilde{\pi}_3, \tilde{\pi}_2, \tilde{\pi}_1) = (4,9,10,11)$. The graph G has an HP $\eta = P_1 = (5,6,7,8,1,2,3,4,9,10,11)$ such that $\eta_1 = 5$, $\eta_8 = \mathbb{j}_1$, $\eta_{11} = \mathrm{rv}(\mathcal{I})$, $\eta_{[8,11]} = \varsigma$ and $\eta[9,11] = R$.

4. Normal vertex ordering. We start from a quite intuitive lemma which says that every normal vertex ordering of an interval graph will list the vertices from "left" to "right". Note that Lemma 4.1 (i) is the so-called "monotone property" of normal paths already known by Damaschke [9, Lemma 4] while Lemma 4.1 (iii) is a slight generalization of another observation of Damaschke [9, Lemma 3].

LEMMA 4.1. *Let G be an n-vertex graph, let $\mathcal{I}$ be an interval representation of G, let a be a vertex from $\mathrm{lc}(\mathcal{I})$ and let π be the output of $\mathrm{NO}(G, \mathcal{I}, a)$.*

- *(i) Suppose π_i and π_j are two vertices of $V(G)$ satisfying $\ell_{\mathcal{I}}(\pi_i) < \ell_{\mathcal{I}}(\pi_j)$ and $r_{\mathcal{I}}(\pi_i) < r_{\mathcal{I}}(\pi_j)$. Then, either $i < j$ or $j = 1$. Consequently, if $a = \mathrm{lv}(\mathcal{I})$ or $\mathcal{I}(\pi_i) < \mathcal{I}(\pi_j)$, then $i < j$.*
- *(ii) If $\pi_n \pi_{n-1} \in E(G)$, then π_n and π_{n-1} belong to $\mathrm{rc}(\mathcal{I})$.*
- *(iii) Take $j, k \in [n]$ with $j \leq k$ and let $H = G[\pi[j,k]]$. Then $\pi_j \in \mathrm{lc}(\mathcal{I}_{V(H)})$ and $\pi_{[j,k]}$ is the output of $\mathrm{NO}(H, \mathcal{I}_{V(H)}, \pi_j)$.*

Proof. **(i).** We only need to prove the first reading as the second one is a direct consequence of it. We assume for a contradiction that $i > j > 1$. By rule 1 of the Normal Ordering Algorithm, it follows from $r_{\mathcal{I}}(\pi_i) < r_{\mathcal{I}}(\pi_j)$ that $\pi_{j-1} \in N_G(\pi_j) \setminus N_G(\pi_i)$. This allows us derive from $\ell_{\mathcal{I}}(\pi_i) < \ell_{\mathcal{I}}(\pi_j)$ that $\mathcal{I}(\pi_i) < \mathcal{I}(\pi_{j-1})$. The same reasoning leads in turn to $\mathcal{I}(\pi_i) < \mathcal{I}(\pi_{j-2})$, $\ldots$, $\mathcal{I}(\pi_i) < \mathcal{I}(\pi_1)$. Since $\pi_1 = a \in \mathrm{lc}(\mathcal{I})$, it is impossible that $\mathcal{I}(\pi_i) < \mathcal{I}(\pi_1)$, as was to be shown.

(ii). Take $\ell \in \{n-1, n\}$. If $\pi_\ell \notin \mathrm{rc}(\mathcal{I})$, then, in view of $\pi_n \pi_{n-1} \in E(G)$, there exists $k \in [n-2]$ such that $\pi_k \in \mathrm{rc}(\mathcal{I}) \setminus N_G(\pi_\ell)$. This instead gives $\mathcal{I}(\pi_\ell) < \mathcal{I}(\pi_k)$, a contradiction with (i).

(iii). By way of induction, what we really need to show is that $\pi_j \in \mathrm{lc}(\mathcal{I}_{V(H)})$. By claim (i), for all $t \in [j+1, k]$, it cannot happen $\mathcal{I}(\pi_t) < \mathcal{I}(\pi_j)$ and hence the result follows. □

LEMMA 4.2. *Let G be an n-vertex graph with an interval representation $\mathcal{I}$ and let π be the normal vertex ordering of G with respect to $\mathcal{I}$. Take $\iota \in [n-1]$.*

- *(i) If $\mathrm{d}_{G,\pi}(\iota) = 1$, then we have $\mathcal{I}(\pi_\iota) < \mathcal{I}(y)$ and $\ell_{\mathcal{I}}(x) < \ell_{\mathcal{I}}(y)$ for each $y \in \pi[\iota+2, n]$ and each $x \in \pi[\iota+1]$. In addition, for any $\pi_i \in N_G(\pi[\iota+2, n])$, it holds $i > 1$ and $\pi[i-1, \iota+1] \subseteq N_G[\pi_i]$.*
- *(ii) If $\mathrm{d}_{G,\pi}(\iota) = 0$, then we have $\mathcal{I}(\pi_\iota) < \mathcal{I}(y)$ and $\ell_{\mathcal{I}}(x) < \ell_{\mathcal{I}}(y)$ for every $y \in \pi[\iota+1, n]$ and every $x \in \pi[\iota]$. In addition, for any $\pi_i \in N_G(\pi[\iota+1, n])$, it holds $i > 1$ and $\pi[i-1, \iota] \subseteq N_G[\pi_i]$.*

Proof. **(i).** Pick $y \in \pi[\iota+2, n]$. Since $\mathrm{d}_{G,\pi}(\iota) = 1$, rule 1 of the Normal Ordering algorithm ensures

$$\pi_{\iota+1} \in N_G(\pi_\iota). \tag{4.1}$$

and $y \notin N_G(\pi_\iota)$. It thus follows from Lemma 4.1 (i) that $\mathcal{I}(\pi_\iota) < \mathcal{I}(y)$. If $\ell_{\mathcal{I}}(x) > \ell_{\mathcal{I}}(y)$, then $\mathcal{I}(\pi_\iota) < \mathcal{I}(x)$. Considering Eq. (4.1), this implies $x \neq \pi_{\iota+1}$; by Lemma 4.1 (i), this yields $x \notin \pi[\iota]$.

Take any $\pi_i \in N_G(\pi[\iota+2,n])$. Since we know above that $\pi_\iota <_{\mathcal{I}} \pi[\iota+2,n]$, it follows from $\pi_i \in N_G(\pi[\iota+2,n])$ that

$$r_{\mathcal{I}}(\pi_i) > r_{\mathcal{I}}(\pi_\iota), \tag{4.2}$$

and hence $i > 1$.

We first show $\pi_{i-1}\pi_i \in E(G)$. From $\pi_i \in N_G(\pi[\iota+2,n])$, we know that $i-1=\iota$ or $i-1<\iota$. When $i-1=\iota$, Eq. (4.1) guarantees what we want. Assume that $i-1<\iota$. This means that, after determining $\pi[i-1]$, the Normal Ordering Algorithm does not choose π_ι as π_i. By Eq. (4.2), this implies $\pi_{i-1}\pi_i \in E(G)$.

We now take $j \in [i,\iota+1]$ and intend to show $\pi_j \in N_G[\pi_i]$. Let us assume $\pi_j \notin N_G[\pi_i]$. By Lemma 4.1 (i), $\mathcal{I}(\pi_i) < \mathcal{I}(\pi_j)$ and hence Eq. (4.2) tells us

$$\mathcal{I}(\pi_\iota) < \mathcal{I}(\pi_j). \tag{4.3}$$

By Lemma 4.1 (i) again, we deduce from $j \in [i,\iota+1]$ that $j=\iota+1$ and so Eq. (4.3) contradicts Eq. (4.1), as desired.

(ii). The proof is similar with the one for (i) and so we omit it. □

LEMMA 4.3. *[9, Theorem 3] Let $\mathcal{I}$ be an interval representation of a graph G. Then G is traceable if and only if G has an HP which starts from* $\mathrm{lv}(\mathcal{I})$ *and ends at* $\mathrm{rv}(\mathcal{I})$.

We now arrive at a key observation for shrinking the search space in the course of solving the 1 HP problem on interval graphs. It suggests that we need only search for Hamiltonian paths going from the specified vertex to the two extreme vertices with respect to a given interval representation.

THEOREM 4.4. *Let $\mathcal{I}$ be an interval representation of a graph G and take $x \in V(G)$. If G has an HP starting from x, then G has an HP which starts at x and ends at* $\mathrm{lv}(\mathcal{I})$ *or* $\mathrm{rv}(\mathcal{I})$.

Proof. Suppose τ is an HP of G satisfying $\tau_1 = x$ and $\{\mathrm{lv}(\mathcal{I}), \mathrm{rv}(\mathcal{I})\} = \{\tau_i, \tau_j\}$ where $i<j$. Let $W = \tau[i, |V(G)|]$ and $\mathcal{I}' = \mathcal{I}_W$. Then $\mathcal{I}'$ is an interval representation of $G[W]$ and $(\mathrm{lv}(\mathcal{I}), \mathrm{rv}(\mathcal{I})) = (\mathrm{lv}(\mathcal{I}'), \mathrm{rv}(\mathcal{I}'))$. Since $\tau_{[i,|V(G)|]}$ is an HP of $G[W]$, Lemma 4.3 tells us that $G[W]$ has an HP σ which starts at τ_i and ends at τ_j. Then $\tau_{[i-1]} + \sigma$ is an HP of G which starts at x and ends at $\tau_j \in \{\mathrm{lv}(\mathcal{I}), \mathrm{rv}(\mathcal{I})\}$, completing the proof. □

The next result, which is a bit surprising at first sight, is very useful for our algorithm as it connects graph theoretic properties (the existence of an HP) with some geometric information (distribution of right endpoints) in a not so trivial way.

LEMMA 4.5. *Let $\mathcal{I}$ be an interval representation of an n-vertex graph G and let π be the normal vertex ordering of G with respect to $\mathcal{I}$. Let X be a k-element subset of $V(G)$. If $G-X$ is traceable, then* $\min\{r_{\mathcal{I}}(\pi_{n+1-j}) :\ j \in [k]\} \geq \min\{r_{\mathcal{I}}(x) :\ x \in X\}$.

Proof. The proof goes by induction on n. When $n \leq k+1$, the result follows directly. We assume $n > k+1$ and the result holds for smaller n. Since π is a normal vertex ordering,

$$\pi_1 = \mathrm{lv}(\mathcal{I}). \tag{4.4}$$

If $\pi_1 \in X$, the result is immediate from Eq. (4.4). We assume $\pi_1 \notin X$ hereafter. If $\pi_{[2,n]}$ is the normal vertex ordering of $G-\pi_1$ with respect to $\mathcal{I}_{V(G-\pi_1)}$, our induction hypothesis on $G-\pi_1$ and X yields the claim. For the remaining case, we have $\pi_1\pi_2 \in E(G)$, $r_{\mathcal{I}}(\pi_3) = \min\{r_{\mathcal{I}}(v) :\ v \in V(G-\pi_1)\}$ and $\pi_{[3,n]}$ is the normal vertex ordering of $G' = G - \pi_1 - \pi_2$ with respect to $\mathcal{I}_{V(G')}$. If $\pi_2 \notin X$, we can finish the proof by invoking the induction assumption on G' and X. We now examine the case of $\pi_2 \in X$. Let τ be an HP of $G - X$. By Eq. (4.4), there exists $y \in N_{G-X}(\pi_1)$ such that both $G - X - \pi_1$ and $G - X - \pi_1 - y$ are traceable. Let $X' = (X \setminus \{\pi_2\}) \cup \{y\}$. Note that $G' - X' = G - X - \pi_1 - y$ and $|X'| = k$. Using induction hypothesis on G' and X', we have

$$\min\{r_{\mathcal{I}}(\pi_{n+1-j}) : \ j \in [k]\} \geq \min\{r_{\mathcal{I}}(x) : \ x \in X'\}. \tag{4.5}$$

As $y \in N_{G-X}(\pi_1)$ and π is a normal vertex ordering, $r_{\mathcal{I}}(y) \geq r_{\mathcal{I}}(\pi_2)$. This combined with Eq. (4.5) establishes the lemma, as desired. □

Theorem 4.6. *Let G be a graph with an interval representation $\mathcal{I}$ and let $\tilde{\pi}$ be the normal vertex ordering of G with respect to $\overleftarrow{\mathcal{I}}$. Take $\mathcal{R} \subseteq V(G)$ such that $\ell_{\mathcal{I}}(v) > \ell_{\mathcal{I}}(u)$ holds for all $v \in \mathcal{R}$ and $u \in V(G) \setminus \mathcal{R}$. Then the following hold.*

(a) If $G[\mathcal{R}]$ is traceable, then $\tilde{\pi}_{[|\mathcal{R}|]}$ is the normal vertex ordering of $G[\mathcal{R}]$ with respect to $\overleftarrow{\mathcal{I}}_{\mathcal{R}}$.

(b) For each $z \in \mathcal{R}$ satisfying $1\,\mathrm{HP}(G[\mathcal{R}], z) = 1$, $N_G(\tilde{\pi}_{|\mathcal{R}|}) \supseteq N_G(z) \setminus \mathcal{R}$.

Proof. **(a).** Applying Lemma 4.5 for $X = V(G)\setminus\mathcal{R}$ yields $\min\{r_{\overleftarrow{\mathcal{I}}}(\tilde{\pi}_i) : \ |\mathcal{R}|+1 \leq i \leq n\} \geq \min\{r_{\overleftarrow{\mathcal{I}}}(x) : \ x \in X\}$. For each $v \in \mathcal{R}$ and $u \in X$, it holds $r_{\overleftarrow{\mathcal{I}}}(v) = -\ell_{\mathcal{I}}(v) < -\ell_{\mathcal{I}}(u) = r_{\overleftarrow{\mathcal{I}}}(u)$. It now follows that $\tilde{\pi}[|\mathcal{R}|] = \mathcal{R}$ and hence the claim.

(b). By claim (a), we can apply Lemma 4.5 for the graph $G[\mathcal{R}]$ and the set $X = \{z\}$ to get $\ell_{\mathcal{I}}(\tilde{\pi}_{|\mathcal{R}|}) = -r_{\overleftarrow{\mathcal{I}}}(\tilde{\pi}_{|\mathcal{R}|}) \leq -r_{\overleftarrow{\mathcal{I}}}(z) = \ell_{\mathcal{I}}(z)$. By assumption, $\ell_{\mathcal{I}}(\tilde{\pi}_{|\mathcal{R}|}) > \ell_{\mathcal{I}}(u)$ for any $u \in V(G) \setminus \mathcal{R}$. It then follows $N_G(\tilde{\pi}_{|\mathcal{R}|}) \supseteq N_G(z) \setminus \mathcal{R}$, as wanted. □

An important ingredient of our algorithm is to make use of the thickness information of the normal vertex ordering to partition the graph vertex set into two parts, its "left" part L and its "right" part R, and then do case analysis accordingly. The next lemma compiles some basic information about such a partition. To prepare for its proof, we need a little more notation. Let S be a set of size n. For any $T \subsetneq S$ and a bijective map $\sigma \in S^{[n]}$, let $\sigma - T$ stand for the ordering of $S \setminus T$ such that for any $(\sigma - T)_i = \sigma_{i'}$ and $(\sigma - T)_j = \sigma_{j'}$, we always have $(i - j)(i' - j') \geq 0$. Note that $\sigma_{[i,j]} = \sigma - T$ where $T = \sigma[i - 1] \cup \sigma[j + 1, n]$.

Lemma 4.7. *Let G be an interval graph with $n \geq 3$ vertices and $\mathcal{I}$ an interval representation of G. Let π be the normal vertex ordering of G with respect to $\mathcal{I}$ and let θ be an HP of G. Take $\iota \in [n - 2]$, put $L = \pi[\iota + 1]$ and $R = \pi[\iota + 2, n]$. If $\mathrm{d}_{G,\pi}(\iota) = 1$, then the following hold.*

(i) $\pi_\iota <_{\mathcal{I}} R$.

(ii) $\ell_{\mathcal{I}}(x) < r_{\mathcal{I}}(\pi_\iota) < r_{\mathcal{I}}(x)$ holds for all $x \in N_G(R)$.

(iii) The ordering π is an HP of G.

(iv) $\emptyset \neq N_G(R) \subseteq \mathrm{rc}(\mathcal{I}_L)$.

(v) The set R appears consecutively in θ.

(vi) If $\theta_1 \in L$, then there exist $y \in N_G(R)$ and $z \in R$ such that y and z appear consecutively in θ and $G[L] - y$ has an HP starting from θ_1.

(vii) If $\iota = \min\{i : \ \mathrm{d}_{G,\pi}(i) = 1\}$, then the following three claims hold.

  (a) Either $\pi_{[\iota+1]}$ is a 2-thick normal vertex ordering of $G[L]$ or $\pi_{[\iota+1]}$ is a length one HP of $G[L]$.

  (b) $r_{\mathcal{I}}(\pi_{\iota+1}) = \max\{r_{\mathcal{I}}(x) : \ x \in L\}$.

  (c) $N_G(\pi_{\iota+1}) \cap R = \cup_{x\in L}(N_G(x) \cap R)$.

Proof. **(i).** This follows directly from Lemma 4.2 (i).

(ii). Pick $x \in N_G(R)$. Lemma 4.2 (i) says that $\pi_\iota \in N_G(x)$. It then suffices to prove $r_{\mathcal{I}}(\pi_\iota) < r_{\mathcal{I}}(x)$, which comes from claim (i).

(iii). By Lemma 2.4, the fact that θ is an HP of G shows that π is also an HP of G.

(iv). By (iii) and $R \neq V(G)$, it is obvious that $N_G(R) \neq \emptyset$. Assume for a contradiction that we can find $x \in N_G(R)$ and $k \leq \iota+1$ such that $x <_{\mathcal{I}} \pi_k$. Claim (ii) then tells us that $\pi_\iota <_{\mathcal{I}} \pi_k$ and so Lemma 4.1 (i) gives $\iota+1 = k$. But $\pi_\iota <_{\mathcal{I}} \pi_k = \pi_{\iota+1}$ has contradicted (iii).

(v). Suppose $R = \cup_{i\in[k]}\theta[s_i, t_i]$, where $1 \leq s_1 \leq t_1 < t_1 + 1 < s_2 \leq t_2 < t_2 + 1 < \cdots < s_k \leq t_k \leq n$. We intend to prove $k = 1$ and this can be reduced to proving the next two statements:

(v.a). For every $S \in \binom{N_G(R)}{\geq 2}$, $G[L] - S$ is not traceable;

(v.b). If $k > 1$, then there exists $S \in \binom{N_G(R)}{\geq 2}$ such that $G[L] - S$ is traceable.

Take any set $S \in \binom{N_G(R)}{\geq 2}$. Obviously, $\pi_{[\iota+1]}$ is the normal vertex ordering of $G[L]$ with respect to $\mathcal{I}_L$. If $G[L] - S$ is traceable, Lemma 4.5 shows that $\min\{r_{\mathcal{I}}(x) :\ x \in S\} \leq r_{\mathcal{I}}(\pi_\iota)$, which is a contradiction with claim (ii). This proves (v.a).

We now turn attention to (v.b). Let $\xi = \theta - R$. As θ is an HP of G, we can deduce from claim (iv) that ξ is an HP of $G - R = G[L]$. For every set $T \subseteq N_G(R)$ which appears consecutively in ξ, we set $\alpha(T) = T$ if $\{\xi_1, \xi_{\iota+1}\} \cap T \neq \emptyset$ and set $\alpha(T) = T \setminus \{t\}$, where t is the element from T such that $\ell_{\mathcal{I}}(t) = \min_{z\in T} \ell_{\mathcal{I}}(z)$, if $\{\xi_1, \xi_{\iota+1}\} \cap T = \emptyset$. Owing to claim (iv), for every two vertices x and y of $N_G(R)$ with $\ell_{\mathcal{I}}(x) < \ell_{\mathcal{I}}(y)$, it holds $N_{G[L]}[y] \subseteq N_{G[L]}[x]$. This means that $\xi - \alpha(T)$ is an HP of $G[L] - \alpha(T)$. Since $k \geq 2$, two possibilities can arise.

Case 1. $t_1 + 1 < s_k - 1$.

We can take the required set S to be $\alpha(T_1) \cup \alpha(T_2)$, where

$$T_1 = \begin{cases} \{\theta_{s_1-1}, \theta_{t_1+1}\}, & \text{if } \ s_1 > 1, \\ \{\theta_{t_1+1}\}, & \text{if } \ s_1 = 1, \end{cases} \quad T_2 = \begin{cases} \{\theta_{s_k-1}, \theta_{t_k+1}\}, & \text{if } \ t_k < n, \\ \{\theta_{s_k-1}\}, & \text{if } \ t_k = n. \end{cases}$$

Case 2. $t_1 + 1 = s_k - 1$.

In this case, we have $k = 2$ and $L = \theta[s_1 - 1] \cup \{\theta_{t_1+1}\} \cup \theta[t_2 + 1, n]$. Since L contains two different points π_ι and $\pi_{\iota+1}$, we find that either $s_1 > 1$ or $t_k = t_2 < n$. Without loss of generality, let us assume that $s_1 > 1$. We now choose

$$S = \begin{cases} \alpha(\{\theta_{s_1-1}, \theta_{t_1+1}, \theta_{t_2+1}\}), & \text{if } \ t_2 < n, \\ \alpha(\{\theta_{s_1-1}, \theta_{t_1+1}\}) = \{\theta_{s_1-1}, \theta_{t_1+1}\}, & \text{if } \ t_2 = n, \end{cases}$$

finishing the proof.

(vi). By claim (v), R appears consecutively in θ. Suppose $\theta[p, q] = R$. Since $\theta_1 \in L$, we have $1 < p \leq q \leq n$.

If $q = n$, then $2 \leq |L| = p - 1$ and we can pick $(y, z) = (\theta_{p-1}, \theta_p)$ and see that $\theta_{[p-2]}$ is an HP of $G[L] - y$ starting from θ_1.

If $q < n$, then both θ_{p-1} and θ_{q+1} belong to $N_G(R)$ and so claim (iv) tells us that θ_{p-1} and θ_{q+1} belong to $\mathrm{rc}(\mathcal{I}_L)$. Without loss of generality, we assume that $\ell_{\mathcal{I}}(\theta_{p-1}) < \ell_{\mathcal{I}}(\theta_{q+1})$ and hence $N_{G[L]}[\theta_{p-1}] \supseteq N_{G[L]}[\theta_{q+1}]$. By now, we can take $(y, z) = (\theta_{q+1}, \theta_q)$ and see that $(\theta_1, \ldots, \theta_{p-1}, \theta_{q+2}, \ldots, \theta_n)$ is an HP of $G[L]-y$ starting from θ_1.

(vii.a). If $\iota = 1$, the claim is direct from Eq. (4.1). Now suppose $\iota > 1$ and let $\xi = \pi_{[\iota+1]}$. By Eq. (4.1), it remains to show that $\mathrm{d}_{G[L],\xi}(i) \geq 2$ for every $i \in [\iota - 1]$.

If $\xi_i \notin N_G(R)$, we have $\mathrm{d}_{G[L],\xi}(i) = \mathrm{d}_{G,\pi}(i) \geq 2$.

If $\xi_i \in N_G(R)$, then the second part of Lemma 4.2 (i) asserts that $\mathrm{d}_{G[L],\xi}(i) = \iota + 1 - i \geq 2$.

(vii.b). If $\iota = 1$, the claim is nothing but $r_{\mathcal{I}}(\pi_2) > r_{\mathcal{I}}(\pi_1)$, the latter being a consequence of the fact that π is the normal vertex ordering with respect to $\mathcal{I}$.

Assume then $\iota > 1$. By Lemma 2.1 and claim (vii.a), $G[L]$ is Hamiltonian and hence $G[L] - x$ is traceable for every $x \in L$. It now follows from Lemma 4.5 that $r_{\mathcal{I}}(\pi_{\iota+1}) = \max\{r_{\mathcal{I}}(x) : x \in L\}$.

(vii.c). Lemma 4.2 (i) shows that $\ell_{\mathcal{I}}(x) < \ell_{\mathcal{I}}(y)$ for all $x \in L$ and $y \in R$. In conjunction with (vii.b), this implies $N_G(\pi_{\iota+1}) \cap R \supseteq N_G(x) \cap R$ for each $x \in L$, finishing the proof. □

As both an interval representation and its adjoint will appear in the main algorithm of this work, it is necessary at this juncture to establish some facts on them.

LEMMA 4.8. *Let G be a graph on n vertices possessing an interval representation $\mathcal{I}$. Let π and $\tilde{\pi}$ be the normal vertex orderings of G with respect to $\mathcal{I}$ and $\overleftarrow{\mathcal{I}}$, respectively. Take $\iota \in [n-2]$ such that $\mathrm{d}_{G,\pi}(\iota) = 1$ and let $R = G[\iota+2, n]$. Then the following hold.*

(i) $\tilde{\pi}_{[n-1-\iota]}$ is the normal vertex ordering of $G[R]$ with respect to $\overleftarrow{\mathcal{I}_R}$.

(ii) If $z \in \pi[\iota+2, n]$ and $y \in \pi[\iota+1]$ are two vertices appeared consecutively in an HP θ of G, then $y \in N_G(\tilde{\pi}_{n-1-\iota})$.

Proof. **(i).** By Lemma 4.2 (i), $\ell_{\mathcal{I}}(u) < \ell_{\mathcal{I}}(v)$ for each $u \in \pi[\iota+1]$ and $v \in \pi[\iota+2, n]$. It then follows from Theorem 4.6 (a) that $\tilde{\pi}_{[n-1-\iota]}$ is the normal vertex ordering of $G[R]$ with respect to $\overleftarrow{\mathcal{I}_R}$.

(ii). By Lemma 4.7 (v), R appears consecutively in θ. As y and z appear consecutively in θ, this implies $1\,\mathrm{HP}(G[R], z) = 1$. Applying Theorem 4.6 (b), $y \in N_G(z) \setminus R \subseteq N_G(\tilde{\pi}_{n-1-\iota})$, as wanted. □

LEMMA 4.9. *Assume that G is an interval graph and π is a normal vertex ordering of G. If G is Hamiltonian, then π is a 2-thick HP of G.*

Proof. Assume that π is the normal vertex ordering of G with respect to an interval representation $\mathcal{I}$. By Lemma 2.4, π is an HP of G. If π is not 2-thick, then there is some integer $i \in [|V(G)| - 2]$ such that $\mathrm{d}_{G,\pi}(i) = 1$, which means $N_G(\pi_i) \cap \pi[i+1, |V(G)|] = \{\pi_{i+1}\}$. Let θ be the normal vertex ordering with respect to $\mathcal{I}_{V(G-\pi_{i+1})}$. Since G is Hamiltonian, $G - \pi_{i+1}$ is traceable, and hence by Lemma 2.4, θ is an HP of $G - \pi_{i+1}$. It is clear that $\pi_{[i]} = \theta_{[i]}$ and so θ cannot be an HP of $G - \pi_{i+1}$. This contradiction then concludes the proof. □

THEOREM 4.10. *An interval graph G is Hamiltonian if and only if it has a 2-thick HP.*

Proof. This follows from Lemmas 2.1 and 4.9. □

5. The $1\,\mathrm{HP}$ algorithm. In this section, we demonstrate an algorithm for solving the $1\,\mathrm{HP}$ problem on interval graphs and prove its correctness. We will need to design two subprograms for related algorithmic problems on interval graphs and here is the first of them.

Procedure $\mathrm{HP}(G, \mathcal{I}, \pi, x)$;
{ Input: A graph G with n vertices, $x \in V(G)$, an interval representation $\mathcal{I}$ of G such that the normal vertex ordering π of G with respect to $\mathcal{I}$ gives a 2-thick HP of G; Output: an HP ϱ of G which starts at x and ends at some vertex $y \in V(G)$ such that $r_{\mathcal{I}}(y) \geq \min\{r_{\mathcal{I}}(\pi_{n-1}), r_{\mathcal{I}}(\pi_n)\}$; }
begin
  If $x = \pi_1$, output $\varrho = \pi$.
  If $x \neq \pi_1$, do Procedure $\mathrm{HC}(G, \pi)$ to get an HC of G, say

$\rho = [\rho_1 = \pi_{n-1}, \rho_2, \ldots, \rho_n = \pi_n]$. Let $x = \rho_j, \pi_1 = \rho_i$.
If $j = 1$, output $\varrho = (\rho_1 = x = \pi_{n-1}, \rho_2, \ldots, \rho_n = \pi_n)$.
If $j = n$, output $\varrho = (\rho_n = x = \pi_n, \rho_{n-1}, \ldots, \rho_1 = \pi_{n-1})$.
Else, let $\phi' = (\rho_j, \rho_{j+1}, \ldots, \rho_{i-1})$ if $1 < j < i$ and let $\phi' = (\rho_j, \rho_{j-1}, \ldots, \rho_{i+1})$ if $i < j < n$. Let $S = V(G) \setminus V(\phi')$. Construct the normal vertex ordering of $G[S]$ with respect to $\mathcal{I}_S$, say ϕ. Output $\varrho = \phi' + \phi$.
end;

THEOREM 5.1. *The algorithm* $\mathrm{HP}(G, \mathcal{I}, \pi, x)$ *is correct.*

Proof. If $x = \pi_1$, the result is trivial. Now suppose $x \neq \pi_1$. By Lemma 2.1, the algorithm $\mathrm{HC}(G, \pi)$ is correct and so ρ is an HC of G in which π_1, π_2 are consecutive and π_{n-1}, π_n are also consecutive. As described in the algorithm, we may assume $\rho_1 = \pi_{n-1}$ and $\rho_n = \pi_n$.

If $x \in \{\pi_{n-1}, \pi_n\}$, namely $j \in \{1, n\}$, the output ϱ is clearly what we want.

Finally, we consider the case that $x \notin \{\pi_1, \pi_{n-1}, \pi_n\}$. Since ρ is an HC and so $\{\rho_{i-1}, \rho_{i+1}\} \subseteq N_G(\rho_i)$, our task is to show that ϕ is an HP of $G[S]$ and that $r_{\mathcal{I}}(\phi_{|S|}) \geq \min(r_{\mathcal{I}}(\pi_{n-1}), r_{\mathcal{I}}(\pi_n))$.

By Lemma 4.1 (ii), π_{n-1} and π_n belong to $\mathrm{rc}(\mathcal{I})$. This implies the existence of w and z such that $\{w, z\} = \{\pi_{n-1}, \pi_n\}$ and $N_G[w] \subseteq N_G[z]$. Thanks to the existence of the HC ρ of G, we now see that both $G[S]$ and $G[S] - w$ are traceable. It follows from Lemma 2.4 that ϕ is an HP of $G[S]$ and from Lemma 4.5 that $r_{\mathcal{I}}(\phi_{|S|}) \geq r_{\mathcal{I}}(w)$, completing the proof. □

LEMMA 5.2. *The algorithm* $\mathrm{HP}(G, \mathcal{I}, \pi, x)$ *can be implemented in linear time.*

Proof. It is straightforward from Lemmas 2.2 and 2.3. □

LEMMA 5.3. *Let G be a graph with $n \geq 3$ vertices and with an interval representation $\mathcal{I}$. Let π be the normal vertex ordering of G with respect to $\mathcal{I}$. Assume that $\mathrm{d}_{G,\pi}(\iota) = 1$ for some $\iota \in [n-2]$. Let $R = \pi[\iota+2, n]$, $L = \pi[\iota+1]$ and $v \in R$. Suppose $N_G(v) \cap L = \{\pi_{\mathbb{j}_1}, \ldots, \pi_{\mathbb{j}_h}\}$ where $\mathbb{j}_1 < \cdots < \mathbb{j}_h$. Let $s \in [h]$ be the index such that $\ell_{\mathcal{I}}(\pi_{\mathbb{j}_s}) = \max\{\ell_{\mathcal{I}}(\pi_{\mathbb{j}_t}) :\ t \in [h]\}$.*

(i) $r_{\mathcal{I}}(\pi_\iota) < \ell_{\mathcal{I}}(v) < r_{\mathcal{I}}(\pi_{\mathbb{j}_i})$ holds for all $i \in [h]$.

(ii) For each $i \in [h]$ and each $j \in [\mathbb{j}_i - 1, \iota + 1]$, it holds $\pi_j \in N_G[\pi_{\mathbb{j}_i}]$.

(iii) It holds

$$N_G[\pi_{\mathbb{j}_s}] \cap L \subseteq \cap_{i \in [h]} N_G[\pi_{\mathbb{j}_i}] \cap L, \tag{5.1}$$

$$\pi[\mathbb{j}_s - 1, \iota + 1] \subseteq \cap_{i \in [h]} N_G[\pi_{\mathbb{j}_i}] \cap L \tag{5.2}$$

and

$$r_{\mathcal{I}}(\pi_{\mathbb{j}_s}) < r_{\mathcal{I}}(\pi_{\mathbb{j}_{s+1}}) < \cdots < r_{\mathcal{I}}(\pi_{\mathbb{j}_h}). \tag{5.3}$$

(iv) It holds $\mathbb{j}_1 > 1$ and $\mathbb{j}_{i+1} > \mathbb{j}_i + 1$ for each $i \in [h-1]$.

(v) For every $i \in [h]$, $N_G(\pi_{\mathbb{j}_i - 1}) \cap \pi[\mathbb{j}_i, n] \subseteq \{\pi_{\mathbb{j}_t} :\ t \in [i, h]\}$.

Proof. **(i).** By Lemma 4.2 (i), $\mathrm{d}_{G,\pi}(\iota) = 1$ gives $\mathcal{I}(\pi_\iota) < \mathcal{I}(v)$; for any $i \in [h]$, from $\pi_{\mathbb{j}_i} \in N_G(v)$ we obtain $\ell_{\mathcal{I}}(v) < r_{\mathcal{I}}(\pi_{\mathbb{j}_i})$. Putting together, we get (i).

(ii). Remember that $\pi_{\mathbb{j}_i} \in N_G(R)$. Thus, the result follows from the second claim of Lemma 4.2 (i).

(iii). By virtue of Lemma 4.7 (iv), Eq. (5.1) is a consequence of $\ell_{\mathcal{I}}(\pi_{\mathbb{j}_s}) = \max\{\ell_{\mathcal{I}}(\pi_{\mathbb{j}_t}) :\ t \in [h]\}$. Claim (ii) shows that $\pi[\mathbb{j}_s - 1, \iota + 1] \subseteq N_G[\pi_{\mathbb{j}_s}] \cap L$ and thus Eq. (5.2) follows. For every $i \in [h]$, claim (ii) asserts $\pi_{\mathbb{j}_i - 1}\pi_{\mathbb{j}_i} \in E(G)$ and hence, by rule 1 of the Normal Ordering Algorithm, we have $r_{\mathcal{I}}(\pi_{\mathbb{j}_i}) = \min\{r_{\mathcal{I}}(z) : z \in \pi[\mathbb{j}_i, n] \cap N_G(\pi_{\mathbb{j}_i-1})\}$. But Eq. (5.2) claims that $\pi_{\mathbb{j}_i-1} \in \cap_{t\in[h]} N_G[\pi_{\mathbb{j}_t}]$ for $i \in [s,h]$. Accordingly, $r_{\mathcal{I}}(\pi_{\mathbb{j}_i}) = \min\{r_{\mathcal{I}}(\pi_{\mathbb{j}_k}) : k \in [i,h]\}$ for $i \in [s,h]$, proving Eq. (5.3).

(iv). The fact that $\mathbb{j}_1 > 1$ is direct from claim (i). Pick $i \in [h-1]$. Claim (i) also ensures $\mathbb{j}_i \neq \iota$. Since $\mathbb{j}_i < \mathbb{j}_h \leq \iota + 1$, we can further obtain $\mathbb{j}_i < \iota$. By claim (ii), $\pi_\iota \in N_G(\pi_{\mathbb{j}_i})$. In view of claim (i), rule 1 of the Normal Ordering Algorithm demonstrates that $\mathbb{j}_i + 1 < \mathbb{j}_{i+1}$.

(v). Take $i \in [h]$ and $k \in [\mathbb{j}_i, n]$ such that $\pi_k \in N_G(\pi_{\mathbb{j}_i-1})$. We aim to prove $\pi_k \in N_G(v) \cap L$. Noting that $\pi_k \in N_G(\pi_{\mathbb{j}_i-1})$ and $k \geq \mathbb{j}_i$, rule 1 of the Normal Ordering Algorithm gives

$$r_{\mathcal{I}}(\pi_{\mathbb{j}_i}) \leq r_{\mathcal{I}}(\pi_k). \tag{5.4}$$

Since $\pi_{\mathbb{j}_i} v \in E(G)$, we have

$$\ell_{\mathcal{I}}(v) < r_{\mathcal{I}}(\pi_{\mathbb{j}_i}). \tag{5.5}$$

Lemma 4.7 (i) implies

$$r_{\mathcal{I}}(\pi_\iota) < \ell_{\mathcal{I}}(v). \tag{5.6}$$

By claim (i), $r_{\mathcal{I}}(\pi_\iota) < r_{\mathcal{I}}(\pi_{\mathbb{j}_i})$ and so rule 1 of the Normal Ordering Algorithm gives $\pi_{\mathbb{j}_i-1}\pi_\iota \notin E(G)$. Accordingly, using Lemma 4.1 (i) yields

$$r_{\mathcal{I}}(\pi_{\mathbb{j}_i-1}) \leq r_{\mathcal{I}}(\pi_\iota). \tag{5.7}$$

It follows from $\pi_k \in N_G(\pi_{\mathbb{j}_i-1})$ that

$$\ell_{\mathcal{I}}(\pi_k) < r_{\mathcal{I}}(\pi_{\mathbb{j}_i-1}). \tag{5.8}$$

In view of Lemma 4.7 (i), we deduce from Eqs. (5.7) and (5.8) that $\pi_k \in L$. Combining Eqs. (5.4), (5.5), (5.6), (5.7), (5.8), we obtain

$$\ell_{\mathcal{I}}(\pi_k) < r_{\mathcal{I}}(\pi_{\mathbb{j}_i-1}) \leq r_{\mathcal{I}}(\pi_\iota) < \ell_{\mathcal{I}}(v) < r_{\mathcal{I}}(\pi_{\mathbb{j}_i}) \leq r_{\mathcal{I}}(\pi_k),$$

which demonstrates $\pi_k \in N_G(v)$. □

LEMMA 5.4. *Let $\mathcal{I}$ be an interval representation of a graph G on $n \geq 3$ vertices. Let π and $\tilde{\pi}$ be the normal vertex orderings of G with respect to $\mathcal{I}$ and $\overleftarrow{\mathcal{I}}$, respectively. Suppose π is 1-thick but not 2-thick. Let*

$$\iota = \min\{i \in [n-1] : \mathrm{d}_{G,\pi}(i) = 1\} \leq n-2, \tag{5.9}$$

$L = \pi[\iota+1]$ and $R = \pi[\iota+2, n]$. Take $v = \tilde{\pi}_{n-1-\iota} = \tilde{\pi}_{|R|}$ and assume that

$$N_G(v) \cap L = \{\pi_{\mathbb{j}_1}, \ldots, \pi_{\mathbb{j}_h}\}, \tag{5.10}$$

where $\mathbb{j}_1 < \cdots < \mathbb{j}_h$. Assume that

$$\ell_{\mathcal{I}}(\pi_{\mathbb{j}_s}) = \max\{\ell_{\mathcal{I}}(\pi_{\mathbb{j}_t}) : t \in [h]\} \tag{5.11}$$

and let $G' = G[L] - \pi_{\mathbb{j}_s}$. Let τ and $\tilde{\tau}$ be the normal vertex orderings of G' with respect to $\mathcal{I}_{V(G')}$ and $\overleftarrow{\mathcal{I}}_{V(G')}$, respectively. Then the following statements hold.

(i) $\tilde{\pi}_{n-\iota} = \pi_{\mathbb{j}_s}$ and $V(G') = \tilde{\pi}[n-\iota+1, n]$.

(ii) $\{\pi_{\mathbb{j}_1}, \ldots, \pi_{\mathbb{j}_h}\} \subseteq N_G(R)$.

(iii) $\mathbb{j}_h = \iota + 1$.

(iv) *The ordering τ is an HP of G'. Moreover, $\tau_i = \pi_i$ for $i \in [\iota] \setminus \{\mathbb{j}_s, \ldots, \mathbb{j}_{h-1}\}$ and $\tau_{\mathbb{j}_k} = \pi_{\mathbb{j}_{k+1}}$ for $k \in [s, h-1]$. In particular, if $h = 1$, then $\tau = \pi_{[\iota]}$.*

(v) *For all $i \in [\iota]$,*

$$\mathrm{d}_{G',\tau}(i) = \begin{cases} \mathrm{d}_{G,\pi}(i), & \text{if } \tau_i \notin N_G(\pi_{\mathbb{j}_s}), \\ \iota - i, & \text{if } \tau_i \in N_G(R) \subseteq N_G(\pi_{\mathbb{j}_s}), \\ \mathrm{d}_{G,\pi}(i) - 1, & \text{if } \tau_i \in N_G(\pi_{\mathbb{j}_s}) \setminus N_G[R]. \end{cases}$$

(vi) *If $h \geq 2$, then $\min\{i : \mathrm{d}_{G',\tau}(i) = 1\} = \mathbb{j}_{h-1} - 1 < \iota - 1 = |V(G')| - 1$ and hence τ is not 2-thick.*

(vii) *There is no HP in G starting from any vertex in $N_G(R)$.*

(viii) *For any $x \in R$, if G has an HP starting from x, then $G[R]$ has an HP starting from x and ending at a vertex $y \in N_G(\pi_{\iota+1})$.*

(ix) *If $x = \pi_p$ for some $p \in [\iota + 1, n]$, then there is no HP of G which starts from x and ends at $\tilde{\pi}_1$.*

(x) *Suppose that $|V(G')| \geq 3$ and τ is not 2-thick. Then $\tilde{\pi}_{n-\iota+1} = \mathrm{rv}(\mathcal{I}_{V(G')})$ and $\tilde{\pi}_{[n-\iota+1,n]} = \tilde{\tau}$.*

Proof. **(i).** We first read from Lemma 4.8 (i) that

$$R = \tilde{\pi}[|R|] = \tilde{\pi}[n - 1 - \iota]. \tag{5.12}$$

In light of Eqs. (5.10) and (5.11), rule 1 of the Normal Ordering Algorithm gives $\tilde{\pi}_{n-\iota} = \pi_{\mathbb{j}_s}$ and so $V(G') = L \setminus \{\pi_{\mathbb{j}_s}\} = \tilde{\pi}[n - \iota + 1, n]$.

(ii). This is a consequence of Eqs. (5.10) and (5.12).

(iii). It is clear that $\mathbb{j}_h \leq \iota + 1$ as $\pi_{\mathbb{j}_h} \in L = \pi[\iota + 1]$. Applying Lemma 4.8 (ii) for $\theta = \pi$, $y = \pi_{\iota+1}$ and $z = \pi_{\iota+2}$, we obtain $\pi_{\iota+1} \in N_G(\tilde{\pi}_{n-1-\iota}) \setminus R$ and so $\mathbb{j}_h \geq \iota + 1$.

(iv). From Lemma 4.7 (vii.a) and Theorem 4.10, we see that G' has an HP. Now, it follows from Lemma 2.4 that τ is an HP of G'.

By Lemma 5.3 (iv), $\mathbb{j}_s \geq \mathbb{j}_1 > 1$. For $i \in [\mathbb{j}_s - 1]$, rule 1 of the Normal Ordering Algorithm implies $\pi_i = \tau_i$. To establish the remaining claimed relationship between π and τ, let us focus on the value of τ_i for $i \in [\mathbb{j}_s, \iota]$ and assume that its relationship with π for smaller i has been verified. By (iii), $\iota = \mathbb{j}_h - 1$ and so we distinguish three cases.

Case 1. $i = \mathbb{j}_k$ for some $k \in [s, h-1]$.

Lemma 5.3 (iv) says that $i - 1 \notin \{\mathbb{j}_t : t \in [h]\}$ and so the induction assumption gives $\tau_{i-1} = \pi_{i-1}$. By Lemma 5.3 (iii) (Eq. (5.2)) and Lemma 5.3 (v), $N_G(\pi_{\mathbb{j}_k-1}) \cap \pi[\mathbb{j}_k, n] = \{\pi_{\mathbb{j}_t} : t \in [k, h]\}$. Therefore, rule 1 of the Normal Ordering Algorithm along with Lemma 5.3 (iii) (Eq. (5.3)) guarantees that $\tau_i = \pi_{\mathbb{j}_{k+1}}$, as desired.

Case 2. $i = \mathbb{j}_k + 1$ for some $k \in [s, h-1]$.

By induction assumption, it holds $\tau_{\mathbb{j}_k} = \pi_{\mathbb{j}_{k+1}}$. In light of Lemma 5.3 (iii) (Eq. (5.2)) and Lemma 5.3 (iv), the normal ordering rule 1 guarantees $\tau_i = \pi_i$.

Case 3. $\mathbb{j}_k + 1 < i < \mathbb{j}_{k+1}$ for some $k \in [s, h-1]$. Recall from (iii) that $\iota = \mathbb{j}_h - 1$.

By induction hypothesis, $\tau_{i-1} = \pi_{i-1}$, $\pi_i \notin \tau[i-1]$ and $\tau[i, \iota] \subseteq \pi[i, n]$. It follows from rule 1 of the Normal Ordering Algorithm that $\tau_i = \pi_i$, as wanted.

(v). We start from the case of $\tau_i \notin N_G(\pi_{\mathbb{j}_s})$. By Lemma 5.3 (ii) and the previous claim (iv), we have $i < \mathbb{j}_s - 1$ and $\tau_i = \pi_i \in L$. According to claim (ii) and Lemma 4.7 (iv), $\tau_i \notin N_G(\pi_{\mathbb{j}_s})$ implies $\tau_i \notin N_G(R)$. Appealing to the second part of claim (iv) yields $\mathrm{d}_{G',\tau}(i) = \mathrm{d}_{G,\pi}(i)$, as desired.

For the next case of $\tau_i \in N_G(R)$, we first deduce from claims (ii) and (iv) that $\pi_i \in N_G(R)$. Then the result follows from Eq. (5.2) and claim (iv).

Finally, let $\tau_i \in N_G(\pi_{\mathbb{j}_s}) \setminus N_G[R]$. By (ii) and (iv), $\tau_i = \pi_i$ and so Lemma 5.3 (iii) (Eq. (5.1)) gives

$$\{\pi_{\mathbb{j}_1}, \ldots, \pi_{\mathbb{j}_h}\} \subseteq N_G(\pi_i) = N_G(\tau_i) \subseteq L. \tag{5.13}$$

By (iii) and (iv), we come to $\mathrm{d}_{G',\tau}(i) = \mathrm{d}_{G,\pi}(i) - 1$.

(vi). By claim (iii) and Lemma 5.3 (iv), we have $\iota - 1 = \mathbb{j}_h - 2 > \mathbb{j}_{h-1} + 1 - 2 = \mathbb{j}_{h-1} - 1 \geq 1$. It remains to check that

(vi.a). $\mathrm{d}_{G',\tau}(\mathbb{j}_{h-1} - 1) = 1$;

(vi.b). $\mathrm{d}_{G',\tau}(i) > 1$ holds for all $i \in [\mathbb{j}_{h-1} - 2]$.

By Eq. (5.9) and Lemma 5.3 (v), $N_G(\pi_{\mathbb{j}_{h-1}-1}) \cap \pi[\mathbb{j}_{h-1}, n] = \{\pi_{\mathbb{j}_{h-1}}, \pi_{\mathbb{j}_h}\}$. This gives $\mathrm{d}_{G,\pi}(\mathbb{j}_{h-1} - 1) = 2$ and $\pi_{\mathbb{j}_{h-1}-1} \notin N_G[R]$. We further claim that $\pi_{\mathbb{j}_{h-1}-1} \in N_G(\pi_{\mathbb{j}_s})$. This is trivial when $s \geq h-1$ and follows from Lemma 5.3 (ii) when $s \leq h-1$. By Lemma 5.3 (iv) and claim (iv), $\tau_{\mathbb{j}_{h-1}-1} = \pi_{\mathbb{j}_{h-1}-1}$. By claim (v), we arrive at $\mathrm{d}_{G',\tau}(\mathbb{j}_{h-1} - 1) = \mathrm{d}_{G,\pi}(\mathbb{j}_{h-1} - 1) - 1 = 2 - 1 = 1$, showing (vi.a).

We now choose $i \in [\mathbb{j}_{h-1} - 2]$ and turn to (vi.b). By claim (v) and Eq. (5.9), it is sufficient to consider the case that $\tau_i \in N_G(\pi_{\mathbb{j}_s}) \setminus N_G(R)$. By Eq. (5.13), $\{\pi_{\mathbb{j}_{h-1}}, \pi_{\mathbb{j}_h}\} \subseteq N_G(\pi_i)$. Since π is 1-thick, we have $\pi_{i+1} \in N_G(\pi_i)$. In all, we see that $\mathrm{d}_{G,\pi}(i) \geq |\{i+1, \mathbb{j}_{h-1}, \mathbb{j}_h\}| = 3$. By (v), we have $\mathrm{d}_{G',\tau}(i) \geq 3 - 1 > 1$, as was to be shown.

(vii). Suppose G has an HP θ starting at $x \in N_G(R) \subseteq L$. By Lemma 4.7 (vi), there exists $y \in N_G(R)$ such that $G[L] - y$ has an HP starting at x. So, $G[L] - \{x, y\}$ is traceable. Accordingly, Lemma 4.5 combined with Lemma 4.7 (vii.b) gives $r_{\mathcal{I}}(\pi_\iota) = \min\{r_{\mathcal{I}}(\pi_\iota), r_{\mathcal{I}}(\pi_{\iota+1})\} \geq \min\{r_{\mathcal{I}}(x), r_{\mathcal{I}}(y)\}$. This contradicts Lemma 4.7 (ii), finishing the proof of (vii).

(viii). Let θ be an HP of G starting at $x \in R$. By Lemma 4.7 (v), R appears consecutively in θ. Since $x \in R$, this implies $\theta[|R|] = \theta[n - 1 - \iota] = R$ and hence $\theta_{n-1-\iota} \in R$ and $\theta_{n-\iota} \in L$. By Lemma 4.7 (vii.c), $\theta_{n-1-\iota} \in N_G(\theta_{n-\iota}) \cap R \subseteq N_G(\pi_{\iota+1}) \cap R$. Consequently, $\theta_{[|R|]}$ is an HP of $G[R]$ starting from x and ending at $y = \theta_{n-1-\iota} \in N_G(\pi_{\iota+1})$.

(ix). When $p = \iota + 1$, (ii) and (iii) show that $x \in N_G(R)$ and hence (vii) gives the result.

Suppose $p > \iota + 1$ and so $x \in R$. Assume that G has an HP θ which starts at x and ends at $\tilde{\pi}_1$. It follows from Lemma 4.8 (i) that $\tilde{\pi}_1 \in \tilde{\pi}[n - 1 - \iota] \subseteq R$. By Lemma 4.7 (v), R must appear consecutively in θ and hence $V(G) = R$ follows, a contradiction.

(x). It follows from (i) that $\pi_{\mathbb{j}_s} = \tilde{\pi}_{n-\iota}$ and hence $G' = G[L] - \tilde{\pi}_{n-\iota}$. By Eqs. (5.10) and (5.12), we obtain

$$\tilde{\pi}_{n-\iota} = \pi_{\mathbb{j}_s} \in N_G(v) \setminus R = N_G(\tilde{\pi}_{n-1-\iota}) \setminus R \subseteq N_G(R). \tag{5.14}$$

Since $|L| = |V(G')| + 1 \geq 4$, Lemma 4.7 (vii.a) says that $\pi_{[\iota+1]}$ is a 2-thick normal vertex ordering of $G[L]$. Also recall that τ is the normal vertex ordering of G' with respect to $\mathcal{I}_{V(G')}$ and τ is not 2-thick. Henceforth, Theorem 4.10 says that $G[L]$ is Hamiltonian while $G' = G[L] - \tilde{\pi}_{n-\iota}$ is not Hamiltonian. Consequently, we can employ Lemma 4.7 (iv), Eq. (5.14) and $|L| \geq 4$ to conclude that $\tilde{\pi}_{n-\iota}$ belongs to the two rightmost maximal cliques of $G[L]$ with respect to $\mathcal{I}_L$. This gives

$$\begin{aligned} &\max\{\ell_{\mathcal{I}}(\tilde{\pi}_j) : \ j \in [n - \iota + 1, n], \tilde{\pi}_j \in N_G(\tilde{\pi}_{n-\iota})\} \\ &= \max\{\ell_{\mathcal{I}}(\tilde{\pi}_j) : \ j \in [n - \iota + 1, n]\}. \end{aligned} \tag{5.15}$$

Recall from (i) that $L = \tilde{\pi}[n-\iota, n]$. Now, rule 1 of the Normal Ordering Algorithm establishes the first claim of (x):

$$\begin{aligned}\ell_{\mathcal{I}}(\tilde{\pi}_{n-\iota+1}) &= \max\{\ell_{\mathcal{I}}(\tilde{\pi}_j) :\ j \in [n-\iota+1, n], \tilde{\pi}_j \in N_G(\tilde{\pi}_{n-\iota})\} \\ &= \max\{\ell_{\mathcal{I}}(\tilde{\pi}_j) :\ j \in [n-\iota+1, n]\} \qquad \text{(By Eq. (5.15))} \\ &= \max\{\ell_{\mathcal{I}}(z) :\ z \in V(G')\}. \qquad \text{(By claim (i))}\end{aligned} \tag{5.16}$$

The second claim of (x) is immediate from (i) and Eq. (5.16). □

Lemma 5.5. *We keep the same assumption as in Lemma 5.4. Take $x \in V(G') = L \setminus \{\pi_{\mathbb{j}_s}\}$ and let $\varsigma = (\pi_{\mathbb{j}_s}, \tilde{\pi}_{n-1-\iota}, \tilde{\pi}_{n-2-\iota}, \dots, \tilde{\pi}_1)$.*

(i) ς is an HP of $G - V(G') = G[R \cup \{\pi_{\mathbb{j}_s}\}]$.

(ii) If $|V(G')| \le 2$, then G' has an HP ρ starting at x and $\rho + \varsigma$ is an HP of G starting at x and ending at $\tilde{\pi}_1$.

(iii) If τ is 2-thick and ρ is the output of Procedure $\mathrm{HP}(G', \mathcal{I}_{V(G')}, \tau, x)$, then $\rho + \varsigma$ is an HP of G starting at x and ending at $\tilde{\pi}_1$.

(iv) Suppose that G has an HP θ with $\theta_1 = x$. Suppose $|V(G')| > 2$ and τ is not 2-thick. By Lemma 5.4 (iv), we can set $\iota' = \min\{i :\ \mathrm{d}_{G',\tau}(i) = 1\}$, $L' = \tau_{[\iota'+1]}$, $R' = \tau_{[\iota'+2,\iota]}$ and $\widehat{R'} = \tau_{[\iota',\iota]} = R' \cup \{\tau_{\iota'}, \tau_{\iota'+1}\}$.

(a) $\widehat{R'} \subseteq N_G(\pi_{\mathbb{j}_s})$.

(b) If $x \in R'$, then $G'[\widehat{R'}]$ has an HP ϱ^2 connecting $\varrho^2_1 = x$ and $\varrho^2_{\iota-\iota'+1} = \tau_{\iota'}$ and every such HP ϱ^2 must satisfy $\varrho^2_{\iota-\iota'} = \tau_{\iota'+1}$, $G'[L']$ has an HP ρ connecting $\rho_1 = \tau_{\iota'+1}$ and $\rho_{\iota'+1} = \tau_{\iota'}$, and G has an HP $\varrho^3 + \varsigma$ where $\varrho^3 = \varrho^2_{[\iota-\iota'-1]} + \rho$.

(c) If $x \in L'$, then G' has an HP ϱ^1 starting from x and ending at $\tilde{\tau}_1$ and $\varrho^1 + \varsigma$ is an HP of G.

(v) If $1\,\mathrm{HP}(G, x) = 1$, then G has an HP η such that $\eta_1 = x$, $\eta_{\iota+1} = \pi_{\mathbb{j}_s}$, $\eta_n = \mathrm{rv}(\mathcal{I})$, $\eta_{[\iota+1,n]} = \varsigma$ and $\eta[\iota+2, n] = R$.

Proof. **(i).** As π is 1-thick, it is indeed an HP of G. We then infer from Lemma 2.4 that $\tilde{\pi}$ is also an HP of G. By Lemma 5.4 (i), $\tilde{\pi}_{n-\iota} = \pi_{\mathbb{j}_s}$ and $V(G') = \tilde{\pi}[n-\iota+1, n]$. This clearly shows that ς is an HP of $G - V(G')$.

(ii). If $|V(G')| = 1$, then $\iota = 1$, $\pi_1 = x$ and $\mathbb{j}_s = \mathbb{j}_h = 2 = \iota + 1$. So, we can take $\rho = (x)$ and $\rho + \varsigma$ is an HP of G starting at x and ending at $\tilde{\pi}_1$.

If $|V(G')| = 2$, then $\iota = 2$ and $R = \pi[4, n]$. Because π is the normal vertex ordering of G with respect to $\mathcal{I}$, it holds $N_G(\pi_1) \subseteq N_G[\pi_2]$. Since π is 1-thick and $\iota = \min\{i :\ \mathrm{d}_{G,\pi}(i) = 1\}$, we find that $G[L]$ is the complete graph on $\pi[3]$ and $N_G(R) = \{\pi_3\}$, which means that $h = s = 1$ and $\mathbb{j}_s = 3$. Letting $\rho = (x, y)$ where $\{x, y\} = \pi[2]$, we can check that both claims are valid.

(iii). Since the normal vertex ordering τ of G' is 2-thick, Theorem 5.1 shows that ρ is an HP of G'. In view of (i), we only need to show that $\rho_\iota = \rho_{|V(G')|} \in N_G(\varsigma_1)$.

By Theorem 5.1, $r_{\mathcal{I}}(\rho_\iota) \ge \min\{r_{\mathcal{I}}(\tau_{\iota-1}), r_{\mathcal{I}}(\tau_\iota)\}$. By Lemma 4.7 (iv) and Lemma 5.4 (ii), $\pi_{\mathbb{j}_s} = \varsigma_1 \in \mathrm{rc}(\mathcal{I}_L)$. This reduces our task to proving $\{\tau_{\iota-1}, \tau_\iota\} \subseteq N_G(\pi_{\mathbb{j}_s})$.

It follows from Lemma 4.7 (i), Lemma 5.4 (ii) and Lemma 5.4 (iv) that $\pi_\iota = \tau_\iota$. Henceforth, Lemma 5.3 (ii) implies $\tau_\iota \in N_G(\pi_{\mathbb{j}_s})$.

By Lemma 5.4 (iv), either $\tau_{\iota-1} \in \{\pi_{\mathbb{j}_1}, \dots, \pi_{\mathbb{j}_h}\} \setminus \{\pi_{\mathbb{j}_s}\}$ or $\tau_{\iota-1} = \pi_{\iota-1}$. For the former case, Lemma 5.3 (ii) demonstrates $\tau_{\iota-1}\pi_{\mathbb{j}_s} \in E(G)$. So, our final task is to show $\pi_{\iota-1}\pi_{\mathbb{j}_s} \in E(G)$. The case of $\mathbb{j}_s \le \iota$ follows from Lemma 5.3 (ii). Therefore, we now assume that $\mathbb{j}_s = \iota + 1$. According to Lemma 4.7 (iv) and the fact that $\pi_{\mathbb{j}_s} \in N_G(R)$ (Lemma 5.4 (ii)), we need only consider the case that $\pi_{\iota-1} \notin N_G(R)$. Since π is 1-thick, the definition of ι (Eq. (5.9)) shows that $2 \le |N_G(\pi_{\iota-1}) \cap \{\pi_\iota, \dots, \pi_n\}| = |N_G(\pi_{\iota-1}) \cap \{\pi_\iota, \pi_{\iota+1}\}|$ and so $\pi_{\iota-1}\pi_{\mathbb{j}_s} = \pi_{\iota-1}\pi_{\iota+1} \in E(G)$, as desired.

(iv.a). By Lemma 4.7 (iv) and Lemma 5.4 (ii), we only need to derive $\mathfrak{R} \subseteq N_G(\pi_{\mathbb{j}_s})$ where $\mathfrak{R} = \widehat{R'} \backslash \{\pi_{\mathbb{j}_1}, \ldots, \pi_{\mathbb{j}_h}\}$. For any $u \in \mathfrak{R} \subseteq \pi[\iota{+}1] \backslash N_G(v)$, Lemma 4.1 (i) and Eq. (5.10) tell us

$$r_{\mathcal{I}}(u) < \ell_{\mathcal{I}}(v) < r_{\mathcal{I}}(\pi_{\mathbb{j}_s}), \tag{5.17}$$

where $v = \tilde{\pi}_{n-1-\iota} \in \pi[\iota+2, n]$. By Lemma 5.4 (iv), τ is a normal HP of G'. Applying the first claim in Lemma 4.2 (i) for $(G, n, \iota) = (G', \iota, \iota')$, this implies

$$r_{\mathcal{I}}(\tau_{\iota'}) < r_{\mathcal{I}}(u) \tag{5.18}$$

for each $u \in \tau[\iota'+1, \iota]$. Since τ is not 2-thick, it holds $\iota' < \iota - 1$. From Lemma 5.4 (v) and the fact that $\mathrm{d}_{G,\pi}(\iota') > 1 = \mathrm{d}_{G',\pi}(\iota')$, we can derive from $\iota' \in [\iota - 2]$ that

$$\tau_{\iota'} \in N_G(\pi_{\mathbb{j}_s}) \setminus N_G[R]. \tag{5.19}$$

Combining Eqs. (5.17), (5.18) and (5.19), we can deduce $\mathfrak{R} \subseteq N_G(\pi_{\mathbb{j}_s})$, as desired.

(iv.b). By Lemma 4.7 (vi), there exists $(y, z) \in N_G(R) \times R$ such that y and z appear consecutively in θ and $G[L] - y$ has an HP κ with $\kappa_1 = x$. Recall that $x \in L \backslash \{\pi_{\mathbb{j}_s}\}$. By Lemma 4.8 (ii), Eq. (5.10) and Lemma 5.3 (iii) (Eq. (5.1)), replacing the vertex $\pi_{\mathbb{j}_s}$ by y in κ, we can get an HP λ of $G' = G[L] - \pi_{\mathbb{j}_s}$ starting from $x = \theta_1$.

Suppose $x \in R'$. For the normal vertex ordering τ of G' with respect to $\mathcal{I}_{V(G')}$ and the HP λ of G' with $\lambda_1 = x$, Lemma 5.4 (viii) applies to show the existence of an HP of $G'[R']$ starting from x and ending at some vertex

$$y' \in N_{G'}(\tau_{\iota'+1}). \tag{5.20}$$

In an obvious way, we can extend this HP of $G'[R']$ to get the HP $\varrho^2 = (x, \varrho^2_2, \ldots, \varrho^2_{\iota-\iota'-1} = y', \tau_{\iota'+1}, \tau_{\iota'})$ of $G'[\widehat{R'}]$, as required. Note that $\mathrm{d}_{G',\tau}(\iota') = 1$ and so $\tau_{\iota'+1}$ is the only neighbour of $\tau_{\iota'}$ in $G'[\widehat{R'}]$. This confirms that every HP ϱ^2 of $G'[\widehat{R'}]$ connecting x and $\tau_{\iota'}$ must satisfy $\varrho^2_{\iota-\iota'} = \tau_{\iota'+1}$.

By Lemma 4.7 (vii.a), $\tau_{[\iota'+1]}$ is either a 2-thick normal vertex ordering of $G'[L']$ or an HP of $G'[L']$ of length 1. By an application of Lemma 2.1, $G'[L']$ has an HP ρ connecting $\rho_1 = \tau_{\iota'+1}$ and $\rho_{\iota'+1} = \tau_{\iota'}$.

By Eq. (5.19) and Eq. (5.20), we have $\{y'\tau_{\iota'+1}, \tau_{\iota'}\pi_{\mathbb{j}_s}\} \subseteq E(G)$ and so G has an HP $\varrho^3 + \varsigma$ where $\varrho^3 = \varrho^2_{[\iota-\iota'-1]} + \rho$. This completes the proof of (iv.b).

(iv.c). and **(v).** We shall proceed by induction with respect to $|V(G)| = n \geq 3$.

We assume $x \in L'$ and focus our attention on (iv.b). It follows from Lemma 5.4 (i) that $\pi_{\mathbb{j}_s} = \tilde{\pi}_{n-\iota}$ and from the second part of Lemma 5.4 (x) that $\tilde{\tau}_1 = \tilde{\pi}_{n-\iota+1}$. Hence, $\pi_{\mathbb{j}_s}\tilde{\tau}_1 = \tilde{\pi}_{n-\iota}\tilde{\pi}_{n-\iota+1} \in E(G)$. Thus, it suffices to find an HP ϱ^1 of G' leading from x to $\tilde{\tau}_1$. By Lemma 5.4 (x), $\tilde{\tau}_1 = \tilde{\pi}_{n-\iota+1} = \mathrm{rv}(\mathcal{I}_{V(G')})$. Accordingly, by further applying the induction hypothesis on claim (v), the existence of ϱ^1 is guaranteed, as desired.

We now aim to prove (v). If $n = 3$, as π is 1-thick but not 2-thick, it holds $E(G) = \{\pi_1\pi_2, \pi_2\pi_3\}$ and so $\eta = \pi$ is what we want. Suppose $n > 3$. By the fact that $\tilde{\pi}$ is the normal vertex ordering of G with respect to $\overleftarrow{\mathcal{I}}$, we have $\varsigma_{n-\iota} = \tilde{\pi}_1 = \mathrm{rv}(\mathcal{I})$.

Case 1. $|V(G')| \leq 2$.

The result follows from claims (i) and (ii).

Case 2. $|V(G')| > 2$, and τ is 2-thick.

By claims (i) and (iii), we can take $\eta = \rho + \varsigma$.

Case 3. $|V(G')| > 2$, and τ is not 2-thick.

If $x \in R'$, then claim (i) and (iv.a) implies that the required HP can be taken as $\eta = \varrho^3 + \varsigma$ where ϱ^3 starts from x. If $x \in L'$, then claim (i) and our inductive assumption on claim (iv.b) demonstrates that $\eta = \varrho^1 + \varsigma$ is what we are searching for. □

Our immediate object is to introduce an algorithm to solve the 2 HP problem when the input interval graph is not Hamiltonian (c.f. Lemma 4.9) and one of the two given endpoints is at the "boundary" of the interval graph. This will be the key subprogram to be used in our main algorithm for the 1 HP problem. Note that it makes use of the subprogram reported at the beginning of this section.

Procedure 2 HP$(G, \mathcal{I}, x = \pi_p, \tilde{\pi}_1, \iota, \pi, \mathrm{d}_{G,\pi}, \tilde{\pi}, \mathrm{d}_{G,\tilde{\pi}})$;
{ Input: a graph G with $n \geq 3$ vertices together with its interval representation $\mathcal{I}$; Let π and $\tilde{\pi}$ be two normal vertex orderings of G with respect to $\mathcal{I}$ and $\overleftarrow{\mathcal{I}}$, respectively; We assume that π is 1-thick but not 2-thick; Let $\iota = \min\{j \in [n] : \mathrm{d}_{G,\pi}(j) = 1\}$; Take $p \in [\iota]$ and let $x = \pi_p$. Output: an HP of G starting from x and ending at $\tilde{\pi}_1$ or "there is no HP of G starting from x"; }
begin
Let $R = \pi[\iota+2, n]$ and $L = \pi[\iota+1]$. Suppose $N_G(\tilde{\pi}_{n-1-\iota}) \setminus R = \{\pi_{\mathbb{j}_1}, \ldots, \pi_{\mathbb{j}_h}\}$ and $\mathbb{j}_1 < \cdots < \mathbb{j}_h$. Assume that $\ell_{\mathcal{I}}(\pi_{\mathbb{j}_s}) = \max\{\ell_{\mathcal{I}}(\pi_{\mathbb{j}_t}) : t \in [h]\}$.
If $x = \pi_{\mathbb{j}_s}$, then output "there is no HP of G starting from x".
Else, let $G' = G[L] - \pi_{\mathbb{j}_s}$. Let τ be the normal vertex ordering of G' with respect to $\mathcal{I}_{L\setminus\{\pi_{\mathbb{j}_s}\}}$. Suppose $x = \tau_q$. Let $\varsigma = (\pi_{\mathbb{j}_s}, \tilde{\pi}_{n-1-\iota}, \tilde{\pi}_{n-2-\iota}, \ldots, \tilde{\pi}_1)$.
If $|V(G')| \leq 2$, then G' has an HP ρ starting at x, return $\rho + \varsigma$.
Else $|V(G')| > 2$. Compute $\mathrm{d}_{G',\tau}$ from $\mathrm{d}_{G,\pi}$.
If τ is 2-thick, then do Procedure HP$(G', \mathcal{I}_{V(G')}, \tau, x)$ to get an HP ρ of G', return $\rho + \varsigma$.
Else τ is 1-thick but not 2-thick. Let $\tilde{\tau}$ be the normal vertex ordering of G' with respect to $\overleftarrow{\mathcal{I}}_{L\setminus\{\pi_{\mathbb{j}_s}\}}$. Compute $\mathrm{d}_{G',\tilde{\tau}}$ from $\mathrm{d}_{G,\tilde{\pi}}$.
Find $\iota' = \min\{j \in [\iota] : \mathrm{d}_{G',\tau}(j) = 1\}$. Let $L' = \tau[\iota' + 1]$, $R' = \tau[\iota' + 2, \iota]$ and $\widehat{R'} = \tau[\iota', \iota]$.
If $x \in L'$, do 2 HP$(G', \mathcal{I}_{V(G')}, x = \tau_q, \tilde{\tau}_1, \iota', \tau, \mathrm{d}_{G',\tau}, \tilde{\tau}, \mathrm{d}_{G',\tilde{\tau}})$.
If the output of this algorithm is an HP of G', say ϱ^1, return $\varrho^1 + \varsigma$.
Else, output "there is no HP of G starting from x", exit.
Else $x \in R'$. Define two orderings σ and $\tilde{\sigma}$ of $\widehat{R'}$ such that $\sigma = \tau_{[\iota',\iota]}$, $\tilde{\sigma}_i = \tilde{\tau}_i$ for all $i \in [\iota - \iota' - 1]$, and $(\tilde{\sigma}_{\iota-\iota'}, \tilde{\sigma}_{\iota-\iota'+1}) = (\tau_{\iota'+1}, \tau_{\iota'})$.
Let $\iota'' = \min\{j : \mathrm{d}_{G[\widehat{R'}],\tilde{\sigma}}(j) = 1\}$ and $x = \tilde{\sigma}_{q'}$.
Do 2 HP$(G[\widehat{R'}], \overleftarrow{\mathcal{I}_{\widehat{R'}}}, x = \tilde{\sigma}_{q'}, \tau_{\iota'}, \iota'', \tilde{\sigma}, \mathrm{d}_{G[\widehat{R'}],\tilde{\sigma}}, \sigma, \mathrm{d}_{G[\widehat{R'}],\sigma})$.
If the output of this algorithm is not an HP of $G[\widehat{R'}]$, then output "there is no HP of G starting from x", exit.
Else, suppose the output of this algorithm is an HP of $G[\widehat{R'}]$ which starts from x and ends at $\tau_{\iota'}$, say $\varrho^2 = (x, \varrho^2_2, \ldots, \varrho^2_{\iota-\iota'-1}, \tau_{\iota'+1}, \tau_{\iota'})$.
If $|L'| = 2$, then let $\rho = (\tau_{\iota'+1}, \tau_{\iota'})$.
Else if $|L'| > 2$, do Procedure HC$(G[L'], \tau)$ to get an HC of $G[L']$ in which $\tau_{\iota'}$ and $\tau_{\iota'+1}$ are consecutive and we can thus get an HP of $G[L']$, say $\rho = (\tau_{\iota'+1}, \rho_2, \ldots, \rho_{\iota'}, \tau_{\iota'})$.
Let $\varrho^3 = (x, \varrho^2_2, \ldots, \varrho^2_{\iota-\iota'-1}, \tau_{\iota'+1}, \rho_2, \ldots, \rho_{\iota'}, \tau_{\iota'})$. Return $\varrho^3 + \varsigma$.
end;

LEMMA 5.6. *We follow the notation in* $2\,\mathrm{HP}(G, \mathcal{I}, x = \pi_p, \tilde{\pi}_1, \iota, \pi, \mathrm{d}_{G,\pi}, \tilde{\pi}, \mathrm{d}_{G,\tilde{\pi}})$. *If* $|V(G')| > 2$ *and* τ *is not* 2*-thick, then* σ *and* $\tilde{\sigma}$ *are the normal vertex orderings of* $G[\widehat{R'}]$ *with respect to* $\mathcal{I}_{\widehat{R'}}$ *and* $\overleftarrow{\mathcal{I}_{\widehat{R'}}}$*, respectively.*

Proof. Recall from Lemma 5.4 (iv) that τ is an HP of G' and so $\tau_{\iota'}\tau_{\iota'+1} \in E(G)$ and

$$N_G(\tau_{\iota'}) \cap R' = \emptyset. \tag{5.21}$$

Moreover, considering that $|V(G')| > 2$, τ is 1-thick but not 2-thick, we conclude that $R' \neq \emptyset$.

Applying the first claim in Lemma 4.2 (i) for $(G, \pi, \iota) = (G', \tau, \iota')$ now yields $\sigma_1 = \tau_{\iota'} <_{\mathcal{I}} R'$. Since τ is 1-thick, $\sigma_2 = \tau_{\iota'+1} \in N_G(R')$. It then follows $r_{\mathcal{I}}(\tau_{\iota'}) = \min\{r_{\mathcal{I}}(v) : v \in \widehat{R'} = R' \cup \{\tau_{\iota'}, \tau_{\iota'+1}\}\}$. This shows that σ is the normal vertex ordering of $G[\widehat{R'}]$ with respect to $\mathcal{I}_{\widehat{R'}}$.

Let $\tilde{\sigma}'$ be the normal vertex ordering of $G[\widehat{R'}]$ with respect to $\overleftarrow{\mathcal{I}_{\widehat{R'}}}$. We proceed to show $\tilde{\sigma} = \tilde{\sigma}'$. By the definition of σ and ι', $\mathrm{d}_{G[\widehat{R'}],\sigma}(1) = 1$. Note that σ is the normal vertex ordering of $G[\widehat{R'}]$ with respect to $\mathcal{I}_{\widehat{R'}}$. By Lemma 4.8 (i) applied to $(G, \pi, n, \iota, R) = (G[\widehat{R'}], \sigma, \iota - \iota' + 1, 1, R')$, $\tilde{\sigma}'_{[\iota-\iota'-1]}$ is the normal vertex ordering of $G[R']$ with respect to $\overleftarrow{\mathcal{I}_{R'}}$. But, Lemma 4.8 (i) applied to $(G, \pi, n, \iota, R) = (G', \tau, \iota, \iota', R')$ says that $\tilde{\tau}_{[\iota-\iota'-1]}$ is the normal vertex ordering of $G[R']$ with respect to $\overleftarrow{\mathcal{I}_{R'}}$. Henceforth,

$$\tilde{\sigma}'_{[\iota-\iota'-1]} = \tilde{\tau}_{[\iota-\iota'-1]} = \tilde{\sigma}_{[\iota-\iota'-1]}. \tag{5.22}$$

Since τ is an HP of G', we know that σ is an HP of $G[\widehat{R'}]$, and so, by Lemma 2.4, $\tilde{\sigma}'$ is an HP of $G[\widehat{R'}]$. Because of Eq. (5.21), for $\tilde{\sigma}'$ to be an HP of $G[\widehat{R'}]$, it must happen $\tilde{\sigma}'_{\iota-\iota'} = \tau_{\iota'+1} = \tilde{\sigma}_{\iota-\iota'}$ and $\tilde{\sigma}'_{\iota-\iota'+1} = \tau_{\iota'} = \tilde{\sigma}_{\iota-\iota'+1}$. This and Eq. (5.22) imply $\tilde{\sigma} = \tilde{\sigma}'$, as wanted. $\square$

THEOREM 5.7. *The algorithm* $2\,\mathrm{HP}(G, \mathcal{I}, x = \pi_p, \tilde{\pi}_1, \iota, \pi, \mathrm{d}_{G,\pi}, \tilde{\pi}, \mathrm{d}_{G,\tilde{\pi}})$ *is correct.*

Proof. We shall proceed by an induction on $n = |V(G)|$. If $n = 3$, we have $V(G) = \pi[3]$, $E(G) = \{\pi_1\pi_2, \pi_2\pi_3\}$ and $\iota = 1$. The correctness of the algorithm is thus trivial. Suppose $n > 3$ and the algorithm is correct for all smaller n.

If $x = \pi_{\mathbb{j}_s}$, Lemma 5.4 (ii) and (vii) claims $1\,\mathrm{HP}(G, x) = 0$. We assume $x \in V(G')$ in the following.

If $|V(G')| \leq 2$ or τ is 2-thick, then by Lemma 5.5 (ii) and (iii), the algorithm is correct.

Suppose $|V(G')| > 2$ and τ is not 2-thick. Recall from Lemma 5.4 (iv) that τ is an HP of G'.

Assume that $x \in L'$, namely $q \leq \iota' + 1$. By the first part of Lemma 5.5 (iv.c), G' has an HP starting from x and ending at $\tilde{\tau}_1$. So, by virtue of our induction hypothesis, the output of the algorithm $2\,\mathrm{HP}(G', \mathcal{I}_{V(G')}, x = \tau_q, \tilde{\tau}_1, \iota', \tau, \mathrm{d}_{G',\tau}, \tilde{\tau}, \mathrm{d}_{G',\tilde{\tau}})$ is an HP of G' starting from x and ending at $\tilde{\tau}_1$, say ϱ^1. By now, the claim follows from the second part of Lemma 5.5 (iv.c).

We now move to the case of $x \in R'$, namely $q > \iota' + 1$. Assume that θ is an HP of G with $\theta_1 = x$. We want to show that our algorithm can really find a required HP of G starting from x and ending at $\tilde{\pi}_1$. The first part of Lemma 5.5 (iv.b) shows that $G[\widehat{R'}]$ has an HP starting from $x = \tilde{\sigma}_{q'}$ and ending at $\tau_{\iota'} = \sigma_1$. By virtue of Lemma 5.4 (ix) and Lemma 5.6, this allows us to reach $q' \leq \iota''$. Making use of the induction hypothesis and Lemma 5.6, we find that the output of

$$2\,\mathrm{HP}(G[\widehat{R'}], \overleftarrow{\mathcal{I}_{\widehat{R'}}}, x = \tilde{\sigma}_{q'}, \tau_{\iota'}, \iota'', \tilde{\sigma}, \mathrm{d}_{G[\widehat{R'}],\tilde{\sigma}}, \sigma, \mathrm{d}_{G[\widehat{R'}],\sigma})$$

is an HP of $G[\widehat{R'}]$ starting from x and ending at $\tau_{\iota'}$, say

$$\varrho^2 = (x, \varrho^2_2, \ldots, \varrho^2_{\iota-\iota'-1}, \tau_{\iota'+1}, \tau_{\iota'}).$$

By Lemma 4.7 (vii.a) for $(G, \pi, L, \iota) = (G[L], \tau, L', \iota')$, $\tau_{[\iota'+1]}$ is either a 2-thick normal vertex ordering of $G[L']$ or a length one HP of $G[L']$. We assert that ρ is an HP of $G[L']$ with endpoints $\tau_{\iota'+1}$ and $\tau_{\iota'}$. When $|L'| = 2$, the result is trivial. We then assume that $|L'| > 2$ and hence $\tau_{[\iota'+1]}$ is a 2-thick HP of $G[L']$. Lemma 2.1 ensures that the output of Procedure $\mathrm{HC}(G[L'], \tau)$ is an HC of $G[L']$ in which $\tau_{\iota'}, \tau_{\iota'+1}$ are consecutive. The required HP ρ is obtained from this cycle by removing the edge $\tau_{\iota'}\tau_{\iota'+1}$.

Finally, the third part of Lemma 5.5 (iv.b) tells us that $\varrho^3 + \varsigma$ is an HP of G starting from x and ending at $\tilde{\pi}_1$, finishing the proof. □

LEMMA 5.8. *The algorithm Procedure* $2\,\mathrm{HP}(G, \mathcal{I}, x = \pi_p, \tilde{\pi}_1, \iota, \pi, \mathrm{d}_{G,\pi}, \tilde{\pi}, \mathrm{d}_{G,\tilde{\pi}})$ *has a linear time implementation.*

Proof. We only need to consider the case of $x \neq \pi_{\mathbb{j}_s}$.

Finding $N_G(\tilde{\pi}_{n-1-\iota}) \setminus R$ and $\ell_{\mathcal{I}}(\pi_{\mathbb{j}_s})$ takes $O(|N_G(\tilde{\pi}_{n-1-\iota})|) \leq O(|N_G[R]|)$ time. Lemma 5.4 (iv) says that $\tau_i = \pi_i$ for $i \in [\iota] \setminus \{\mathbb{j}_s, \ldots, \mathbb{j}_{h-1}\}$ and $\tau_{\mathbb{j}_s} = \pi_{\mathbb{j}_{s+1}}, \tau_{\mathbb{j}_{s+1}} = \pi_{\mathbb{j}_{s+2}}, \ldots, \tau_{\mathbb{j}_{h-1}} = \pi_{\mathbb{j}_h}$. It follows

$$\begin{aligned}\{\pi_i :\ \tau_i \neq \pi_i\} &= \{\pi_{\mathbb{j}_{s+1}}, \pi_{\mathbb{j}_{s+2}}, \ldots, \pi_{\mathbb{j}_h}\}\\ &\subseteq \pi[\mathbb{j}_s - 1, \iota + 1]\\ &\subseteq N_G[\pi_{\mathbb{j}_s}]. \qquad \text{(By Lemma 5.3 (ii))}\end{aligned} \tag{5.23}$$

So it takes $O(|N_G(\pi_{\mathbb{j}_s})|)$ time to get τ from π. Applying Lemma 5.4 (iv) again, the fact that $x = \pi_p = \tau_q$ together with Eq. (5.23) ensures that we can determine q in $O(|N_G(\pi_{\mathbb{j}_s})|)$ time. Obviously, it costs $O(n - \iota) = O(|R| + 1) = O(|R|)$ time to get ς.

The case of $|V(G')| \leq 2$ is trivial. We now assume $|V(G')| > 2$.

By Lemma 5.4 (v), it takes $O(|N_G(\pi_{\mathbb{j}_s})|)$ time to calculate $\mathrm{d}_{G',\tau}$ from $\mathrm{d}_{G,\pi}$. Lemma 5.4 (v) also says that $\mathrm{d}_{G',\tau}(i) = \mathrm{d}_{G,\pi}(i) > 1$ when $\tau_i \notin N_G(\pi_{\mathbb{j}_s})$. This means that we can decide if τ is 0-, 1- or 2-thick in $O(|N_G(\pi_{\mathbb{j}_s})|)$ time.

According to Lemma 5.2, Procedure $\mathrm{HP}(G', \mathcal{I}_{V(G')}, \tau, x)$ costs $O(|V(G')| + |E(G')|)$ time. Therefore, we can proceed with the assumption that $|V(G')| > 2$ and τ is 1-thick but not 2-thick.

By the last assertion in Lemma 5.4 (x), it takes $O(|R|)$ time to get $\tilde{\tau}$ from $\tilde{\pi}$ and get $\mathrm{d}_{G',\tilde{\tau}}$ from $\mathrm{d}_{G,\tilde{\pi}}$. Thanks to Lemma 5.4 (v), to determine $\iota' = \min\{j :\ \mathrm{d}_{G',\tau}(j) = 1\}$ we only need to check those $\tau_i \in N_G(\pi_{\mathbb{j}_s})$ and hence it costs us $O(|N_G(\pi_{\mathbb{j}_s})|)$ time.

Case 1. $x \in L'$, that means $q \leq \iota' + 1$.

It follows by induction that $O(|V(G')| + |E(G')|)$ time is enough to finish $2\,\mathrm{HP}(G'$, $\mathcal{I}_{V(G')}$, $x = \tau_q$, $\tilde{\tau}_1$, ι', τ, $\mathrm{d}_{G',\tau}$, $\tilde{\tau}$, $\mathrm{d}_{G',\tilde{\tau}})$. So we need $O(|N_G[R]|) + O(|N_G(\pi_{\mathbb{j}_s})|) + O(|V(G')| + |E(G')|) \leq O(|V(G)| + |E(G)|)$ time to complete the whole algorithm.

Case 2. $x \in R'$, or equivalently, $q > \iota' + 1$.

By Lemma 5.5 (iv.a), it takes $O(|N_G(\pi_{\mathbb{j}_s})|)$ time to get σ from τ and $\tilde{\sigma}$ from $\tilde{\tau}$. Since $\sigma = \tau_{[\iota',\iota]}$, we have $\mathrm{d}_{G[\widehat{R'}],\sigma}(i) = \mathrm{d}_{G',\tau}(i + \iota' - 1)$ for all $i \in [\iota - \iota' + 1]$, and so, thanks to Lemma 5.5 (iv.a) again, $\mathrm{d}_{G[\widehat{R'}],\sigma}$ can be determined in $O(|N_G(\pi_{\mathbf{j}_s})|)$ time based on the known information of $\mathrm{d}_{G',\tau}$. Because $\tilde{\sigma}_i = \tilde{\tau}_i$ for all $i \in [\iota - \iota' - 1]$, we only need $O(|R'|) + O(|E_G(N_{G'}(R'), R')|)$ time to compute $\mathrm{d}_{G[\widehat{R'}],\tilde{\sigma}}$ from $\mathrm{d}_{G',\tilde{\tau}}$. By Lemma 5.5 (iv.a), we can obtain q' and ι'' in $O(|N_G(\pi_{\mathbf{j}_s})| + |E_G(N_{G'}(R'), R')|)$ time.

The induction assumption says that running Procedure $2\,\mathrm{HP}(G[\widehat{R'}], \overleftarrow{\mathcal{I}_{\widehat{R'}}}, x = \tilde{\sigma}_{q'}, \tau_{\iota'}, \iota'', \tilde{\sigma}, \mathrm{d}_{G[\widehat{R'}]}, \tilde{\sigma}, \sigma, \mathrm{d}_{G[\widehat{R'}],\sigma})$ takes $O(|\widehat{R'}| + |E(G[\widehat{R'}])|)$ time. We already know that Procedure $\mathrm{HC}(G[L'], \tau)$ can be finished in $O(|L'| + |E(G[L'])|)$ time. So we need $O(|N_G[R]|) + O(|N_G(\pi_{\mathbf{j}_s})|) + O(|E_G(N_{G'}(R'), R')|) + O(|L'| + |E(G[L'])|) + O(|\widehat{R'}| + |E(G[\widehat{R'}])|) = O(|V(G)| + |E(G)|)$ time to complete the whole algorithm. □

We are ready to describe our linear time algorithm for solving the 1 HP problem on interval graphs.

Procedure $1\,\mathrm{HP}(G, \mathcal{I}, x)$;
{ Input: a graph G with n vertices and its interval representation $\mathcal{I}$, a vertex $x \in V(G)$; Output: an HP of G which starts at x or the statement "there is no HP of G which starts at x"; }
begin
  Let π and $\tilde{\pi}$ be two normal vertex orderings of G with respect to $\mathcal{I}$ and $\overleftarrow{\mathcal{I}}$, respectively.
  Suppose $x = \pi_p = \tilde{\pi}_{\tilde{p}}$.
    If $n \le 2$ and G is traceable, Output a Hamiltonian path of G.
    If $n \le 2$ and G is not traceable, Output "there is no HP of G starting from x".
    If π is 2-thick, then do Procedure $\mathrm{HC}(G, \pi)$ to get an HC ρ of G.
      Suppose $\rho = [\rho_1, \rho_2, \ldots, \rho_n]$ and $x = \rho_j$. Return an HP of G which starts at x, say $\varrho = (x = \rho_j, \rho_{j-1}, \ldots, \rho_1, \rho_n, \rho_{n-1}, \ldots, \rho_{j+1})$.
    Else if π is not a path, then return "there is no HP of G starting at x".
    Else, find $\iota = \min\{j \in [n] : \mathrm{d}_{G,\pi}(j) = 1\}$, $\tilde{\iota} = \min\{j \in [n] : \mathrm{d}_{G,\tilde{\pi}}(j) = 1\}$.
      If $p \le \iota$, do $2\,\mathrm{HP}(G, \mathcal{I}, x = \pi_p, \tilde{\pi}_1, \iota, \pi, \mathrm{d}_{G,\pi}, \tilde{\pi}, \mathrm{d}_{G,\tilde{\pi}})$.
      If $p > \iota$ and $\tilde{p} \le \tilde{\iota}$, do $2\,\mathrm{HP}(G, \overleftarrow{\mathcal{I}}, x = \tilde{\pi}_{\tilde{p}}, \pi_1, \tilde{\iota}, \tilde{\pi}, \mathrm{d}_{G,\tilde{\pi}}, \pi, \mathrm{d}_{G,\pi})$.
      Else, $p > \iota$ and $\tilde{p} > \tilde{\iota}$. Output "there is no HP of G which starts at x".
  end;

It may be helpful to go back to Example 2.5 at this moment. The reader is invited to apply the algorithm Procedure $1\,\mathrm{HP}(G, \mathcal{I}, 5)$ and reproduce the sequence P_1 there, which is surely an HP of G.

THEOREM 5.9. *The algorithm Procedure* $1\,\mathrm{HP}(G, \mathcal{I}, x)$ *is correct.*

Proof. The case of $n \le 2$ is trivial and so we assume that $n \ge 3$. If π is 2-thick, then the correctness of Algorithm $1\,\mathrm{HP}(G, \mathcal{I}, x)$ follows from Lemma 2.1; if the normal vertex ordering π is not a path and hence not 1-thick, Lemma 2.4 implies that G is not traceable.

Suppose π is 1-thick but not 2-thick. By Lemma 4.9, G is not Hamiltonian. In view of Lemma 2.4 and Theorem 4.10, we derive that $\tilde{\pi}$ is an HP of G but is not 2-thick. So, we can find $\iota = \min\{j : \mathrm{d}_{G,\pi}(j) = 1\}$ and $\tilde{\iota} = \min\{j : \mathrm{d}_{G,\tilde{\pi}}(j) = 1\}$. We now proceed under the assumption that G has an HP starting from x.

If $p \le \iota$ or $\tilde{p} \le \tilde{\iota}$, Theorem 5.7 establishes the result. Suppose $p > \iota$ and $\tilde{p} > \tilde{\iota}$. Then Lemma 5.4 (ix) yields $2\,\mathrm{HP}(G, x, \tilde{\pi}_1) = 0$ and $2\,\mathrm{HP}(G, x, \pi_1) = 0$. As we know

π and $\tilde{\pi}$ are the normal vertex orderings of G with respect to $\mathcal{I}$ and $\overleftarrow{\mathcal{I}}$, it holds $\pi_1 = \mathrm{lv}(\mathcal{I})$ and $\tilde{\pi}_1 = \mathrm{rv}(\mathcal{I})$. As a consequence of Theorem 4.4, $1\,\mathrm{HP}(G, x) = 0$ follows, finishing the proof. ☐

LEMMA 5.10. *It takes linear time to implement the algorithm Procedure* $1\,\mathrm{HP}(G, \mathcal{I}, x)$.

Proof. If $n \leq 2$, the result is trivial. We thus assume that $n > 2$.

By Lemma 2.3, it costs linear time to get π and $\tilde{\pi}$. For any $i \in [n]$, computing $\mathrm{d}_{G,\pi}(i) = |N_G(\pi_i) \cap \{\pi_{i+1}, \dots, \pi_n\}|$ needs $O(|N_G(\pi_i)|)$ time. Therefore, $d_{G,\pi}$ can be calculated in $O(\sum_{i \in [n]}(|N_G(\pi_i)|)) = O(|V(G)| + |E(G)|)$ time. This means that we can decide if π is 0-, 1- or 2-thick in $O(|V(G)| + |E(G)|)$ time. Now, the result follows from Lemmas 2.2 and 5.8. ☐

6. The $1\,\mathrm{PC}$ algorithm. Let x be a vertex of a graph G. For any set U, let $G*U$ be the graph whose vertex set is the disjoint union of $V(G)$ and U and whose edge set is the union of $E(G)$ and the set of all elements uv with $u \in U$ and $v \in V(G*U) \setminus \{u\}$. The following facts are obvious and so their proofs are safely skipped:

$$\begin{cases} \mathrm{pc}(G, x) = \mathrm{pc}(G * [i], x) + i \text{ holds for every } i \in [\mathrm{pc}(G) - 1]. \\ \mathrm{pc}(G) = \mathrm{pc}(G * [i]) + i \text{ holds for every } i \in [\mathrm{pc}(G) - 1]. \end{cases} \tag{6.1}$$

Let G be an interval graph on n vertices. Let $\mathcal{I}$ be an interval representation of G and π be the normal vertex ordering of G with respect to $\mathcal{I}$. We assume Eq. (1.2) and let the path cover of G corresponding to π be as displayed in Eq. (1.3). Let $U = \{u_1, \dots, u_{r-1}\}$ be a set disjoint from $V(G)$. Let $G^* = G * U$ be the interval graph equipped with the interval representation $\mathcal{I}^*$ that fulfils

$$\mathcal{I}^*(v) = \mathcal{I}(v)$$

for each $v \in V(G)$ and

$$\mathcal{I}^*(u_j) = [\min\{\ell_{\mathcal{I}}(v) : \ v \in V(G)\} - j, \ \max\{r_{\mathcal{I}}(v) : \ v \in V(G)\} + j] \tag{6.2}$$

for each $j \in [r-1]$. Let

$$\pi^* = (\pi_{[\mathbb{s}_1, \mathbb{t}_1]}, u_1, \pi_{[\mathbb{s}_2, \mathbb{t}_2]}, u_2, \dots, \pi_{[\mathbb{s}_{r-1}, \mathbb{t}_{r-1}]}, u_{r-1}, \pi_{[\mathbb{s}_r, \mathbb{t}_r]})$$

be an ordering of $V(G^*)$. It is easy to see that $\pi^* - U = \pi$. We reserve the notation ι^* for $\min\{i \in [n + r - 1] : \ \mathrm{d}_{G^*, \pi^*}(i) = 1\}$. Let $R^* = \pi^*[\iota^* + 2, n + r - 1]$ and $L^* = \pi^*[\iota^* + 1]$. The parameter r in Eq. (1.2) may be 1 or greater than 1. The two cases correspond to the two claims in Lemma 4.2. For the former case, we have Lemma 5.4; for the latter, the subsequent Lemma 6.1 summarizes what we know by looking at G^* and so reducing it to the former case. Especially, one can compare Lemma 5.4 (iv) and Lemma 6.1 (vii).

LEMMA 6.1. *We follow the notation introduced above and assume $r \geq 2$. Let $R = \pi[\mathbb{s}_r, \mathbb{t}_r]$ and $L = V(G) \setminus R = \pi[\mathbb{t}_{r-1}]$. Let $\tilde{\pi}'$ be the normal vertex ordering of $G[R]$ with respect to $\overleftarrow{\mathcal{I}_R}$. Suppose*

$$N_G(\tilde{\pi}'_{n-\mathbb{s}_r+1}) \cap L = N_G(\tilde{\pi}'_{n-\mathbb{s}_r+1}) \setminus R = \{\pi_{\mathbb{j}_1}, \dots, \pi_{\mathbb{j}_h}\} \neq \emptyset, \tag{6.3}$$

where $1 \leq \mathbb{j}_1 < \cdots < \mathbb{j}_h \leq \mathbb{t}_{r-1} - 1 = \mathbb{s}_r - 2$. Let s be the index such that

$$\ell_{\mathcal{I}}(\pi_{\mathbb{j}_s}) = \max\{\ell_{\mathcal{I}}(\pi_{\mathbb{j}_k}) : \ k \in [h]\} \tag{6.4}$$

and let $G' = G[L \setminus \{\pi_{\mathbb{j}_s}\}]$. Let τ be the normal vertex ordering of G' with respect to $\mathcal{I}_{L \setminus \{\pi_{\mathbb{j}_s}\}}$. Let $\varsigma = (\pi_{\mathbb{j}_s}, \tilde{\pi}'_{n-\mathbb{s}_r+1}, \tilde{\pi}'_{n-\mathbb{s}_r}, \dots, \tilde{\pi}'_1)$. Then the following hold.

(i) π^* *is a normal HP of* G^* *with respect to* $\mathcal{I}^*$, π^* *is not* 2*-thick and* $\mathbb{t}_{r-1}+r-2=\iota^*$.
(ii) $\pi[\mathbb{j}_k,\mathbb{t}_{r-1}]\subseteq N_G[\pi_{\mathbb{j}_k}]$ *for each* $k\in[h]$.
(iii) *There exists* $q\in[r-1]$ *such that* $\{\mathbb{j}_s,\dots,\mathbb{j}_h\}\subseteq[\mathbb{s}_q,\mathbb{t}_q-1]$.
(iv) *The sequence* ς *is a path in* G.
(v) $\mathrm{pc}(G)=\mathrm{pc}(G')=r$.
(vi) $\mathrm{pc}(G,\pi_{\mathbb{j}_s})=r+1$.
(vii) *For all* $i\in[\mathbb{t}_{r-1}-1]=[|V(G')|]$, *it occurs*

$$\tau_i=\begin{cases}\pi_i, & \text{if } i\in[\mathbb{j}_h]\setminus\{\mathbb{j}_s,\dots,\mathbb{j}_h\},\\ \pi_{\mathbb{j}_{s'+1}}, & \text{if } i=\mathbb{j}_{s'},s'\in[s,h-1],\\ \pi_{i+1}, & \text{if } i\in[\mathbb{j}_h,\mathbb{t}_{r-1}-1].\end{cases}\tag{6.5}$$

(viii) *Let* q *be the number appeared in claim (iii). Let* $P'_i=\tau_{[\mathbb{s}_i,\mathbb{t}_i]}=\pi_{[\mathbb{s}_i,\mathbb{t}_i]}$ *for* $i\in[q-1]$, $P'_q=\tau_{[\mathbb{s}_q,\mathbb{j}_h-1]}$, $P'_{q+1}=\tau_{[\mathbb{j}_h,\mathbb{t}_q-1]}=\pi_{[\mathbb{j}_h+1,\mathbb{t}_q]}$, $P'_i=\tau_{[\mathbb{s}_{i-1}-1,\mathbb{t}_{i-1}-1]}=\pi_{[\mathbb{s}_{i-1},\mathbb{t}_{i-1}]}$ *for* $i\in[q+2,r]$, *and*

$$P'_r=\begin{cases}\tau_{[\mathbb{s}_{r-1}-1,\mathbb{t}_{r-1}-1]}=\pi_{[\mathbb{s}_{r-1},\mathbb{t}_{r-1}]}, & \text{if } q<r-1,\\ \tau_{[\mathbb{j}_h,\mathbb{t}_{r-1}-1]}=\pi_{[\mathbb{j}_h+1,\mathbb{t}_{r-1}]}, & \text{if } q=r-1.\end{cases}\tag{6.6}$$

Then, $P'_1,\dots,P'_r$ *is the path cover of* G' *corresponding to* τ. *In addition, it holds* $V(P'_r)\subseteq N_G(\pi_{\mathbb{j}_s})$.

Proof. **(i).** By Eqs. (1.2) and (6.2), $\mathrm{d}_{G^*,\pi^*}(i)\geq 2$ for every $i\in[\mathbb{t}_{r-1}+r-3]$, $\mathrm{d}_{G^*,\pi^*}(\mathbb{t}_{r-1}+r-2)=1$, while $\mathrm{d}_{G^*,\pi^*}(i)\geq 1$ for every $i\in[n+r-2]$.

(ii). Follows from Lemma 4.2 (ii).

(iii). Take $q\in[r-1]$ so that $\mathbb{j}_s\in[\mathbb{s}_q,\mathbb{t}_q]$. It is trivial that $\pi_{\mathbb{t}_q}\notin N_G(\tilde{\pi}'_{n-\mathbb{s}_r+1})$ and thus our task is to show that $\mathbb{j}_i\in[\mathbb{s}_q,\mathbb{t}_q]$ for every $i\in[s+1,h]$. If there were some $i\in[s+1,h]$ such that $\mathbb{j}_i\notin[\mathbb{s}_q,\mathbb{t}_q]$, then it follows from $\mathbb{j}_i>\mathbb{j}_s$ that $\mathbb{j}_i>\mathbb{t}_q$. By Eq. (1.2) and Lemma 4.2 (ii), we obtain

$$\mathcal{I}(\pi_{\mathbb{t}_q})<\mathcal{I}(\pi_{\mathbb{j}_i}).\tag{6.7}$$

As we have $\ell_{\mathcal{I}}(\pi_{\mathbb{j}_s})=\max\{\ell_{\mathcal{I}}(\pi_{\mathbb{j}_k}):\ k\in[h]\}$, Eq. (6.7) implies $\mathcal{I}(\pi_{\mathbb{t}_q})<\mathcal{I}(\pi_{\mathbb{j}_s})$. From Lemma 4.1 (i) we now derive $\mathbb{t}_q<\mathbb{j}_s$, contradicting $\mathbb{j}_s\in[\mathbb{s}_q,\mathbb{t}_q]$.

(iv). Since $\pi_{[\mathbb{s}_r,\mathbb{t}_r]}$ is an HP of $G[R]$, we deduce from Lemma 2.4 that so is $\tilde{\pi}'$. The claim now follows from Eq. (6.3).

(v). Since π is a normal vertex ordering of G, Lemma 2.4 combined with Eq. (1.2) yields $\mathrm{pc}(G)=r$.

The path cover of G' corresponding to $\pi_{[\mathbb{t}_{r-1}]}-\pi_{\mathbb{j}_s}$ is of size at most r, demonstrating $\mathrm{pc}(G')\leq r$. To finish the proof, it suffices to verify $\mathrm{pc}(G')>r-1$.

By Eq. (6.3), $\pi_{\mathbb{j}_s}\in N_G(\tilde{\pi}'_{n-\mathbb{s}_r+1})\setminus R\subseteq N_{G^*}(R^*)$. By claim (i), we can apply Lemma 5.4 (vii) for $(G,\pi,\iota,R)=(G^*,\pi^*,\iota^*,R^*)$ and thus obtain

$$\mathrm{pc}(G^*,\pi_{\mathbb{j}_s})\geq 2.\tag{6.8}$$

If $\mathrm{pc}(G')>r-1$ were not true, then G' has a path cover $W_1,\dots,W_{r'}$ where $r'\leq r-1$. Due to claim (iv), we see that $\varsigma+u_1+W_1+\cdots+u_{r'}+W_{r'}+(u_{r'+1}\cdots u_{r-1})$ is an HP of G^* starting from $\pi_{\mathbb{j}_s}$, which is a contradiction with Eq. (6.8).

(vi). By Eq. (6.1), claim (v) and Eq. (6.8), $\mathrm{pc}(G,\pi_{\mathbb{j}_s})\geq r-1+2=r+1$. By claim (v) and Eq. (1.1), $\mathrm{pc}(G,\pi_{\mathbb{j}_s})\leq r+1$.

(vii). Obviously, $L^* = L \cup U$, $R^* = R$, and $\pi^*_{\iota^*+1} = u_{r-1}$. By Eqs. (6.2) and (6.3), we can assume that

$$N_{G^*}(\tilde{\pi}'_{n-\mathbb{s}_r+1}) \cap L^* = \{\pi^*_{\mathbb{j}^*_1}, \dots, \pi^*_{\mathbb{j}^*_{h+r-1}}\} = U \cup \{\pi_{\mathbb{j}_1}, \dots, \pi_{\mathbb{j}_h}\}, \tag{6.9}$$

where $1 \le \mathbb{j}^*_1 < \cdots < \mathbb{j}^*_{h+r-1} = \iota^* + 1$. Let $s^* \in [h+r-1]$ be the index such that

$$\ell_{\mathcal{I}^*}(\pi^*_{\mathbb{j}^*_{s^*}}) = \max\{\ell_{\mathcal{I}^*}(\pi_{\mathbb{j}^*_i}) : \ i \in [h+r-1]\} \tag{6.10}$$

and let

$$G'^* = G^*[L^*] - \pi^*_{\mathbb{j}^*_{s^*}}. \tag{6.11}$$

Note that, according to claim (i), $|V(G'^*)| = \mathbb{t}_{r-1} + r - 2 = \iota^*$. Also observe that $G'^* = G[L \setminus \{\pi^*_{\mathbb{j}^*_{s^*}}\}] * U$ and hence, by Eq. (1.2), G'^* is traceable. Let τ^* be the normal vertex ordering of G'^* with respect to $\mathcal{I}^*{}_{L^* \setminus \{\pi^*_{\mathbb{j}^*_{s^*}}\}}$, which must be an HP of G'^* according to Lemma 2.4. From Eqs. (6.2), (6.3), (6.4), (6.9) and (6.10), we derive

$$\pi^*_{\mathbb{j}^*_{s^*}} = \pi_{\mathbb{j}_s}. \tag{6.12}$$

and so it is clear that

$$G'^* = G' * U. \tag{6.13}$$

By claim (iii) and Eq. (6.12),

$$\pi^*_{\mathbb{j}^*_{s^*}} = \pi_{\mathbb{j}_s}, \pi^*_{\mathbb{j}^*_{s^*+1}} = \pi_{\mathbb{j}_{s+1}}, \dots, \pi^*_{\mathbb{j}^*_{s^*+h-s}} = \pi_{\mathbb{j}_h}. \tag{6.14}$$

By Lemma 5.4 (iv) applied to $(G, \pi, v, \iota) = (G^*, \pi^*, \tilde{\pi}'_{n-\mathbb{s}_r+1}, \iota^*)$, for all $i \in [|V(G'^*)|] = [\iota^*]$ it holds

$$\tau^*_i = \begin{cases} \pi^*_i, & \text{if } i \in [\iota^*] \setminus \{\mathbb{j}^*_{s^*}, \dots, \mathbb{j}^*_{h+r-2}\}, \\ \pi^*_{\mathbb{j}^*_{k+1}}, & \text{if } i = \mathbb{j}^*_k \text{ and } k \in [s^*, h+r-2]. \end{cases} \tag{6.15}$$

By Eqs. (6.2) and (6.13), it holds

$$\tau = \tau^* - U. \tag{6.16}$$

We can now put together Eqs. (6.14), (6.15) and (6.16) to establish claim (vii).

(viii). By Eqs. (1.2), (1.3) and (6.5), all those P'_i for $i \in [r] \setminus \{q\}$ are paths in both G and G' and can be read as subsequences of both π and τ as indicated. By (v), $\mathrm{pc}(G') = r$ and so Lemma 2.4 says that the path cover of G' corresponding to τ has size r. In light of Eq. (1.2), the first part of the claim will follow once we can show $\tau_{\mathbb{j}_h-1}\tau_{\mathbb{j}_h} \notin E(G')$.

Applying Lemma 5.3 (iv) for $(G, \mathcal{I}, \pi) = (G^*, \mathcal{I}^*, \pi^*)$, we see that the set displayed in Eq. (6.9) does not contain consecutive integers. This along with Eq. (6.14) implies

$$\{\mathbb{j}_h - 1, \mathbb{j}_h + 1\} \cap \mathbb{j}[h] = \emptyset. \tag{6.17}$$

It follow from Eqs. (6.5) and (6.17) that $(\tau_{\mathbb{j}_h-1}, \tau_{\mathbb{j}_h}) = (\pi_{\mathbb{j}_h-1}, \pi_{\mathbb{j}_h+1})$. Assume for a contradiction that $\tau_{\mathbb{j}_h-1}\tau_{\mathbb{j}_h} \in E(G')$, namely $\pi_{\mathbb{j}_h-1}$ and $\pi_{\mathbb{j}_h+1}$ are adjacent in $G^*[L^*]$. Note that, after visiting the vertex $\pi_{\mathbb{j}_h-1}$, the sequence π^* visits some vertex π^*_i for

$i \in \mathbb{j}^*[h+r-1]$ instead of the vertex $\pi_{\mathbb{j}_h+1}$. Since π^* is a normal vertex ordering of G^*, it must hold

$$r_{\mathcal{I}^*}(\pi_i^*) < r_{\mathcal{I}^*}(\pi_{\mathbb{j}_h+1}). \tag{6.18}$$

By Lemma 4.7 (iv) for $(G,R) = (G^*, R^*)$, Eq. (6.3) and Eq. (6.17) show that Eq. (6.18) is impossible, thus proving that $\tau_{\mathbb{j}_h-1}\tau_{\mathbb{j}_h} \notin E(G')$.

By claim (ii), $\pi[\mathbb{j}_s, \mathbb{t}_{r-1}] \subseteq N_G(\pi_{\mathbb{j}_s})$. It is always true that $\mathbb{j}_s < \mathbb{j}_h + 1$ while when $q < r-1$ it also holds $\mathbb{j}_s < \mathbb{t}_q < \mathbb{s}_{r-1}$ (claim (iii)). Thus, the second part of (viii) follows from Eq. (6.6). □

Procedure 1 PC($G, \mathcal{I}, x = \pi_p, r, \pi_{[\mathbb{s}_1,\mathbb{t}_1]}, \ldots, \pi_{[\mathbb{s}_r,\mathbb{t}_r]}$);
{ Input: Let G, $\mathcal{I}$, π, r, $P_1 = \pi_{[\mathbb{s}_1,\mathbb{t}_1]}, \ldots, P_r = \pi_{[\mathbb{s}_r,\mathbb{t}_r]}$ be as described before Lemma 6.1; Take $x = \pi_p \in V(G)$; Output: a minimum $\{x\}$-fixed-endpoint path cover of G; }
begin
  If $r = 1$, then we do Procedure 1 HP($G, \mathcal{I}, x$);
    If the output of 1 HP($G, \mathcal{I}, x$) is an HP of G, then return this path and exit; Else, output two paths of G, say $\pi_{[p]}$ and $\pi_{[p+1,n]}$, exit;
  Else $r > 1$. Let $R = \pi[\mathbb{s}_r, \mathbb{t}_r]$.
    If $p \geq \mathbb{s}_r$, then do Procedure 1 HP($G[R], \mathcal{I}_R, x$);
      If the output of 1 HP($G[R], \mathcal{I}_R, x$) is an HP of $G[R]$, say P', then we return $P_1, \ldots, P_{r-1}, P'$, exit;
      Else, output $P_1, \ldots, P_{r-1}, \pi_{[\mathbb{s}_r,p]}, \pi_{[p+1,\mathbb{t}_r=n]}$, exit;
    Else $p < \mathbb{s}_r$. Let $\tilde{\pi}'$ be the normal vertex ordering of $G[R]$ with respect to $\overleftarrow{\mathcal{I}_R}$.
      If $N_G(\tilde{\pi}'_{n-\mathbb{s}_r+1}) \setminus R = \emptyset$. Do Procedure 1 PC($G - R, \mathcal{I}_{V(G)\setminus R}, x = \pi_p$, $r-1$, $P_1, \ldots, P_{r-1}$); Suppose the output of the algorithm is $W_1, \ldots, W_\ell$. Output $W_1, \ldots, W_\ell, P_r$, exit;
      Else $N_G(\tilde{\pi}'_{n-\mathbb{s}_r+1}) \setminus R \neq \emptyset$. Let $N_G(\tilde{\pi}'_{n-\mathbb{s}_r+1}) \setminus R = \{\pi_{\mathbb{j}_1}, \ldots, \pi_{\mathbb{j}_h}\}$ where $h \geq 1$ and $\mathbb{j}_1 < \cdots < \mathbb{j}_h$. Let $\ell_{\mathcal{I}}(\pi_{\mathbb{j}_s}) = \max\{\ell_{\mathcal{I}}(\pi_{\mathbb{j}_k}) :\ k \in [h]\}$ and $G' = G[\pi[\mathbb{s}_r - 1] \setminus \{\pi_{\mathbb{j}_s}\}]$.
        If $p = \mathbb{j}_s \in [\mathbb{s}_q, \mathbb{t}_q]$. Output $P_1, \ldots, P_{q-1}, \pi_{[\mathbb{s}_q,\mathbb{j}_s]}, \pi_{[\mathbb{j}_s+1,\mathbb{t}_q]}, P_{q+1}, \ldots, P_r$.
        Else $p \neq \mathbb{j}_s$. Let q be the index such that $\mathbb{s}_q \leq \mathbb{j}_h \leq \mathbb{t}_q$ and let $\varsigma = (\pi_{\mathbb{j}_s}) + \tilde{\pi}'$. Define an ordering τ of $V(G')$ by setting $\tau_i = \pi_i$ for $i \in [\mathbb{j}_h] \setminus \{\mathbb{j}_s, \ldots, \mathbb{j}_h\}$, $\tau_{\mathbb{j}_{s'}} = \pi_{\mathbb{j}_{s'+1}}$ for every $s' \in [s, h-1]$ and $\tau_i = \pi_{i+1}$ for every $i \in [\mathbb{j}_h, \mathbb{s}_r - 2]$. Suppose $x = \tau_{p'}$. Do Procedure 1 PC($G', \mathcal{I}_{V(G')}$, $x = \tau_{p'}$, $r, P'_1, \ldots, P'_r$) where $P'_1, \ldots, P'_r$ are listed in Lemma 6.1 (viii); Suppose the output of the algorithm is $Q_1, \ldots, Q_\ell$ in this order. Output $Q_1, \ldots, Q_{\ell-1}, Q_\ell + \varsigma$, exit;
end;

THEOREM 6.2. *The algorithm* 1 PC($G, \mathcal{I}, x = \pi_p, r, \pi_{[\mathbb{s}_1,\mathbb{t}_1]}, \ldots, \pi_{[\mathbb{s}_r,\mathbb{t}_r]}$) *is correct. Furthermore, assuming that the output of* 1 PC($G, \mathcal{I}, x = \pi_p$, $r, \pi_{[\mathbb{s}_1,\mathbb{t}_1]}$, $\ldots$, $\pi_{[\mathbb{s}_r,\mathbb{t}_r]}$) *is* $Q_1, \ldots, Q_{\mathrm{pc}(G,x)}$ *in this order, then*

- x *does not appear in* $Q_{\mathrm{pc}(G,x)}$ *unless it is the first vertex of* $Q_{\mathrm{pc}(G,x)}$, *and*
- *the last vertex of* $Q_{\mathrm{pc}(G,x)}$ *belongs to* $\pi[\mathbb{s}_r, \mathbb{t}_r]$.

Proof. By Eq. (1.1) and Lemma 6.1 (v), we have

$$\mathrm{pc}(G,x) \in \{r, r+1\}. \tag{6.19}$$

If $r = 1$, the theorem follows from Theorem 5.9. We assume $r > 1$ and the algorithm is correct for interval graphs with less number of vertices from now on. Let $\{\pi^*_{\mathbb{j}^*_1}, \dots, \pi^*_{\mathbb{j}^*_{h+r-1}}\}$, s^* and G'^* be as specified in Eqs. (6.9), (6.10) and (6.11), respectively.

Case 1. $x \in R$, namely $p \geq \mathbb{s}_r$.

If the output of $1\,\mathrm{HP}(G[R], \mathcal{I}_R, x)$ is an HP of G, say P', then $P_1, \dots, P_{r-1}, P'$ form an $\{x\}$-fixed-endpoint path cover of G of size r. Note that $x = \pi_p \in \pi[\mathbb{s}_r, \mathbb{t}_r]$. Thanks to Eq. (6.19), it must be a required minimum path cover.

If the output of $1\,\mathrm{HP}(G[R], \mathcal{I}_R, x)$ is not an HP of G, then Theorem 5.9 implies $\mathrm{pc}(G[R], x) = 2$. By Lemma 4.7 (v) applied to $(G, \pi, \iota, R) = (G^*, \pi^*, \iota^*, R^*)$, $\mathrm{pc}(G^*, x) = 1$ only if $\mathrm{pc}(G^*[R^*], x) = \mathrm{pc}(G[R], x) = 1$. Henceforth, $\mathrm{pc}(G^*, x) \geq 2$, which together with Eq. (6.1) demonstrating $\mathrm{pc}(G, x) \geq r+1$. This then ensures that

$$P_1, \dots, P_{r-1}, \pi_{[\mathbb{s}_r, p]}, \pi_{[p+1, \mathbb{t}_r = n]}$$

is a required minimum size $\{x\}$-fixed-endpoint path cover of G.

Case 2. $x \in L = \pi[\mathbb{t}_{r-1}] \subseteq \pi[\mathbb{t}_{r-1}] \cup U = L^*$, namely $p < \mathbb{s}_r$.

Case 2.1. $N_G(\tilde{\pi}'_{n-\mathbb{s}_r+1}) \setminus R^* = \emptyset$.

We do Procedure $1\,\mathrm{PC}(G - R, \mathcal{I}_{V(G)\setminus R}, x = \pi_p, r-1, P_1, \dots, P_{r-1})$ to find paths $W_1, \dots, W_\ell$. Our induction assumption shows that $\ell = \mathrm{pc}(G - R, x)$. If we can get

$$\mathrm{pc}(G - R, x) = \mathrm{pc}(G, x) - 1, \tag{6.20}$$

then it is clear that $W_1, \dots, W_\ell, P_r$ is a required minimum size $\{x\}$-fixed-endpoint path cover of G. Our remaining task is to prove Eq. (6.20).

First observe that $N_{G^*}(\tilde{\pi}'_{n-\mathbb{s}_r+1}) \setminus R^* = U$. By Eq. (6.2), we obtain

$$\pi^*_{\mathbb{j}^*_{s^*}} = u_1 \in U. \tag{6.21}$$

It is an immediate consequence of Lemma 2.4 that

$$\mathrm{pc}(G - R) = r - 1 \tag{6.22}$$

and hence Eq. (1.1) leads to

$$\mathrm{pc}(G - R, x) \in \{r-1, r\}. \tag{6.23}$$

By Eq. (6.19), according to the value of $\mathrm{pc}(G, x)$, we treat two cases in turn.

Case 2.1.1. $\mathrm{pc}(G, x) = r + 1$.

On account of Eq. (6.23), to derive Eq. (6.20), we just need to exclude the possibility of $\mathrm{pc}(G - R, x) = r - 1$. But P_r is an HP of $G[R]$ and so $\mathrm{pc}(G - R, x) = r - 1$ will give $\mathrm{pc}(G, x) \leq r$, which is absurd.

Case 2.1.2. $\mathrm{pc}(G, x) = r$.

Eq. (6.1) implies $\mathrm{pc}(G^*, x) = 1$. By Lemma 5.5 (v) applied to

$$(G, \pi, x, \iota, L, R) = (G^*, \pi^*, x, \iota^*, L^*, R^*),$$

we arrive at

$$\mathrm{pc}(G^* - R^* - \pi^*_{\mathbb{j}^*_{s^*}}, x) = 1. \tag{6.24}$$

So, we have

$$\begin{aligned} & \mathrm{pc}(G-R,x) \\ &= \mathrm{pc}(G^*-R^*-\pi^*_{\mathbb{j}^*_{s^*}},x)+r-2 && \text{(By Eqs. (6.21), (6.22) and Eq. (6.1))} \\ &= r-1 && \text{(By Eq. (6.24))} \\ &= \mathrm{pc}(G,x)-1, \end{aligned}$$

which verifies Eq. (6.20), as wanted.

Case 2.2. $N_G(\tilde{\pi}'_{n-\mathbb{s}_r+1}) \setminus R \neq \emptyset$.

Case 2.2.1. $p = \mathbb{j}_s$.

By Lemma 6.1 (vi), $\mathrm{pc}(G,x) = r+1$. So, the path cover consisting of $P_1, \ldots, P_{q-1}, \pi_{[\mathbb{s}_q,\mathbb{j}_s]}, \pi_{[\mathbb{j}_s+1,\mathbb{t}_q]}, P_{q+1}, \ldots, P_r$ is what we want.

Case 2.2.2. $p \neq \mathbb{j}_s$.

By Lemma 6.1 (vii), the ordering τ constructed by the algorithm is the normal vertex ordering of G' with respect to $\mathcal{I}_{V(G')}$. Moreover, as a result of the first part of Lemma 6.1 (viii), the sequence $P'_1, \ldots, P'_r$ is the path cover of G' corresponding to τ. Suppose the output of Procedure $1\,\mathrm{PC}(G', \mathcal{I}_{V(G')}, x = \tau_{p'}, r, P'_1, \ldots, P'_r)$ is $Q_1, \ldots, Q_\ell$ in this order. By the induction hypothesis, $Q_1, \ldots, Q_\ell$ form a path cover of G' of size $\ell = \mathrm{pc}(G', x)$, x is either the first vertex of Q_ℓ or an element from $\cup_{i\in[\ell-1]} V(Q_i)$, and the last vertex of Q_ℓ, say z, belongs to $V(P'_r)$. It follows from the second part of Lemma 6.1 (viii) that $z \in N_G(\pi_{\mathbb{j}_s})$. Consequently, Lemma 6.1 (iv) shows that $Q_\ell + \varsigma$ is a path in G. Since $Q_1, \ldots, Q_\ell + \varsigma$ is an $\{x\}$-fixed-endpoint path cover of G, it follows

$$\mathrm{pc}(G', x) = \ell \geq \mathrm{pc}(G, x). \tag{6.25}$$

Now, to show that $Q_1, \ldots, Q_\ell + \varsigma$ is a required minimum size $\{x\}$-fixed-endpoint path cover of G, it remains to prove $\mathrm{pc}(G', x) = \mathrm{pc}(G, x)$. By virtue of Eq. (6.25), Lemma 6.1 (v) and Eq. (1.1), our goal is to show $\mathrm{pc}(G', x) = r$ under the assumption of $\mathrm{pc}(G, x) = r$. By Lemma 6.1 (v) and Eq. (6.1), $\mathrm{pc}(G, x) - \mathrm{pc}(G^*, x) = \mathrm{pc}(G', x) - \mathrm{pc}({G'}^*, x) = r - 1$. From $\mathrm{pc}(G, x) = r$ we obtain $\mathrm{pc}(G^*, x) = 1$. By Lemma 5.5 (v) applied to $(G, \pi, x, \iota, L, R) = (G^*, \pi^*, x, \iota^*, L^*, R^*)$, $\mathrm{pc}({G'}^*, x) = 1$ follows, hence $\mathrm{pc}(G', x) = r$, as wanted. ☐

Lemma 6.3. *The algorithm Procedure* $1\,\mathrm{PC}(G, \mathcal{I}, x = \pi_p, r, \pi_{[\mathbb{s}_1,\mathbb{t}_1]}, \ldots, \pi_{[\mathbb{s}_r,\mathbb{t}_r]})$ *can be implemented in linear time.*

Proof. By Lemma 5.10, Procedure $1\,\mathrm{HP}(G[R], \mathcal{I}_R, x)$ costs $O(|R| + |E(G[R])|)$ time. Thus, we just need to consider the case of $r > 1$ and $p < \mathbb{s}_r$. By Lemma 2.3, computing $\tilde{\pi}'$ takes $O(|R| + |E(G[R])|)$ time.

If $N_G(\tilde{\pi}_{n-\mathbb{s}_r+1}) \setminus R = \emptyset$, the induction assumption guarantees that Procedure $1\,\mathrm{PC}(G - R, \mathcal{I}_{V(G)\setminus R}, x = \pi_p, r - 1, P_1, \ldots, P_{r-1})$ takes $O(|V(G - R)| + |E(G - R)|)$ time. So the whole algorithm takes $O(|R| + |E(G[R])|) + O(|V(G-R)| + |E(G-R)|) \leq O(|V(G)| + |E(G)|)$ time.

Assume that $N_G(\tilde{\pi}_{n-\mathbb{s}_r+1}) \setminus R \neq \emptyset$. Finding $N_G(\tilde{\pi}_{n-\mathbb{s}_r+1}) \setminus R$ and $\ell_{\mathcal{I}}(\pi_{\mathbb{j}_s})$ requires $O(|N_G(\tilde{\pi}_{n-\mathbb{s}_r+1})|) \leq O(|N_G[R]|)$ time. We only consider the case of $p \neq \mathbb{j}_s$. By Lemma 6.1 (ii), $\pi[\mathbb{j}_s, \mathbb{t}_{r-1}] \subseteq N_G[\pi_{\mathbb{j}_s}]$, which along with Lemma 6.1 (vii) implies τ, q and p' can be computed in $O(|N_G(\tilde{\pi}_{n-\mathbb{s}_r+1})|) \leq O(|N_G[R]|)$ time. By the induction hypothesis, the algorithm $1\,\mathrm{PC}(G', \mathcal{I}_{V(G')}, x = \tau_{p'}, r, P'_1, \ldots, P'_r)$ costs $O(|V(G')| + |E(G')|)$ time. In all, Procedure $1\,\mathrm{PC}(G, \mathcal{I}, x = \pi_p, r, \pi_{[\mathbb{s}_1,\mathbb{t}_1]}, \ldots, \pi_{[\mathbb{s}_r,\mathbb{t}_r]})$ can be implemented in $O(|R| + |E(G[R])| + |E_G(R, N_G(R))|) + O(|V(G')| + |E(G')|) = O(|V(G)| + |E(G)|)$ time. ☐

Finally, we arrive at the main algorithm of our paper.

Procedure $1\,\mathrm{PC}(G, \mathcal{I}, x)$;
{ Input: an interval graph G, an interval representation $\mathcal{I}$ of G and $x \in V(G)$;
Output: a minimum size $\{x\}$-fixed-endpoint path cover of G; }
begin
Let π be the normal vertex ordering of G with respect to $\mathcal{I}$. Suppose $x = \pi_p$ and the path cover of G corresponding to π is $\pi_{[\mathbb{s}_1, \mathbb{t}_1]}, \ldots, \pi_{[\mathbb{s}_r, \mathbb{t}_r]}$ in this order.
Do Procedure $1\,\mathrm{PC}(G, \mathcal{I}, x = \pi_p, r, \pi_{[\mathbb{s}_1, \mathbb{t}_1]}, \pi_{[\mathbb{s}_2, \mathbb{t}_2]}, \ldots, \pi_{[\mathbb{s}_r, \mathbb{t}_r]})$;
end;

THEOREM 6.4. *The algorithm Procedure* $1\,\mathrm{PC}(G, \mathcal{I}, x)$ *is correct and can be implemented in* $O(|V(G)| + |E(G)|)$ *time. Consequently, the* $1\,\mathrm{PC}$ *problem for interval graphs can be solved in linear time.*

Proof. By Lemma 2.3, it costs linear time to obtain the normal vertex ordering π. Hence, the claim on Procedure $1\,\mathrm{PC}(G, \mathcal{I}, x)$ follows from Theorem 6.2 and Lemma 6.3. For an interval graph, it is well-known that an interval representation can be computed in linear time [5, 7, 14]. This completes the proof. □

7. Concluding remarks. We summarize briefly the approach of our algorithm. Theorem 4.4 allows us reduce the $1\,\mathrm{HP}$ problem for interval graphs to a special $2\,\mathrm{HP}$ problem for interval graphs. Namely, given any vertex x, we need only either try to find an HP connecting x to a leftmost vertex or an HP connecting x to a rightmost vertex. The theme of our algorithm is to divide cases according to the values of the forward degrees, 0, 1, or at least 2. Especially, the main recursive step of our algorithm partitions the graph into "left" part and "right" part in conformity with the forward degree information and continue recursively by adopting different left/right directions depending on which part the fixed endpoint lies in. Recall here that the normal vertex orderings with respect to an interval representation and its adjoint give us two closely related left/right directions on the interval graph. The analysis of our algorithm builds on the analysis of the relationship between these two orderings whilst our algorithm has the flavor of a multi-sweep graph search algorithm [7, 14].

We expect that forward degree sequence will play more role on solving various restricted spanning connectivity problems for interval graphs and other graphs [2, 6, 15]. We have extended the arguments in this paper to come up with a polynomial time algorithm for solving the $2\,\mathrm{HP}$ problem on interval graphs. However, the complexity of the general $k\,\mathrm{PC}$ problem on interval graphs seems to remain a challenging open problem.

In recent years, there is a great success of combining LDFS and a generalization of the normal vertex ordering (called "rightmost neighbour algorithm" in [8]) to solve algorithmic problems for cocomparability graphs by adapting relevant good algorithms on interval graphs [8, 13, 16]. It looks to be a promising direction to determine the complexity of the fixed endpoint minimum path cover problem on cocomparability graphs. On the other hand, there have been nice algorithms for finding a longest path in an interval graph and even in a cocomparability graph [10, 11, 16]. If we fix one or two endpoints, the complexity of the longest path problem on interval graphs seems to be still unknown.

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