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Spectra and elementary cycles of the digraphs with unique paths of fixed length

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Abstract

A digraph G, whose adjacency matrix A satisfies $A^k = J_n - I_n$, where J_n is the $n \times n$ matrix of all ones, is called a digraph with unique paths of fixed length k, or simply a UPFL-k digraph. We prove that all the UPFL-k digraphs of the same order are cospectral and have the same number of elementary cycles of length l for each $l \leqslant k$. We also provide some techniques helpful for computing the spectrum and the numbers of short elementary cycles of a UPFL digraph, including the determination of the numbers of reentrant paths of every fixed length in a UPFL digraph. At the end of the paper we point out an interesting relation between the number of elementary cycles of the UPFL digraphs and the number of circular sequences with equal length and period. Our theorems generalize corresponding results of Lam and Van Lint. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

We follow Tutte [5] for most of our graph theory terminology.

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PII: S 0 0 2 4 - 3 7 9 5 (9 9) 0 0 0 2 8 - 2

A path in a digraph G is a sequence $P = (D_1, \ldots, D_n)$ of n darts D_j of G, not necessarily all distinct. It is required that the head of D_j shall be the tail of D_{j+1} whenever $1 \leqslant j < n$. The number n is the length $l(P)$ of P. A path is reentrant if its origin and terminus coincide, and this common vertex is called the base point of P. A path P is head-simple (tail-simple) if no vertex occurs twice as head (tail) of a term of it. It is simple if it is both head-simple and tail-simple. Sometimes, the base point of a simple reentrant path is of no concern, and we use the term of an elementary cycle to indicate the set of darts in it.

In [3], Lam and Van Lint proposed a problem as a generalization of the well known Friendship theorem. The question is to determine, for each given k, digraphs for which every ordered pair of its vertices p, q $(p \neq q)$ has a unique path of length k joining p to q and no vertex has a path of length k to itself. They called such a digraph a digraph with unique paths of fixed length (k). We name it a UPFL-k digraph. The definition of the UPFL-k digraph leads us to consider the matrix equation

$$A^k = J_n - I_n, \tag{1.1}$$

where J_n is the $n \times n$ matrix of all one's, I_n is the identity matrix of order n, and A is an unknown nonnegative integer matrix. We remark that the nonnegative integer solutions A of (1.1) are in fact $(0, 1)$ matrices. Consequently, when speaking of a solution A of (1.1), we shall always mean an $n \times n$ $(0, 1)$ matrix A. There is a simple connection between this equation and the concept of UPFL-k digraph. The adjacency matrix of a UPFL-k digraph with n vertices is a solution to (1.1); conversely, if A is a solution to (1.1), then its digraph is a UPFL-k digraph of order n. In other words, to determine all the nonisomorphic UPFL digraphs amounts to searching for all the nonpermutation-similar solutions of (1.1). Some results relevant to this problem of Lam and Van Lint can be found in [3,4,7–10].

The problem can be tackled in two directions. On the one hand, we can try to construct as many UPFL digraphs as possible. On the other hand, we can try to ascertain more and more characteristic properties a UPFL digraph must possess, and so narrow the range of possible UPFL digraphs. For the second direction, a basic result is that a UPFL-k digraph on $n > 1$ vertices can exist only if k is odd and $n = d^k + 1$ for some nonnegative integer d [3]. Since a one vertex UPFL digraph is trivial, we will only consider the UPFL-k digraphs on n vertices for some odd k and $n = d^k + 1 > 1$ throughout the paper. As for the first direction, we have constructed a class of UPFL digraphs [9], which contains the UPFL digraphs described by Lam and Van Lint [3] as a special subfamily. We note that for any fixed order n, the number of nonisomorphic UPFL-k digraphs in this class is $\varphi(k)$ [8,9], where $\varphi(k)$ is the number of positive integers less than k that are relatively prime to k.

In this paper, we tackle the problem in the second direction. We present two results which generalize the corresponding ones of Lam and Van Lint. Roughly speaking, we claim the uniqueness of two sets of parameters, the eigenvalues and the numbers of short elementary cycles of the UPFL-k digraphs of fixed order n. But for each odd k and each $n = d^k + 1$, a UPFL-k digraph of order n has been found [3]. So these parameters really make sense and can be determined completely. In fact, we will give in Section 4 a sketch of a procedure to compute them. Therefore our theorems provide some characteristic parameters of the UPFL digraphs.

In [3], Lam and Van Lint observed that all the UPFL-3 digraphs of the same order must be co-spectral. We find that much more can be said. Let us state here:

Theorem 1.1. *All UPFL-k digraphs of the same order n must be co-spectral.*

A fish is defined to be an elementary 2-cycle. It is proved in [3] that all the UPFL-3 digraphs of the same order have the same number of fishes. We make the comment that this property plays a fundamental role in the work of Lam and Van Lint in determining all the UPFL-3 digraphs of order 9 [3]. Following a suggestion by Xinmao Wang [6], we use Möbius inversion to deduce Theorem 1.2 from Theorem 1.1.

Theorem 1.2. *For each $l \leqslant k$, the number of elementary cycles of length l is the same for all UPFL-k digraphs of the same order.*

Of course, this strengthens the result of Lam and Van Lint about fishes.

Theorem 1.1 also enables us to get the exact values of the numbers of reentrant paths of every fixed length in UPFL digraphs in a quite simple manner. We will present this result in Section 4 as a step in computing the number of short elementary cycles. Finally, we will establish an interesting result which relates the number of elementary cycles of a UPFL digraph to the number of circular sequences of equal length and period, i.e., the number of primitive circular sequences.

Let us conclude this section by explaining some notations. If p is a prime number and $p^r \mid a$ but $p^{r+1} \nmid a$ then we write $p^r \| a$. Let B and C be two sets of numbers and h a number. We write $B \equiv C \pmod{h}$ to mean $\{x \pmod{h} \mid x \in B\} = \{y \pmod{h} \mid y \in C\}$ and use hB to denote the multiset in which every element of B is repeated h times. The following are some other conventions.

$$B \times C := \{xy \mid x \in B, y \in C\}, \qquad h \times B := \{hx \mid x \in B\},$$

$$-B := (-1) \times B, \qquad B + C := \{x + y \mid x \in B,\ y \in C\},$$

$$M_s := \{x \mid 0 < x < s, \gcd(x, s) = 1\}.$$

2. Proof of Theorem 1.1

We only need to prove that all the solutions A to (1.1) are co-spectral, or the spectrum of A is uniquely determined by the relation (1.1).

Suppose $A^k = J_n - I_n$. It is known that d is an eigenvalue of A [3]. Note in addition that k is odd and the eigenvalues of $J - I$ are $n - 1 = d^k$ with multiplicity 1 and -1 with multiplicity $n - 1$. Hence we can list the spectrum of A as $d, -\xi_1, \ldots, -\xi_{n-1}$, where the ξ_i's are kth roots of unity.

Let the characteristic polynomial of A be

$$f(x) = \det(xI - A) = (x - d)(x + \xi_1)\cdots(x + \xi_{n-1}).$$

Since A is a matrix on the integer ring Z, we have $f(x) \in Z[x]$, the ring of polynomials over Z in the indeterminate x. Then the remainder theorem [2, p. 125] implies $f(x) = (x - d)q(x) + f(d) = (x - d)q(x)$ for some unique $q(x) = x^{n-1} - a_1 x^{n-2} + \cdots + (-1)^{n-1} a_{n-1} \in Z[x]$. This gives

$$(x + \xi_1)\cdots(x + \xi_{n-1}) = q(x) = x^{n-1} - a_1 x^{n-2} + \cdots + (-1)^{n-1} a_{n-1} \in Z[x],$$

and hence $s(x) = (x - \xi_1)\cdots(x - \xi_{n-1}) = x^{n-1} + a_1 x^{n-2} + \cdots + a_{n-1} \in Z[x]$ follows.

Let $s(x) = s_1(x)\cdots s_t(x)$ be the factorization of $s(x)$ as a product of irreducible polynomials in $Z[x]$. Suppose $s_1(x) = (x - \eta_1)\cdots(x - \eta_q)$. Then η_1 is a primitive mth root of 1 for some divisor m of k. (The term divisor always means positive divisor here and in what follows.) Let W_m be the set of all mth primitive roots of 1 and $\lambda_m(x) = \prod_{\eta \in W_m}(x - \eta)$. It is well known that $\lambda_m(x)$, the so called cyclotomic polynomial, is an irreducible polynomial in $Z[x]$ of degree $\varphi(m)$ [2, p. 148, Ex. 1, p. 264–265]. Since η_1 is a common root of $\lambda_m(x)$ and $s_1(x)$, Euclid's algorithm [2, p. 144, Ex. 11] tells us that η_1 is also a root of $\gcd(s_1(x), \lambda_m(x))$. But both $s_1(x)$ and $\lambda_m(x)$ are monic and irreducible. It then follows that $s_1(x) = \lambda_m(x)$ and thus the set $\{\eta_1, \ldots, \eta_q\}$ is exactly the set W_m. We repeat this argument for $s_2(x), \ldots, s_t(x)$. Thus there are nonnegative integers $c_m, m \mid k$, such that the spectrum of A consists of a single d together with c_m copies of $-W_m$, where m ranges over all divisors of k.

Write k as $p_1^{e_1}\cdots p_r^{e_r}$ where $p_1 < \cdots < p_r$ are primes. Then the number of divisors of k is $\tau(k) = (1 + e_1)\cdots(1 + e_r)$. The preceding results indicate that our task now is reduced to investigating the $\tau(k)$ numbers c_m's for all divisors m of k.

Let us look at Eq. (1.1) again.

Given a divisor of k, say s, we can write (1.1) in the form $(A^s)^{k/s} = J - I$. Because A^s is a nonnegative matrix and all the diagonal entries of $J - I$ are zeros, it immediately follows that

$$\operatorname{Tr} A^s = 0. \tag{2.1}$$

This observation gives us $\tau(k)$ relations, one for each divisor of k. Recall that the trace of a matrix is equal to the sum of its eigenvalues. Thus if we let $T_m^s = \sum_{x\in w_m} x^s$, then (2.1) implies that

$$-\sum_{m|k} T_m^s c_m + d^s = 0$$

holds for all divisors s of k. Here the fact that s must be odd has been used. We can display these $\tau(k)$ relations in matrix form as

$$(T_m^s)_{\substack{s|k\\ m|k}} (c_m)_{m|k} = (d^s)_{s|k}. \tag{2.2}$$

We pause to clarify the notations. Denote the three matrices in (2.2) by $T(k), C(k)$ and $D(k)$. For the vector $C(k)$, we define its subscript vector to be $SC(k) = (m)_{m|k}$ where the number m appears in the ith position of $SC(k)$ iff c_m appears in the ith position of $C(k)$. Similarly we can define the superscript vector $SD(k)$ of $D(k)$. For definiteness, we fix the notations here by requiring that the vectors $C(k)$ and $D(k)$ have, respectively, the subscript vector

$$SC(k) = (p_1, p_1^2, \ldots, p_1^{e_1}, 1)^{\mathrm{T}} \otimes \cdots \otimes (p_r, p_r^2, \ldots, p_r^{e_r}, 1)^{\mathrm{T}},$$

and superscript vector

$$SD(k) = (1, p_1, \ldots, p_1^{e_1},)^{\mathrm{T}} \otimes \cdots \otimes (1, p_r, \ldots, p_r^{e_r},)^{\mathrm{T}},$$

where we have used the notation α^{T} for the transpose of a vector α and $\otimes$ for the Kronecker product (also called the tensor product). After giving the explicit definitions of $C(k)$ and $D(k)$, there remains only one choice for the definition of the matrix $T(k) = (T_m^s)$ with rows indexed by s and columns by m such that (2.2) is in conformity with the $\tau(k)$ relations above. In fact, the (i, j) position of $T(k)$ should contain the element T_m^s if the ith position of $SD(k)$ is occupied by s and the jth position of $SC(k)$ is occupied by m.

Let us continue with the proof. At this point, we see that the spectrum of A will only depend on (1.1) provided that $T(k)$ can be proved to be nonsingular. We shall demonstrate that this is indeed the case.

First, let us set up an auxiliary result.

Lemma 2.1. *If m and b are relatively prime integers, then $W_m \times W_b = W_{mb}$.*

Proof. We deduce from the Chinese remainder theorem that

$$b \times M_m + m \times M_b \equiv M_{mb} \pmod{mb}.$$

So for any (mb)th root of 1, say η,

$$\{\eta^{bx} \mid x \in M_m\} \times \{\eta^{mx} \mid x \in M_b\} = \{\eta^x \mid x \in M_{mb}\}. \tag{2.3}$$

In particular, if we take η to be a primitive (mb)th root of 1, then η^b is a primitive mth root of 1 and η^m a primitive bth root of 1, and thus (2.3) becomes $W_m \times W_b = W_{mb}$. □

This lemma implies that $T^s_{mb} = T^s_m T^s_b$ when $\gcd(m, b) = 1$. But for any t prime to m, we can deduce from $t \times M_m \equiv M_m \pmod{m}$ that $T^{qt}_m = T^q_m$. Hence we get

$$T^s_m = T^s_{p_1^{g_1}} \cdots T^s_{p_r^{g_r}} = T^{p_1^{f_1}}_{p_1^{g_1}} \cdots T^{p_r^{f_r}}_{p_r^{g_r}},$$

where $m = \prod_{i=1}^r p_i^{g_i}$ and $s = \prod_{i=1}^r p_i^{f_i}$. This, in turn, implies that

$$T(k) = T(p_1^{e_1}) \otimes \cdots \otimes T(p_r^{e_r}). \tag{2.4}$$

To proceed further, notice that it is a simple property of the Kronecker product that

$$(B \otimes C)(D \otimes E) = (BD) \otimes (CE).$$

So $T(k)$ is nonsingular if and only if $T(p_i^{e_i})$ is nonsingular for $i = 1, \ldots, r$.

Now the proof of Theorem 1.1 can be completed by using the following lemma.

Lemma 2.2. *Let p be a prime and $e \geqslant 0$. Then $T(p^e)$ is nonsingular.*

Proof. We start by collecting here some identities to be used later. The first one is

$$T^{p^j}_{p^i} = p^j T^1_{p^{i-j}} \quad \text{for } i \geqslant j. \tag{2.5}$$

The reason is that $p^j \times M_{p^i} \equiv p^j M_{p^{i-j}} \pmod{p^{i-j}}$ for $i \geqslant j$.

The second is very simple:

$$T^{p^j}_{p^i} = \varphi(p^i) \quad \text{for } i \leqslant j. \tag{2.6}$$

Here are some others. Let x_i be a primitive (p^i)th root of 1 for $i = 1, 2, \ldots, e$. Then we have

$$T^1_1 + T^1_p = 1 + x_1 + \cdots + x_1^{p-1} = \frac{1 - x_1^p}{1 - x_1} = 0, \tag{2.7.1}$$

$$T^1_1 + T^1_p + T^1_{p^2} = 1 + x_2 + \cdots + x_2^{p^2-1} = \frac{1 - x_2^{p^2}}{1 - x_2} = 0, \tag{2.7.2}$$

$$\vdots$$

$$T^1_1 + T^1_p + \cdots + T^1_{p^e} = 1 + x_e + \cdots + x_e^{p^e-1} = \frac{1 - x_e^{p^e}}{1 - x_e} = 0. \tag{2.7.e}$$

Clearly $T_1^1 = 1$. So (2.7.1) gives

$$T_p^1 = -1. \tag{2.8}$$

By subtracting (2.7.i) from (2.7.$i+1$) in turn for $i = 1, \ldots, e-1$, we get

$$T_{p^2}^1 = T_{p^3}^1 = \cdots = T_{p^e}^1 = 0. \tag{2.9}$$

Finally, combining (2.5) with (2.8) yields

$$T_{p^{j+1}}^{p^j} = -p^j, \tag{2.10}$$

while by (2.5) together with (2.9) we obtain

$$T_{p^{j+a}}^{p^j} = 0 \quad \text{for } a \geqslant 2. \tag{2.11}$$

We are now ready to evaluate the determinant of $T(p^e)$. Employing (2.6), (2.10) and (2.11), we know that $T(p^e)$ is of the form

$$\begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 1 \\ p-1 & -p & 0 & \cdots & 0 & 1 \\ p-1 & p(p-1) & -p^2 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p-1 & p(p-1) & p^2(p-1) & \cdots & -p^{e-1} & 1 \\ p-1 & p(p-1) & p^2(p-1) & \cdots & p^{(e-1)}(p-1) & 1 \end{pmatrix}, \tag{2.12}$$

where the ith row corresponds to p^{i-1}, while the jth column corresponds to p^j when $j \leqslant e$ and the $(e+1)$th column the number 1.

Observe that $p^j = \sum_{i=0}^{j-1} p^i(p-1) + 1$. This implies that all but one row sum of $T(p^e)$ equals zero, the only exception being the last row, which sums to p^e. Hence if we add the columns $1, 2, \ldots, e$ to the last column, then $T(p^e)$ becomes

$$T = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ p-1 & -p & 0 & \cdots & 0 & 0 \\ p-1 & p(p-1) & -p^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p-1 & p(p-1) & p^2(p-1) & \cdots & -p^{e-1} & 0 \\ p-1 & p(p-1) & p^2(p-1) & \cdots & p^{e-1}(p-1) & P^e \end{pmatrix}. \tag{2.13}$$

So $\det T(p^e) = \det\ T = (-1)^e p^{e(e+1)/2} \neq 0$, and thus the nonsingularity of $T(p^e)$ is established. □

3. Proof of Theorem 1.2

Let $l \leqslant k$ and G be a UPFL-k digraph of order n. A parameter of G is called characteristic if its value is unchanged when G varies over all UPFL-k digraphs of order n. Our aim then is to show that the number of elementary cycles of length l in G, which shall be denoted by E_l, is characteristic.

Let A be the adjacency matrix of G. Theorem 1.1 asserts that the spectrum of A is a set of characteristic parameters of G. Consequently, we have $\mathrm{Tr}\,A^s$ is characteristic for each s. We remark that $\mathrm{Tr}\,A^s$ is just equal to the number of reentrant paths of length s in G.

For two paths P and Q such that the terminus of P coincides with the origin of Q, there is a path PQ formed by writing down the terms of P, in their order in P, and continuing with the terms of Q, in their order in Q. We say that PQ is the product of P and Q in that order.

The following is a special property of the UPFL-k digraphs.

Lemma 3.1. *Assume R_1 and R_2 are reentrant paths of G with the same base point v. If $l(R_1) + l(R_2) \leqslant k$ and $l(R_2) \geqslant l(R_1)$, then there is a reentrant path R_3 with base point v such that $R_2 = R_1R_3$.*

Proof. The case $l(R_2) = 0$ is trivial and thus we may let $l(R_2) > 0$. Pick a path P of length $k - l(R_1) - l(R_2)$ with origin v and terminus some vertex w. We can achieve this, for example, just by traversing along the darts in R_2. Since the number of paths joining every ordered pair of vertices in G with length k is either zero or one, we know that the two paths R_1R_2P and R_2R_1P from v to w must be the same. Hence, comparing the first $l(R_1)$ darts in R_1R_2P and R_2R_1P gives the result. □

For any elementary cycle R of length l and any number t, one can derive l distinct reentrant paths of length lt by choosing one of the l vertices on R as the base point and then moving along the darts of R t times. We assert that if $m \leqslant k$, then all the reentrant paths of length m can be thus generated. If this is not the case, then there is a reentrant path $R' = (D_1, \ldots, D_m)$ with length m not greater than k such that it has two darts D_i and D_j with a common tail v but different heads. Let $R_1 = (D_i, D_{i+1}, \ldots, D_{j-1})$ and $R_2 = (D_j, D_{j+1}, \ldots, D_{i-1})$, in which the subscripts are computed modulo m. Then they are both reentrant paths with base point v. Further, we have $l(R_1) + l(R_2) = m \leqslant k$. So it follows from Lemma 3.1 that D_i and D_j must have the same head, which is a contradiction.

In view of the above analysis, we have $\mathrm{Tr}\,A^s = \sum_{l|s} lE_l$ for $s \leqslant k$. Now the classical Möbius inversion formula [2, p. 145, Ex. 18] says that

$$E_l = \frac{1}{l} \sum_{s|l} \mu\left(\frac{l}{s}\right) \mathrm{Tr}\,A^s, \tag{3.1}$$

holds for $l \leqslant k$. Accordingly, E_l is characteristic for each $l \leqslant k$. This finishes the proof of Theorem 1.2. □

4. Computations

For feasible k and n, once the values of such characteristic parameters as the eigenvalues and the numbers of elementary cycles of every fixed length l not greater than k are known, then only the digraphs with these parameter values need be examined when searching for UPFL digraphs. Sometimes this may help a lot as [3] has shown. This section will describe some techniques for computing these parameters.

In the former two sections, we have acquired some knowledge about these parameters of a UPFL-k digraph of order n on condition that there really exists one such digraph. However, the existence question has been settled affirmatively [3]. Therefore we certainly can apply those results hereafter.

We shall first carry out the calculation of the number of reentrant paths of some fixed length in any UPFL digraph. Denote the number of reentrant paths of length l in a digraph G by $N_l(G)$. For a digraph G with adjacency matrix A, let G^l be the digraph with adjacency matrix A^l. It is not difficult to see that $N_l(G) = N_1(G^l)$. Let G be a UPFL-k digraph of order n. According to our analysis at the beginning of Section 3, we can conclude from Theorem 1.1 that $N_l(G)$ is characteristic. Hence we only need to calculate these parameters for a particular UPFL- k digraph of order n, and they are surely the common parameters for all UPFL-k digraphs of order n. We do this by choosing G to be the digraph with the adjacency matrix A of order $n = d^k + 1$ that was given in [3, Section 4]. That is, A is the matrix that has $(0, 1, 1, \ldots, 1, 0, 0, \ldots, 0)$ as its first row where there are d consecutive ones after the initial 0 and then followed by $(n - 1 - d)$ 0's and every successive row is obtained by shifting the row before it by d positions to the left. From [7, Section 3], we have:

(i) If l is odd, then G^l is the digraph on Z_n with darts D_{ij} directed from $i \pmod n$ to $(-d^l i + j) \pmod n$ for $i = 1, \ldots, n$ and $j = 1, \ldots, d^l$.

(ii) If l is even, then G^l is the digraph on Z_n with darts D_{ij} directed from $i \pmod n$ to $(d^l i - j) \pmod n$ for $i = 1, \ldots, n$ and $j = 0, \ldots, d^l - 1$.

Let us denote $1 - (-d)^{\gcd(k,l)}$ by g. Note that

$$g = \gcd(1 - (-d)^k, 1 - (-d)^l). \tag{4.1}$$

Now we shall go to see what is the value of $N_l(G)$ in case of odd l. Note that (i) asserts

$$\begin{aligned} N_l(G) &= N_1(G^l) \\ &= \mathrm{Card}\{(i,j) \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant d^l,\ i \equiv -d^l i + j \pmod n\} \\ &= \mathrm{Card}\{(i,j) \mid 1 \leqslant i \leqslant 1 - (-d)^k,\ 1 \leqslant j \leqslant d^l, \\ &\qquad (1 - (-d)^l)i \equiv j \pmod{(1 - (-d)^k)}\}. \end{aligned}$$

By noting (4.1), we have $N_l(G)/g$ is just the number of multiples of g in the set $\{1, 2, \ldots, d^l\}$.However, $d^l + 1 = 1 - (-d)^l$ is divisible by g. It then follows

$$N_l(G) = g\left(\frac{1-(-d)^l}{g} - 1\right) = d^l - d^{\gcd(k,l)}. \tag{4.2}$$

In the case that l is even, we deduce from (ii) in a similar way as above that

$$N_l(G) = \mathrm{Card}\Big\{(i,j) \mid 1 \leqslant i \leqslant 1-(-d)^k,\ 0 \leqslant j \leqslant d^l - 1,$$
$$(1-(-d)^l)i \equiv -j \pmod{(1-(-d)^k)}\Big\}.$$

Using (4.1) again, we find at this time that $N_l(G)/g$ equals to the number of multiples of g in the set $\{0, -1, \ldots, 1-d^l\}$. But 0 and $1 - d^l = 1 - (-d)^l$ are both multiples of g. So

$$N_l(G) = g\left(\frac{d^l - 1}{g} + 1\right) = d^l + d^{\gcd(k,l)}. \tag{4.3}$$

Combining (4.2) and (4.3), we infer that it holds for all l and all UPFL-k digraphs of order $n = d^k + 1$ that

$$N_l(G) = d^l + (-1)^l d^{\gcd(k,l)}. \tag{4.4}$$

Substituting (4.4) into (3.1), we obtain an expression for $E_l, l \leqslant k$, for any UPFL-k digraph of order n, namely

$$E_l = h_l + g_l, \tag{4.5}$$

where

$$h_l = \frac{1}{l}\sum_{s|l}\mu\left(\frac{l}{s}\right)d^s \quad \text{and} \quad g_l = \frac{1}{l}\sum_{s|l}\mu\left(\frac{l}{s}\right)(-1)^s d^{\gcd(k,s)}.$$

Let us examine the E_l's in more detail. We always use l to mean a number not greater than k in what follows.

(1) If $l \mid k$, then since k is odd, one can see that $g_l = -h_l$ and thus it holds $E_l = 0$. Of course this also follows from (2.1).

(2) If l is odd and l does not divide k. Let $l = \gcd(k,l)q_1^{f_1} \cdots q_t^{f_t}$, where the q_i's are distinct odd primes and $f_i > 0$ for $i = 1, \ldots, t$. The set $\{s \mid s \mid l\}$ can be divided into three parts. The first part consists of those s with $q_1^2 \mid l/s$. Clearly the numbers in it contribute nothing to the summation in the expression of g_l. The second part includes those s with $q_1 \| l/s$, and the third includes those s for which q_1 does not divide l/s. We can establish a bijection between the numbers in the second part and those in the third part. In fact, we just let a number x in the second part pair with the number $q_1 x$ in the third part. One can check for the above mentioned x and $q_1 x$ that

$$\gcd(k,x) = \gcd(k,q_1,x), \quad \mu\left(\frac{l}{x}\right) = -\mu\left(\frac{\ell}{q_1 x}\right), \quad (-1)^x = (-1)^{q_1 x}. \tag{4.6}$$

Hence the contributions of x and $q_1 x$ in g_l cancel each other. Therefore we find that

$$E_l = h_l. \tag{4.7}$$

(3) Suppose $4 \mid l$. It is not difficult to see that we can replace q_1 by 2 in the above argument to yield (4.7) again.

(4) Finally, we consider the case $2\|l$. Let $l = 2j$. We use the same technique as in case 3. The first two identities of (4.6) remain valid, but the third becomes

$$(-1)^x = -(-1)^{2x}.$$

So $2x$ and x have the same contribution to g_l. It then follows that

$$g_l = 2\left(\frac{1}{2j}\right)\left(\sum_{s|j}\mu\left(\frac{2j}{s}\right)(-1)^s d^{\gcd(k,s)}\right) = -g_j.$$

Since j is odd, we can determine g_j from the preceding results. Now we shall distinguish two subcases.

(4.i) If j does not divide k, then (2) implies that $g_j = 0$ and hence $E_l = h_l$ follows.

(4.ii) If $j \mid k$, then (1) implies $g_j = -h_j$. So we have

$$E_l = h_l + g_l = h_l - g_j = h_l + h_{l/2}.$$

To sum up, for $l \leqslant k$, we get

$$E_l = \begin{cases} 0 & \text{if } l \mid k, \\ h_l + h_{l/2} & \text{if } \frac{l}{2} \mid k \text{ and } 2\|l, \\ h_l & \text{otherwise.} \end{cases}$$

We remark that h_l is just the number of circular sequences on d letters of length and period l [1, p. 12]. We also note that all the UPFL digraphs known to us belong to the class of consecutive-d digraphs, which is introduced by D-Z Du et al. and contains many important network models defined on letters [8,9]. We think that there is something behind this fact and the investigation of it may be helpful to a better understanding of the structure of the UPFL digraphs.

Let us deal with the spectra next. From the work in Section 2, we see that it suffices to obtain the vector $C(k)$ in order to determine these parameters. Furthermore one can pay attention to $T(k)^{-1}$ instead because (2.2) says $C(k) = T(k)^{-1}D(k)$. But we have still (2.4). So we only need to get $T(p^e)^{-1}$ for a prime p and $e > 0$, which will turn out to have a simple expression. Note that our discussion in Section 2 about the relationship between $T(p^e)$ and the matrix T defined there implies that

$$T(p^e)\begin{pmatrix} 1 & & & 1 \\ & 1 & & 1 \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}_{(e+1)\times(e+1)} = T. \tag{4.8}$$

Let

$$H = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{pmatrix}_{(e+1)\times(e+1)}.$$

Then it follows from (2.13) that

$$T(p^e)\begin{pmatrix} 1 & & & & 1 \\ & 1 & & & 1 \\ & & \ddots & & \vdots \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}\begin{pmatrix} -1 & & & & \\ & -p & & & \\ & & \ddots & & \\ & & & -p^{e-1} & \\ & & & & p^e \end{pmatrix}^{-1}$$
$$= I - (p-1)W, \tag{4.9}$$

where

$$W = H + H^2 + \cdots + H^e. \tag{4.10}$$

Since $H^{e+1} = 0$, we also obtain $W^{e+1} = 0$. So

$$(I - (p-1)W)^{-1} = I + (p-1)W + \cdots + (p-1)^e W^e. \tag{4.11}$$

Substituting (4.10) into (4.11), we obtain

$$(I - (p-1)W)^{-1} = I + f_1 H + \cdots + f_e H^e,$$

where f_i is the coefficient of x^i in the polynomial

$$1 + (x + x^2 + \cdots + x^e)(p-1) + \cdots + (x + x^2 + \cdots + x^e)^e (p-1)^e.$$

Clearly

$$f_i = \sum_{j=1}^{i} (p-1)^j \binom{i-1}{j-1} \quad \text{for } e \geqslant i \geqslant 1$$

for the coefficient of x^i in $(x + x^2 \cdots + x^e)^j$ is the number of ordered partitions of i into j parts, which instead is the number of ways of putting $j-1$ markers in the $i-1$ spaces between i balls in a line, and hence is

$$\binom{i-1}{j-1}.$$

It in turn shows

$$f_i = (p-1)\sum_{j=0}^{i-1}(p-1)^j\binom{i-1}{j} = (p-1)p^{i-1} = \varphi(p^i).$$

Thus $(I-(p-1)W)^{-1} = I + \sum_{i=1}^{e}\varphi(p^i)H^i$. Finally (4.9) gives the formula

$$T(p^e)^{-1} = \begin{pmatrix} 1 & & & & 1 \\ & 1 & & & 1 \\ & & \ddots & & \vdots \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}\begin{pmatrix} -1 & & & & \\ & -p & & & \\ & & \ddots & & \\ & & & -p^{e-1} & \\ & & & & p^e \end{pmatrix}^{-1} \times \left(I + \sum_{i=1}^{e}\varphi(p^i)H^i\right).$$

We do not know if our procedure above will be of any help in the investigation of UPFL-k digraphs and if there is any method to simplify the remaining computations in the general setting. We also remark that there is another way to compute the spectrum of a UPFL- k digraph. One should only notice that for each feasible k and n, there exists a UPFL-k digraph with a $(-d)$-circulant as its adjacency matrix [3]; while a description of the eigenvalues of a g-circulant has been given in [11]. But it seems that we cannot learn any more from this method either. Perhaps the distribution of the eigenvalues of UPFL digraphs will behave rather regularly and some structural properties of the UPFL digraphs may be detected from it. It may be of value to pursue a more satisfactory description of the distribution of the spectra of UPFL digraphs.

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