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LINEAR ALGEBRA AND ITS APPLICATIONS

On the matrix equation $A^l + A^{l+k} = J_n$

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Abstract

This paper studies the matrix equation $A^l + A^{l+k} = J_n$, where l, k are nonnegative integers, J_n is the $n \times n$ matrix of all 1's, and A is an unknown (0,1)-matrix. We shall provide a solution for every odd k and every n which is feasible, i.e. $n = d^l + d^{l+k}$ for some nonnegative integer d, and show that the equation has no solution in other cases with some trivial cases excluded. We also show that for any solution A to this equation there must be a (0,1)-matrix C satisfying $I + C^k = J_{d^k+1}$, and $\Gamma(A)$, the associated digraph of A, is the lth iterated line digraph of $\Gamma(C)$. In particular, the well-known Kautz digraph $K(d, l+1)$ can be characterized as $\Gamma(A)$, where A satisfies $A^l + A^{l+1} = J_n$ for $n = d^l + d^{l+1}$. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

This paper studies the matrix equation

$$A^l + A^{l+k} = J_n, \tag{1.1}$$

where l, k are nonnegative integers, J_n is the $n \times n$ matrix of all 1's, and A is an unknown $n \times n$ (0,1)-matrix.

Let $[m]$ denote the set $\{0, 1, \ldots, m\}$ for any nonnegative integer $m, e_i^{(m)}$ the $1 \times m$ row vector with 1 in the $(i+1)$th position and zeros elsewhere, and

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I_m the $m \times m$ identity matrix. We often freely omit some indices if no confusion should be made. We always let α^{T} stand for the transpose of a row vector α.

Suppose A is a solution to (1.1). It is obvious that $AJ = JA$, and $e_iAJe_j^{\mathrm{T}} = e_iAJe_s^{\mathrm{T}} = e_iJAe_s^{\mathrm{T}} = e_rJAe_s^{\mathrm{T}}$ for any $i, j, r, s \in [n-1]$. Hence $AJ = JA = dJ$ for some nonnegative integer d. It follows that $nJ_n = J_nJ_n = J_n(A^l + A^{l+k}) = (d^l + d^{l+k})J_n$ (we adopt the convention that $0^\circ = 1$). Therefore $n = d^l + d^{l+k}$, which is a necessary condition for the solvability of (1.1). One can easily verify that the case $d = 0$ happens if and only if $n = 1, A = (0)_{1\times 1}$, $l = 0$ and $k \geqslant 1$. In the rest of this paper, we always exclude this trivial case, and assume that A is a (0,1)-matrix of order $n = d^l + d^{l+k} > 1$, $d > 0$, and $AJ = JA = dJ$.

In [1], Bosàk considered the (0,1)-matrix equation

$$B^a + B^{a+1} + \cdots + B^b = J_m,$$

where $a \leqslant b$ are nonnegative integers. He proved that this equation has a solution if and only if $b = a$ or $b = a + 1$ and m is of some special forms correspondingly. A simple proof of this fact is contained in [2]. Note that Bosàk's equation coincides with (1.1) when $b = a + 1$ and $k = 1$. In this case, Bosàk pointed out that the adjacency matrix of the Kautz digraph $K(d, l+1)$ is a solution to (1.1). The Kautz digraph $K(d, l+1)$ is defined to be $L^l(K_{d+1})$, the lth iterated line digraph of the complete digraph K_{d+1}. By the equivalent definition of the Kautz digraph from an alphabet [3], the reader should easily verify that the adjacency matrix of the Kautz digraph is a solution to (1.1) when $k = 1$. We shall use the notation $\Gamma(B)$ for the associated digraph of a nonnegative integer matrix B. Hoffman indicated that if $A + A^2 = J_n$, then $\Gamma(A) = K(d, 2)$ [4]. As a consequence of our main result, we show that if $A^l + A^{l+1} = J$ then $\Gamma(A) = K(d, l+1)$, which is a generalization of Hoffman's assertion.

The case $l = 0$ of (1.1) was studied by Lam and Van Lint in [5]. They showed that the equation $A^k = J - I$ has no solution for any even k and provided a solutions for every odd k and every feasible n (see also [6]). We shall show that the same thing holds for all $l \geqslant 0$ based on similar techniques.

Our main result of this paper will reveal that the case $l = 0$ is essential to the investigation of (1.1). In fact, we show that for any solution A of (1.1) there must be a (0,1)-matrix C satisfying $I + C^k = J_{d^k+1}$ with $\Gamma(A) = L^l(\Gamma(C))$, the lth iterated line digraph of $\Gamma(C)$. As its consequence, a new characterization of the Kautz digraph is given.

2. The case of even k

Theorem 2.1. *Eq. (1.1) has no solution when k is even.*

Proof. Let $k = 2h$. Assume that A is a solution to (1.1). Then A has eigenvalue d and satisfies

$$A^l(I + A^{2h})(A - dI) = 0.$$

So the multi-set of nonzero eigenvalues of the real matrix A^h consists of d^h with positive multiplicity and $\pm\sqrt{-1}$ with equal may be zero multiplicities (note that we use the assumption $d > 0$ here), and $\operatorname{Tr} A^h > 0$ follows. Since A^h is a nonnegative integer matrix, we have

$$\operatorname{Tr} A^{2h} = \operatorname{Tr}(A^h)^2 \geqslant \operatorname{Tr} A^h > 0.$$

Suppose $e_i A^{2h} e_i^{\mathrm{T}} > 0$. Then

$$e_i(A^{2h} + I)e_i^{\mathrm{T}} \geqslant 2$$

and so

$$\sum_{j=0}^{n-1} e_j^{\mathrm{T}} = Je_i^{\mathrm{T}} = A^l(I + A^{2h})e_i^{\mathrm{T}} = A^l \sum_{j=0}^{n-1} e_j^{\mathrm{T}}(e_j(I + A^{2h})e_i^{\mathrm{T}})$$
$$\geqslant A^l e_i^{\mathrm{T}}(e_i(I + A^{2h})e_i^{\mathrm{T}}) \geqslant 2A^l e_i^{\mathrm{T}}.$$

But the nonnegative integer vector $A^l e_i^{\mathrm{T}}$ cannot be zero. A contradiction. □

3. A solution to (1.1) of odd k

Let

$$N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}$$

and Q_p be the unique $n \times n$ matrix satisfying

$$e_0 Q_p = e_0$$

and

$$NQ_p = Q_p N^p$$

for any integer p. From

$$e_0 Q_p Q_q = e_0$$

and

$$NQ_pQ_q = Q_pQ_qN^{pq}$$

we have

$$Q_pQ_q = Q_{pq}.$$

Denote $Q_{-d}f(N)$ by $M = M(d,n)$ where

$$f(x) = x\left(\frac{1-x^d}{1-x}\right).$$

Now we prove the following theorem by a simple computation.

Theorem 3.1. *The matrix $M(d,n)$ satisfies (1.1) when k is odd.*

Proof. Since

$$(-d)^l \equiv (-d)^{k+l} (\mathrm{mod}\ n)$$

when k is odd, we obtain

$$Q_{(-d)^l} = Q_{(-d)^{k+l}}.$$

For any nonnegative integer s, we have

$$\begin{aligned} M^s &= Q_{-d}f(N)Q_{-d}f(N)M^{s-2} \\ &= Q_{-d}(Q_{-d}f(N^{-d})f(N)M^{s-2} \\ &= Q_{(-d)^2}f(N^{-d})f(N)M^{s-2} \\ &\ \vdots \\ &= Q_{(-d)^s}\prod_{i=0}^{s-1} f(N^{(-d)^i}) \end{aligned}$$

(we regard $\prod_{i=0}^{-1} f(N^{(-d)^i})$ as I_n). But

$$\begin{aligned} \prod_{i=0}^{s-1} f(x^{(-d)^i}) &= \prod_{i=0}^{s-1}\left[x^{(-d)^i}(1 - x^{(-1)^i d^{i+1}})(1 - x^{(-d)^i})^{-1}\right] \\ &= (-1)^s \prod_{i=0}^{s-1}\left[(x^{(-1)^{i+2}d^{i+1}} - 1)(x^{(-1)^{i+1}d^i} - 1)^{-1}\right] \\ &= \begin{cases} 1 + x^{-1} + (x^{-1})^2 + \cdots + (x^{-1})^{d^s-1}, & \text{s even}, \\ x + x^2 + \cdots + x^{d^s}, & \text{s odd}. \end{cases} \end{aligned}$$

Consequently,

$$M^l + M^{l+k} = Q_{(-d)^l}J_{d^l+d^{l+k}} = J_{d^l+d^{l+k}}. \qquad \square$$

4. Main result

Theorem 4.1. *For any solution A of (1.1) there must be a (0,1)-matrix C satisfying $I + C^k = J_{d^k+1}$ and $\Gamma(A) = L^l(\Gamma(C))$.*

Proof. The case $l = 0$ is trivial. Now suppose $l \geqslant 1$. We first give some observations.

(1) $A^l + A^{l+k} = J \Longrightarrow A$ is irreducible; $\sum_{i=0}^{n-1} e_i$ is a positive eigenvector of A corresponding to the eigenvalue d. Hence, d is a simple eigenvalue of A by the Perron–Frobenius theorem.

(2) $JA^l = d^l J \Longrightarrow$ None of the column vectors of A^l can be zero vector; $A^l(I + A^k) = J \Longrightarrow$ none of the entries of $I + A^k$ can be greater than 1, and Tr $(A^k) = 0$ follows.

(3) $J(A - dI) = 0 \Longrightarrow x^l(x^k + 1)(x - d)$ is a polynomial which annihilates A. So we can assume that the nonzero eigenvalues of A are d, $x_1, \ldots, x_t$, where $x_i^k + 1 = 0$ for $i = 1, \ldots, t$. Since Tr $(A^k) = 0$, we must have $t = d^k$.

Note that the order of the largest Jordan block of A corresponding to the zero eigenvalue is at most l, so we have

$$\text{rank } (A^l) = d^k + 1.$$

For a row vector α, we set

$$P(\alpha) = \{i \mid \alpha e_i^{\mathrm{T}} \neq 0\}.$$

Let $\alpha_0, \alpha_1, \ldots, \alpha_{d^k}$ be a set of maximal linear independent row vectors of A^l. From $(A^k + I)A^l = J_n$ we know that

$$\sum_{i=0}^{n-1} e_i \in \text{Span } \{\alpha_0, \ldots, \alpha_{d^k}\}.$$

Hence

$$\bigcup_{i=0}^{d^k} P(\alpha_i) \supseteq P\left(\sum_{i=0}^{n-1} e_i\right) = [n-1]. \tag{4.1}$$

We denote $P(e_s A^j)$ by $P(s, j)$. It can be easily seen that A^j must be a (0,1)-matrix for all $j \in [l + k]$. Owing to this fact and that $A^j J = d^j J$, we get

$$|P(s, j)| = d^j \quad \text{for } s \in [n-1] \text{ and } j \in [l+k] \ , \tag{4.2}$$

where $|P(s, j)|$ means that cardinal number of the set $P(s, j)$.

Let us say that a set system $\mathscr{A}$ has property L provided that any two members of $\mathscr{A}$ are either identical or disjoint.

Comparing the cardinality of the sets on the two sides of (4.1), we find that $P(\alpha_i)$ must be disjoint mutually for $i \in [d^k]$. But we also have $|P(i, l)| = d^l$ and $e_i A^l \in \text{Span } \{\alpha_0, \ldots, \alpha_{d^k}\}$ for all $i \in [n-1]$. So it follows

$$\{P(i, l) \mid i \in [n-1]\} \text{ is a set system with property } L. \tag{4.3}$$

Now we claim that $\{P(i, 1) \mid i \in [n-1]\}$ has property L too. If $l = 1$, there is nothing to do. If $l \geqslant 2$, choose $i \neq j \in [n-1]$ arbitrarily. Suppose

$P(i, l-1) \bigcap P(j, l-1) \neq \emptyset$. Since $Ae_i^{\mathrm{T}} \neq 0$ and $Ae_j^{\mathrm{T}} \neq 0$, we have $e_q A e_i^{\mathrm{T}} = e_s A e_j^{\mathrm{T}} = 1$ for some $q, s \in [n-1]$. Hence

$$P(q, l) \supseteq P(i, l-1),$$

$$P(s, l) \supseteq P(j, l-1)$$

and

$$P(q, l) \bigcap P(s, l) \supseteq P(i, l-1) \bigcap P(j, l-1) \neq \emptyset.$$

We also have

$$P(i, l) \bigcap P(j, l) \neq \emptyset,$$

since $A^l = A^{l-1}A$ and any row vector of A cannot be a zero vector. By (4.3), we get

$$P(q, l) = P(s, l)$$

and

$$P(i, l) = P(j, l).$$

Let

$$e_q A = e_i + \sum_{t=1}^{d-1} e_{i_t}.$$

Then

$$P(j, l-1) \subseteq P(s, l) = P(q, l) = P(i, l-1) \bigcup \left(\bigcup_{t=1}^{d-1} P(i_t, l-1) \right), \tag{4.4}$$

$$P(q, l+1) = P(i, l) \bigcup \left(\bigcup_{t=1}^{d-1} P(i_t, l) \right) = P(j, l) \bigcup \left(\bigcup_{t=1}^{d-1} P(i_t, l) \right). \tag{4.5}$$

Estimating the cardinality of the sets on the two sides of (4.5) by (4.2), we obtain

$$P(j, l) \bigcap \left(\bigcup_{t=1}^{d-1} P(i_t, l) \right) = \emptyset.$$

Applying the fact that $A^l = A^{l-1}A$ and A has no zero row vector again, we get

$$P(j, l-1) \bigcap \left(\bigcup_{t=1}^{d-1} P(i_t, l-1) \right) = \emptyset. \tag{4.6}$$

Now, the equality

$$P(i, l-1) = P(j, l-1)$$

follows from (4.2), (4.4) and (4.6). Therefore $\{P(i, l-1) \mid i \in [n-1]\}$ has property L too. Noting that this statement is based on only Eqs. (4.2) and (4.3), the assertion that $\{P(i, 1) \mid i \in [n-1]\}$ has property L then follows by induction on l.

By the characterization theorem of line digraph [4], Theorem 8.4, the last assertion implies that $\Gamma(A)$ is the line digraph of some d-diregular digraph Δ. If we can prove that

$$B^{l-1} + B^{l-1+k} = J_p, \tag{4.7}$$

where B is the adjacency matrix of Δ and $p = n/d = d^{l-1} + d^{l-1+k}$ is the number of vertices in $\Delta = \Gamma(B)$, then it is obvious that our proof can be completed by induction on l.

We prove (4.7) as follows.

For any two vertices p, q in $\Gamma(B)$, we can choose two vertices r, s of $\Gamma(B)$ such that $\overrightarrow{rp}$ and $\overrightarrow{qs}$ are edges of $\Gamma(B)$, hence vertices of $\Gamma(A)$, since $d \geqslant 1$. Let $\mathscr{A} = \{$walks in $\Gamma(A)$ from $\overrightarrow{rp}$ to $\overrightarrow{qs}\}$, $\mathscr{B} = \{$walks in $\Gamma(B)$ from p to $q\}$. For any walk $w = (\overrightarrow{rp}, \overrightarrow{pw_1}, \overrightarrow{w_1w_2}, \ldots, \overrightarrow{w_{m-1}q}, \overrightarrow{qs}) \in \mathscr{A}$, let $\pi(w) = (p, w_1, \ldots, w_{m-1}, q) \in \mathscr{B}$. It is easy to verify that actually the mapping π from $\mathscr{A}$ to $\mathscr{B}$ defined above is a bijection, and w has length $m+1$ in $\Gamma(A)$ if and only if $\pi(w)$ has length m in $\Gamma(B)$.

Now observe that (1.1) ((4.7)) can be interpreted graph-theoretically as there is a unique walk of length either l or $l+k$ (either $l-1$ or $l-1+k$) from any given vertex to any other, may be the same, given vertex in $\Gamma(A)$ ($\Gamma(B)$).

Due to the existence of the bijection π constructed above, we see that $\Gamma(B)$ has similar property as $\Gamma(A)$ with the only distinction that the parameter l is decreased to $l-1$. Therefore, (4.7) follows from (1.1). □

Consider the case $k = 1$ for (1.1), namely

$$A^l + A^{l+1} = J_n. \tag{4.8}$$

Since $I + C = J_{d+1}$ implies $\Gamma(C) = K_{d+1}$, the complete digraph on $d+1$ vertices, it follows immediately by Theorem 4.1 the fact that $A^l + A^{l+1} = J_n$ implies $\Gamma(A) = L^l(K_{d+1})$, the well-known Kautz digraph $K(d, l+1)$ [7,8,3]. On the other hand, (4.8) must be solvable by Theorem 3.1. So the following characterization theorem is established, and its validity even for the case $n = 1$ and $d = 0$ can be easily verified.

Theorem 4.2. *An $n \times n$ $(0,1)$-matrix A satisfies (4.8) if and only if $n = d^l + d^{l+1}$ for some nonnegative integer d and $\Gamma(A) = K(d, l+1)$.* □

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