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Invariant subspace, determinant and characteristic polynomials[☆]

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Abstract

Making use of an elementary fact on invariant subspace and determinant of a linear map and the method of algebraic identities, we obtain a factorization formula for a general characteristic polynomial of a matrix. This answers a question posed in [A. Deng, I. Sato, Y. Wu, Characteristic polynomials of ramified uniform covering digraphs, European J. Combin. 28 (2007) 1099–1114]. The approach of this work can be used to supply alternative proofs of several other earlier results, including some results of [Y. Teranishi, Equitable switching and spectra of graphs, Linear Algebra Appl. 359 (2003) 121–131].

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1. Introduction

Divisor, equitable partition and digraph homomorphism are closely related concepts [3,5] and the recognition of them indicates some structure of the digraphs involved and often helps to expose to us some secret of the digraph spectra. In this regard, let us quote a famous saying from [3, p. 116]: “It is shown in which way the divisor, on the one hand, is connected with structural, in particular (generalized) symmetry properties and, on the other hand, is connected with spectral properties of a graph. The divisor is utilized for factoring the characteristic polynomial…”

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It is discovered by Haynsworth [7], and independently by Petersdorf and Sachs [13], that the characteristic polynomial of a graph has as a factor the characteristic polynomial of a divisor of it. Indeed, this result holds for the generalized characteristic polynomial of a graph, a multi-variables polynomial which generalizes the concept of characteristic polynomial; for details, see [3, Theorem 0.12, 4.7]. But can we say anything on the quotient of these two polynomials? Can we further factor it? This paper is some effort to answer these questions.

When the divisor possesses certain property, some work have been carried out in answering the above questions. Wang [20] observes that the characteristic polynomial of a voltage graph decompose into two parts and provides some observations on them. Using Fourier transform, Mizuno and Sato [12] obtain an explicit decomposition formula for such a characteristic polynomial. To go a step further, one can consider those digraphs whose vertex set can be partitioned into several parts each of which induces a voltage digraph. To take the Fourier transform approach, we need to establish some good properties for the Fourier matrix involved so that we can diagonalize the matrix in a good manner and thus get a good decomposition formula. As a first attempt, Lee and Kim [9] calculate the characteristic polynomial of a graph with a semi-free action by an Abelian group, taking advantage of the good knowledge of all irreducible representations of an Abelian group. Deng and Wu work out a key lemma on the Fourier matrix; see [4, Lemma 4.2]. This is extended by Deng et al. later; see [5, Lemma 6.3]. With this kind of work, Deng et al. [4,5,6] obtain some factorization results for the characteristic polynomials of a series of digraphs with certain symmetries, including a generalization of the result of Lee and Kim to general digraphs with any semi-free action by a general finite group. Making use of the lemma of Deng and Wu on Fourier matrix, Kim and Lee [8] further consider the corresponding factorization formula for the generalized characteristic polynomial of a graph. Note that the approach of this line of research is to first focus on representing the digraph with group actions and then invoke basic properties of group representations to calculate the characteristic polynomials. Some other related research utilizing similar techniques can be found in [1,10,16,19].

In this note, we generalize and utilize the above series of work by adopting another viewpoint. Our starting point is some really easy observations on determinant and invariant subspaces of a linear map. They are addressed in the next section. In Section 3, we proceed with a discussion of equitable partitions which, combined with the so-called method of algebraic identities, then leads to our main result on characteristic polynomials. We conclude this paper with some comments in Section 4.

2. Determinant and invariant subspace

In this section, we review some basic linear algebra facts.

Throughout this note, we let $\mathbb{F}$ be a field and let V be a linear space over $\mathbb{F}$. We use $Ł(V)$ to designate the set of linear transformations from V to itself. For any $f \in Ł(V)$ and any basis $\mathscr{B} = \{e_1, e_2, \ldots, e_n\}$ of V, there is a matrix $A_{f,\mathscr{B}} \in \mathbb{F}^{n\times n}$ such that

$$f(\overline{\mathscr{B}}) = \overline{\mathscr{B}} A_{f,\mathscr{B}},$$

where $f(\overline{\mathscr{B}}) = (f(e_1)f(e_2)\ldots f(e_n))$ and $\overline{\mathscr{B}} = (e_1e_2\ldots e_n)$.

Assume that V is a direct sum of V_1 and V_2, denoted $V = V_1 \oplus V_2$. Then there is a unique element in $Ł(V)$, denoted $p_{V_1}^{V_2}$ and called the *projection* along V_2 to V_1, which satisfies $p_{V_1}^{V_2}(x) = x$ for $x \in V_1$ and $p_{V_1}^{V_2}(x) = 0$ for $x \in V_2$. For any $f \in Ł(V)$, we use the notation $f_{V_1}^{V_2}$ for the map $p_{V_1}^{V_2} \circ f|_{V_1}$, which will be viewed as an element of $Ł(V_1)$.

Lemma 1. *Let $f \in Ł(V)$ and let W be an invariant subspace of f. For any subspace W' of V which is complementary to W, the quantity* $\det\left(f^W_{W'}\right)$ *is independent of the choice of W'.*

Proof. Let us look at the quotient space V/W of V modulo W and the contraction $f^W \in Ł(V/W)$ of f defined by $f^W(x+W) = f(x) + W$. Consider the isomorphism g from W' to V/W which sends x to $x+W$. It is immediate that $f^W_{W'} = g^{-1} f^W g$ and hence $\det\left(f^W_{W'}\right) = \det(f^W)$ is independent of the choice of W'. $\square$

Remark 2. We follow the notation used in the proof of Lemma 1. It is well known that $\det(f) = \det(f|_W)\det(f^W)$ when W is invariant under f. Accordingly, we will write $\frac{\det(f)}{\det(f|_W)}$ for the W'-independent parameter $\det\left(f^W_{W'}\right) = \det(f^W)$ hereafter, even if $\det(f|_W)$ might vanish. Similar convention will be used when we discuss determinants of matrices.

Lemma 3. *Assume that $W \leqslant V = V_1 \oplus V_2$, $V_1 = W_1 \oplus (V_1 \cap W)$ and $f \in Ł(V)$. Then, in case that both W and $V_1 + W$ are invariant subspaces of f, we have*

$$\frac{\det(f)}{\det(f|_W)} = \frac{\det\left(f^{V_2}_{V_1}\right)}{\det\left(f^{V_2}_{V_1}|_{V_1\cap W}\right)} \times \frac{\det\left(f^{V_1}_{V_2}\right)}{\det\left(f^{V_1}_{V_2}|_{V_2\cap W}\right)}.$$

Proof. Let W_2 be a subspace of V_2 being complementary to $W \cap V_2$. Choose bases for the four subspaces, $V_1 \cap W$, $V_2 \cap W$, W_1 and W_2, say, $\mathscr{B}_1$, $\mathscr{B}_2$, $\mathscr{B}_3$ and $\mathscr{B}_4$, respectively. Now let $\mathscr{B}$ be the basis of V such that $\overline{\mathscr{B}} = (\overline{\mathscr{B}_1}\overline{\mathscr{B}_2}\overline{\mathscr{B}_3}\overline{\mathscr{B}_4})$. Considering that W and $V_1 + W = W \oplus W_1$ are both invariant under f, we can write down

$$A_{f,\mathscr{B}} = \begin{pmatrix} X & Y & Z & W \\ E & F & G & H \\ 0 & 0 & P & Q \\ 0 & 0 & 0 & R \end{pmatrix}, \tag{1}$$

where the lines are collected into four blocks which correspond to $\mathscr{B}_1$, $\mathscr{B}_2$, $\mathscr{B}_3$ and $\mathscr{B}_4$ in that order.

Now, picking the basis $\mathscr{B}^1$ of V_1 such that $\overline{\mathscr{B}}^1 = (\overline{\mathscr{B}_1}\overline{\mathscr{B}_3})$ and the basis $\mathscr{B}^2$ of V_2 such that $\overline{\mathscr{B}}^2 = (\overline{\mathscr{B}_2}\overline{\mathscr{B}_4})$, we have

$$A_{f^{V_2}_{V_1},\mathscr{B}^1} = \begin{pmatrix} X & Z \\ 0 & P \end{pmatrix} \quad \text{and} \quad A_{f^{V_1}_{V_2},\mathscr{B}^2} = \begin{pmatrix} F & H \\ 0 & R \end{pmatrix}. \tag{2}$$

In view of Eqs. (1) and (2), we find that

$$\frac{\det(f)}{\det(f|_W)} = \det(P)\det(R) = \frac{\det(f^{V_2}_{V_1})}{\det(f^{V_2}_{V_1}|_{V_1\cap W})}\frac{\det(f^{V_1}_{V_2})}{\det(f^{V_1}_{V_2}|_{V_2\cap W})},$$

as desired. $\square$

Remark 4. We mention that the condition in Lemma 3 that $V_1 + W$ is invariant under f is actually equivalent to any of the following three statements: (i) $f(W_1) \leqslant W_1 \oplus W$; (ii) $f(V_1) \leqslant V_1 + W = V_1 \oplus (V_2 \cap W)$; (iii) W_1 is invariant under $f^W_{W_1 \oplus W_2}$.

For any matrix A, any set R of row labels and any set C of column labels of A, we set $A[R,C]$ to be the submatrix of A obtained by deleting all those rows whose labels are outside R and deleting all those columns whose labels are outside C. For simplicity, when both the row label set and the column label set of A have S as a subset, we use $A[S]$ for the principal submatrix of A whose lines are indexed by S.

Corollary 5. *Let S_1, S_2, T_1, T_2 be four disjoint sets and $S = S_1 \cup S_2, T = T_1 \cup T_2$. Let $A \in \mathbb{F}^{S\times S}$, $P \in \mathbb{F}^{S\times T}$, $U \in \mathbb{F}^{T\times T}$. Assume that*

(1) $P[S_1, T_2] = 0, P[S_2, T_1] = 0$;
(2) $AP = PU$;
(3) *There is* $C \in \mathbb{F}^{T_2\times S_1}$ *such that* $A[S_2, S_1] = P[S_2, T_2]C$;
(4) P *is of full column rank.*

Then, it holds

$$\frac{\det(A)}{\det(U)} = \frac{\det(A[S_1])}{\det(U[T_1])} \times \frac{\det(A[S_2])}{\det(U[T_2])}. \tag{3}$$

Proof. Take $V = \mathbb{F}^S$. The matrix A corresponds to an element $f \in Ł(V)$ such that $f(x) = Ax$ for any $x \in V$. Clearly, V is the direct sum of $V_1 = \mathbb{F}^{S_1}$ and $V_2 = \mathbb{F}^{S_2}$. Denote the column space of P by W. It follows from condition (1) that $V_1 \cap W$ is the column space of $P[S_1, T_1]$ while $V_2 \cap W$ is the column space of $P[S_2, T_2]$.

Condition (2) implies that W is invariant under f and condition (3) shows that $f(V_1) \leqslant V_1 \oplus (V_2 \cap W)$. Taken together, we derive from Lemma 3 and Remark 4 that

$$\frac{\det(f)}{\det(f|_W)} = \frac{\det(f_{V_1}^{V_2})}{\det(f_{V_1}^{V_2}|_{V_1\cap W})} \times \frac{\det(f_{V_2}^{V_1})}{\det(f_{V_2}^{V_1}|_{V_2\cap W})}. \tag{4}$$

Note that f, $f_{V_1}^{V_2}$ and $f_{V_2}^{V_1}$ have as basis representations A, $A[S_1]$ and $A[S_2]$, respectively. Moreover, it can be gleaned from condition (4) that $f|_W$, $f_{V_1}^{V_2}|_{V_1\cap W}$ and $f_{V_2}^{V_1}|_{V_2\cap W}$ have as basis representations U, $U[T_1]$ and $U[T_2]$, respectively. This then means that Eq. (4) is nothing but Eq. (3), finishing the proof. □

3. Equitable partition and characteristic polynomial

The use of equitable partition dates back to Sachs [14,15] while the notation of equitable partition is proposed by Schwenk [17].

The *characteristic matrix* $P(\pi)$ of a partition $\pi = (C_1, \ldots, C_t)$ of a set of n elements is the $n \times t$ matrix whose ith column consists of the characteristic vector of the ith *cell* of π, namely C_i. Suppose $[n] = \{1, 2, \ldots, n\}$ is a disjoint union of $S_1, S_2, \ldots,$ and S_m. Let π_i be a partition of S_i, $i \in [m]$. We denote by $\bigvee_{i=1}^m \pi_i$ the partition of $[n]$ whose cells are the cells of π_i, $i \in [m]$.

Following Stadler and Tinhofer [18], we say that a partition π of $[n]$ is *row-equitable* with respect to $A \in \mathbb{F}^{n\times n}$ if there is $C \in \mathbb{F}^{t\times t}$ such that

$$AP(\pi) = P(\pi)C, \tag{5}$$

and say that π is *column-equitable* with respect to $A \in \mathbb{F}^{n\times n}$ if π is row-equitable with respect to $A^\top$. If Eq. (5) holds, we write A/π^+ for C and call it the *front divisor* of A with respect to the row-equitable partition π. Similarly, we can define the *rear divisor* A/π^- of a matrix A

with respect to one of its column-equitable partition π [6]. If π is the trivial partition such that $P(\pi) = I$, then π is both row-equitable and column-equitable and $A = A/\pi^+ = A/\pi^-$.

Observe that π is row-equitable with respect to A if and only if the column space of $P(\pi)$ is invariant under the action of A. Moreover, if we assume that $A = A_{f,\mathscr{B}}$ for a basis $\mathscr{B}$ of an n-dimensional space V and an $f \in Ł(V)$ and let U be the space with a basis $\mathscr{B}_1$ where $\overline{\mathscr{B}_1} = \overline{\mathscr{B}}P(\pi)$, we can see that U is invariant under f. As a matter of fact, Eq. (5) reveals that $C = A_{f|_U,\mathscr{B}_1}$ and thus

$$\det(f|_U) = \det(C) = \det(A/\pi^+).$$

Lemma 6. *Consider $A \in \mathbb{F}^{n\times n}$ whose lines are indexed by $[n]$. Suppose $[n]$ is the disjoint union of S_1 and S_2 and π_1 and π_2 are two partitions of S_1 and S_2, respectively. Assume that*

(1) *$\pi = \pi_1 \bigvee \pi_2$ is a row-equitable partition of $[n]$ with respect to A;*
(2) *For each $a \in S_1$ and for any b, c from the same cell of π_2, we have $A(b, a) = A(c, a)$.*

Let $A_1 = A[S_1]$ and $A_2 = A[S_2]$. Then it holds

$$\frac{\det(A)}{\det(A/\pi^+)} = \frac{\det(A_1)}{\det(A_1/\pi_1^+)} \times \frac{\det(A_2)}{\det(A_2/\pi_2^+)}.$$

Proof. By putting additionally T_1 to be the set of cells of π_1, T_2 to be the set of cells of π_2, and P to be $P(\pi)$, we can apply Corollary 5 and the definition of front divisors to arrive at the desired claim. □

Theorem 7. *Consider $A \in \mathbb{F}^{n\times n}$ whose lines are indexed by $[n]$. Assume that $[n]$ is a disjoint union of $S_1, S_2, \ldots,$ and S_m. Let π_i be a partition of S_i, $i \in [m]$. Suppose*

(1) *$\pi = \bigvee_{i=1}^m \pi_i$ is a row-equitable partition of $[n]$ with respect to A;*
(2) *For any $1 \leqslant i < j \leqslant m$, any $a \in S_i$ and any b, c from the same cell of π_j, we have $A(b, a) = A(c, a)$.*

Let A_i be the principal submatrix $A[S_i]$ of A for each $i \in [m]$. Then it holds

$$\frac{\det(A)}{\det(A/\pi^+)} = \prod_{i=1}^m \frac{\det(A_i)}{\det(A_i/\pi_i^+)}.$$

Proof. Follows from Lemma 6 by induction on m. □

Here comes our main result.

Theorem 8. *Let $\mathbb{F}$ be an infinite field. Consider $A \in \mathbb{F}^{n\times n}$ whose lines are indexed by $[n]$. Assume that $[n]$ is a disjoint union of $S_1, S_2, \ldots,$ and S_m. Let $\pi_i = (S_i^1, \ldots, S_i^{\nu_i})$ be a partition of S_i, $i \in [m]$. Let $A_i = A[S_i]$. Suppose:*

(i) *$\pi = \bigvee_{i=1}^m \pi_i$ is a row-equitable partition of $[n]$ with respect to A;*
(ii) *For any $1 \leqslant i < j \leqslant m$, any $a \in S_i$ and any b, c from the same cell of π_j, we have $A(b, a) = A(c, a)$.*

Let $\Lambda = \{\lambda_{i,t} : i \in [m], t \in [\nu_i]\}$, be a set of independent indeterminates. Let

- $A^{[\tilde{\lambda}]} \doteq \mathrm{diag}[\tilde{\lambda_1}, \ldots, \tilde{\lambda_n}] - A$, *where* $\tilde{\lambda_j} = \lambda_{i,t}$ *provided* $j \in S_i^t$; *and its determinant is denoted* $\chi(A; [\tilde{\lambda}])$;
- $A^{[\lambda]} \doteq \mathrm{diag}[\lambda_1, \ldots, \lambda_{\sum_{i=1}^m \nu_i}] - A/\pi^+$, *where* $\lambda_j = \lambda_{i,t}$ *provided the* j*th line of* A/π^+ *is indexed by* S_i^t; *and its determinant is denoted* $\chi(A/\pi^+; [\lambda])$;
- $A_i^{[\tilde{\lambda}]_i} \doteq A^{[\tilde{\lambda}]}[S_i]$; *and its determinant is denoted* $\chi(A_i; [\tilde{\lambda}]_i)$;
- $A_i^{[\lambda]_i} \doteq A^{[\lambda]}[\{S_i^1, \ldots, S_i^{\nu_i}\}]$; *and its determinant is denoted* $\chi(A_i/\pi_i^+; [\lambda]_i)$.

Then, in the polynomial ring $\mathbb{F}[\lambda_{i,t}]$ *we have*

$$\chi(A; [\tilde{\lambda}]) = \chi(A/\pi^+; [\lambda]) \prod_{i=1}^{m} \frac{\chi(A_i; [\tilde{\lambda}]_i)}{\chi(A_i/\pi_i^+; [\lambda]_i)}. \tag{6}$$

Proof. Since $\mathbb{F}$ is infinite, by the "principle of irrelevance of algebraic inequalities" [11, Exercise I.A.6; 21, p. 4], two polynomials are equal if and only if they are equal as polynomial functions. Therefore, we only need to prove Eq. (6) as an equality where all $\lambda_{i,t}$ are viewed as elements of $\mathbb{F}$.

It is easy to see that

(1′) $\bigvee_{i=1}^m \pi_i$ is a row-equitable partition of $[n]$ with respect to $A^{[\tilde{\lambda}]}$;
(2′) For any $1 \leqslant i < j \leqslant m$, any $a \in S_i$ and any b, c from the same cell of π_j, we have $A^{[\tilde{\lambda}]}(b, a) = A^{[\tilde{\lambda}]}(c, a)$;
(3′) $A^{[\lambda]} = A^{[\tilde{\lambda}]}/\pi^+$; $A_i^{[\lambda]_i} = A_i^{[\tilde{\lambda}]_i}/\pi_i^+ \ \forall i \in [m]$.

The truth of Eq. (6) as a relation over $\mathbb{F}$ now comes directly from Theorem 7, as wanted. □

Remark 9. Ref. [6, Theorem 3.4] gives a decomposition formula similar to Eq. (6), which is deduced under the additional assumption that the variables in Λ are all equal and the condition 2 listed in Theorem 8 holds for all i, j rather than just for $i < j$. Prompted by [3, Theorem 4.7] and the conjecture made in the remark immediately following it there [3, p. 124], Deng et al. put forward a question, [6, Question 3.6], to ask how to get a multi-variables generalization of [6, Theorem 3.4]. Theorem 8 may be viewed as an answer to [6, Question 3.6].

4. Further comments

Theorem 8 should be helpful in factoring certain kind of generalized characteristic polynomials of some large matrix of high regularity. We mention that various digraph invariants are connected to generalized characteristic polynomials and thus it also helps to understand such invariants. In Eq. (6), the polynomial $\frac{\chi(A_i;[\tilde{\lambda}]_i)}{\chi(A_i/\pi_i^+;[\lambda]_i)}$ can be further factorized in the event that $A_i^{[\tilde{\lambda}]_i}$ admits a group action such that the quotient matrix becomes $A_i^{[\lambda]_i} = A_i^{[\tilde{\lambda}]_i}/\pi_i^+$. This is achieved by some simple application of group characters; see [3, Chapter 5].

Our approach of establishing Theorem 8 consists of three main steps, first building connections between invariant subspace and determinant factorization, then establishing the relationship between matrix structure and invariant subspace, and finally making use of the principle of irrelevance of algebraic inequalities. We notice that several earlier results in the relevant directions can be treated and generalized under this framework as well. These include [3, Theorem 4.7] CDS, [19, Lemma 1.1, Theorem 2.1, Proposition 3.1], just to name a few. We point out that the principle of irrelevance of algebraic inequalities is implicitly used by Kim and Lee [8]. Indeed, it is Lee who suggests to us that an answer to [6, Question 3.6] follows readily from [6, Theorem 3.4] and his comment initiates the work of this note.

Comparing with the earlier work in [4,5,6,9], the derivation of the stronger result here looks really trivial. But in those former work, the aim is to find a decomposition formula and only after rearranging the formula obtained then via a long sequence of technical steps we get the insight that equitable partition and invariant subspace are two key concepts in understanding those results. Similar facts on equitable partitions of complex matrix or real matrix are also used in [2,19]. But the arguments there often make use of some special facts which is valid only over some special fields, e.g., two orthogonal nonzero vectors are linearly independent. In contrast, apart from Theorem 8, all results in this paper pose no restriction on the underlying field.

In the proof of Theorem 8, we rely on the method of algebraic identities. Recall that for the proof of its prototype, [6, Theorem 3.4], no such principle is used and the whole proof there is some concrete combinatorial demonstration. It would be interesting to find a combinatorial proof of Theorem 8.

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