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Homomorphisms, representations and characteristic polynomials of digraphs ☆

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Abstract

The existence of a homomorphism between two digraphs often implies many structural properties. We collect in this paper some characterizations of various digraph homomorphisms using matrix equations and fiber partitions. We also survey the relationship among the characteristic polynomials of a digraph and its divisors. This includes an introduction of the concept of branched coverings of digraphs, their voltage assignment representations, and a decomposition formula for the characteristic polynomial of a branched cover with branch index 1. Some open problems are included.

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1. Introduction

For a digraph Γ we denote its vertex set and arc set by $V(\Gamma)$ and $E(\Gamma)$, respectively. A *symmetric involution* β of Γ is an involution map on $E(\Gamma)$ such that e and $\beta(e)$ connect the same pair of

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vertices u and v and are of opposite directions when $u \neq v$. We only consider finite digraphs and regard graphs as symmetric digraphs, namely digraphs possessing a symmetric involution. We write $\mathbf{A}(\Gamma)$ for the adjacency matrix of Γ whose lines correspond to some orderings of $V(\Gamma)$ which may be clear from the context or may be of no relevance to our interest. Let I_n be the identity matrix of order n and let J_n represent the all ones row vector of length n. We often suppress the subscript if no confusion arises. The *characteristic polynomial* of Γ with indeterminate λ is $\chi(\Gamma; \lambda) = \det(\lambda I - \mathbf{A}(\Gamma))$. For any positive integer k we use the notation $[k]$ for $\{1, 2, \ldots, k\}$ hereafter.

There have been lots of work on the relationships among the characteristic polynomials of digraphs, both in the spectral graph theory field [2,4,7,9,12,19–21,23,24,27,28] and in the field of symbolic dynamics [1,3,18,22,25,30]. Many of them tell us how the relations among a set of digraphs (often just two digraphs) will be reflected in the relations among their characteristic polynomials. Clearly, the existence of a digraph homomorphism is the most important relationship between two digraphs. This note is an attempt to organize sparse results from different areas into the framework of digraph homomorphisms, its matrix representations and implications on the relationship of the characteristic polynomials of involving digraphs.

2. Digraph homomorphism, equitable partition and their matrix characterizations

We introduce in this section some notions related to digraph homomorphism and equitable partition. We also give some characterizations of them in terms of matrix equations.

2.1. Map and digraph

For any finite set X, let $\#X$ denote its cardinality. The *multiplicity set* of a not necessarily surjective map ϕ from a set X to a set Y is defined to be $M(\phi) = \{\#\phi^{-1}(y) : y \in Y\}$. With any fixed ordering of X and Y, the map $\phi : X \to Y$ has a matrix representation M_ϕ whose rows correspond to X and columns to Y and is specified by

$$M_\phi(x, y) = \begin{cases} 1, & \text{if } y = \phi(x), \\ 0, & \text{otherwise.} \end{cases}$$

Note that when ϕ and φ are both maps from X to Y and when we are using M_ϕ and M_φ at the same time, we always implicitly assume that the lines of M_ϕ and M_φ are indexed according to the same orderings of X and of Y. A *subamalgamation matrix* is a rectangular (0, 1) matrix with exactly one 1 in each row while an *amalgamation matrix* is a rectangular (0, 1) matrix with exactly one 1 in each row and at least one 1 in each column [22,30]. Here comes an easy and well-known result.

Lemma 2.1

(i) *A matrix is a subamalgamation matrix if and only if it can be written as M_ϕ for some map ϕ.*

(ii) *The following are equivalent*: (a) M_ϕ *is of full column rank*; (b) $M_\phi^\top M_\phi$ *is a nonsingular diagonal matrix*; (c) ϕ *is onto*; (d) M_ϕ *is an amalgamation matrix*.

A *partition* of a set X is a collection $X_1, \ldots, X_t$ of disjoint subsets of X whose union is X. Note that we do not require here that each X_i, which will be called a *cell* of the partition, to be nonempty. For any partition $\pi : X = \cup_{i=1}^t X_i$ of the set X, we associate with it a map p_π from X

to $\{X_1, \ldots, X_t\}$ such that $p_\pi(x) = X_i$ whenever $x \in X_i$. Let ϕ be a map from a set X to a set Y. For each $y \in Y$, $\phi^{-1}(y)$ is referred to as the *fiber* of ϕ over $y \in Y$ and the partition of X consisting of all those fibers of ϕ will be called the *fiber partition* induced by ϕ and will be denoted $\pi(\phi)$.

Lemma 2.2. *For any map $\phi : X \to Y$, we have $M_{p_{\pi(\phi)}} = M_\phi$.*

Proof. By identifying $y \in Y$ with $\phi^{-1}(y)$ we see that $p_{\pi(\phi)}$ and ϕ are essentially the same map and so the result follows. □

If an arc e goes from a vertex u to a vertex v, we say that u is the *initial vertex* (or the *tail*) of e, and v is the *terminal vertex* (or the *head*) of e or sometimes just say that u and v are the *endpoints* of e. The incidence structure of a digraph Γ is characterized by two maps from $E(\Gamma)$ to $V(\Gamma)$, the *tail operator* $\mathtt{i}_\Gamma$ that sends an arc to its initial vertex and the *head operator* $\mathtt{t}_\Gamma$ that sends an arc to its terminal vertex. For any $v \in V(\Gamma)$, the set $\mathtt{i}_\Gamma^{-1}(v) = \{e \in E(\Gamma) : \mathtt{i}_\Gamma(e) = v\}$ will be referred to as the *out-arcs* of v in Γ and denoted by $E_\Gamma^+(v)$; analogously, we call the set $\mathtt{t}_\Gamma^{-1}(v) = \{e \in E(\Gamma) : \mathtt{t}_\Gamma(e) = v\}$ the *in-arcs* of v and represent it as $E_\Gamma^-(v)$. For any $V_0 \subseteq V(\Gamma)$, we write $E_\Gamma^+(V_0)$ for $\cup_{v \in V_0} E_\Gamma^+(v)$ and $E_\Gamma^-(V_0)$ for $\cup_{v \in V_0} E_\Gamma^-(v)$. Observe that for an involution β of a symmetric digraph it holds $E_\Gamma^+ = \beta E_\Gamma^-$. Since both $\mathtt{i}_\Gamma$ and $\mathtt{t}_\Gamma$ are maps from $E(\Gamma)$ to $V(\Gamma)$, we can fix an ordering of $V(\Gamma)$ and an ordering of $E(\Gamma)$ and use this same pair of orderings to produce the *initial incidence matrix* $M_{\mathtt{i}_\Gamma}$ and the *terminal incidence matrix* $M_{\mathtt{t}_\Gamma}$.

Lemma 2.3 [22, Theorem 2.4.12; 31, Corollary 5.26]. $M_{\mathtt{i}_\Gamma}^\top M_{\mathtt{t}_\Gamma} = \mathbf{A}(\Gamma)$.

2.2. *Digraph homomorphism*

Let Γ and Σ be two digraphs. A *homomorphism* from Γ to Σ is a pair of maps $\phi = (\phi_0, \phi_1) : (V(\Gamma), E(\Gamma)) \to (V(\Sigma), E(\Sigma))$ for which we have

$$\mathtt{i}_\Sigma(\phi_1(e)) = \phi_0(\mathtt{i}_\Gamma(e)) \tag{2.1}$$

and

$$\mathtt{t}_\Sigma(\phi_1(e)) = \phi_0(\mathtt{t}_\Gamma(e)), \tag{2.2}$$

i.e., the diagrams in Fig. 1 commute.

A homomorphism ϕ is an *isomorphism* if both ϕ_0 and ϕ_1 are bijections. If two digraphs Γ and Σ are isomorphic, we do not distinguish between them and often just write $\Gamma = \Sigma$.

$$\begin{tikzcd} e \arrow[r, "\phi_1"] \arrow[d, "\mathbf{i}_\Gamma"'] & \phi_1(e) \arrow[d, "\mathbf{i}_\Sigma"] \\ u \arrow[r, "\phi_0"] & \phi_0(u) \end{tikzcd} \qquad \begin{tikzcd} e \arrow[r, "\phi_1"] \arrow[d, "\mathbf{t}_\Gamma"'] & \phi_1(e) \arrow[d, "\mathbf{t}_\Sigma"] \\ v \arrow[r, "\phi_0"] & \phi_0(v) \end{tikzcd}$$

Fig. 1. Digraph homomorphism.

Lemma 2.4. *$\phi = (\phi_0, \phi_1) : (V(\Gamma), E(\Gamma)) \to (V(\Sigma), E(\Sigma))$ is a digraph homomorphism from Γ to Σ if and only if it holds*

$$M_{\phi_1} M_{\mathtt{i}_\Sigma} = M_{\mathtt{i}_\Gamma} M_{\phi_0} \tag{2.3}$$

and

$$M_{\phi_1} M_{\mathtt{t}_\Sigma} = M_{\mathtt{t}_\Gamma} M_{\phi_0}. \tag{2.4}$$

Proof. Eq. (2.3) is just Eq. (2.1) while Eq. (2.4) is just Eq. (2.2). □

A digraph homomorphism ϕ from Γ to Σ is a *right covering projection* (*left covering projection*) if ϕ_0 is onto and ϕ_1 induces a bijection from $E_\Gamma^+(v)$ ($E_\Gamma^-(v)$) to $E_\Sigma^+(\phi_0(v))$ ($E_\Sigma^-(\phi_0(v))$) for each $v \in V(\Gamma)$ [22, p. 275]. We remark that the concept of right/left covering projection arose independently under different names in different contexts, including symbolic dynamics [1,18,22,25], graph divisors [7,16], and a categorical definition of fibration motivated by the study of sense of direction in the distributed algorithm field [3]. A *covering projection* ϕ from Γ to Σ is a homomorphism which is both left covering and right covering. We assert that Γ is a *cover* of Σ when there exists a covering projection ϕ from Γ to Σ. A *covering digraph* of a given digraph Σ consists of a digraph Γ together with a covering projection ϕ from Γ to Σ. Moreover, we call Γ a *k-fold cover* of Σ and ϕ a *k-fold covering projection* in case that $M(\phi_0) = \{k\}$. For any $V_0 \subseteq V(\Gamma)$, we write $\langle V_0\rangle_\Gamma$ for the subdigraph of Γ induced by V_0. For any two digraphs Γ and Σ, a *k-fold branched covering projection* ϕ from Γ to Σ with *branch index* ℓ [15, p. 174] is a homomorphism $\phi : \Gamma \to \Sigma$ such that

(i) $M(\phi_0) = \{k, \ell\}$ and we write $V_1 = \{v \in V(\Sigma) : \#\phi_0^{-1}(v) = \ell\}$;
(ii) ϕ is a covering projection when restricted to $\langle\phi_0^{-1}(V_1)\rangle_\Gamma$ and when restricted to $\langle\phi_0^{-1}(V(\Sigma)\setminus V_1)\rangle_\Gamma$;
(iii) ϕ_1 induces a bijection from $E_\Gamma^+(v)$ to $E_\Sigma^+(\phi_0(v))$ and a bijection from $E_\Gamma^-(v)$ to $E_\Sigma^-(\phi_0(v))$ for each $v \in \phi_0^{-1}(V(\Sigma)\setminus V_1)$.

We will speak of the set V_1 appeared above as the *branch set* of ϕ and Γ a k-fold *branched cover* of Σ with branch set V_1 and branch index ℓ.

Lemma 2.5. *Let Γ and Σ be two digraphs. Then $\phi : \Gamma \to \Sigma$ is a covering projection if and only if it holds*

(i) *ϕ_0 is onto*;

(ii) $M_{\mathtt{i}_\Gamma}^\top M_{\phi_1} = M_{\phi_0} M_{\mathtt{i}_\Sigma}^\top$; (2.5)

(iii) $M_{\mathtt{t}_\Gamma}^\top M_{\phi_1} = M_{\phi_0} M_{\mathtt{t}_\Sigma}^\top$. (2.6)

Proof. Eq. (2.5) (Eq. (2.6)) means that ϕ_1 induces a bijection from $E_\Gamma^+(v)$ ($E_\Gamma^-(v)$) to $E_\Sigma^+(\phi_0(v))$ ($E_\Sigma^-(\phi_0(v))$). On the other hand, Eq. (2.3) follows from Eq. (2.5) and Eq. (2.4) follows from Eq. (2.6). □

2.3. Equitable partition

Let Γ be a digraph and $\pi : V(\Gamma) = \cup_{i=1}^{r} V_i$ a partition of $V(\Gamma)$ into nonempty sets. π is *out-equitable* if for every $i, j \in [r]$ there is a number $d_{ij}^{+}(\pi)$ such that for any $u \in V_i$ we have $\#(E_{\Gamma}^{+}(u) \cap (E_{\Gamma}^{-}(V_j))) = d_{ij}^{+}(\pi)$; correspondingly, π is *in-equitable* if for every $i, j \in [r]$ there is a number $d_{ij}^{-}(\pi)$ such that for any $u \in V_j$ we have $\#(E_{\Gamma}^{-}(u) \cap (E_{\Gamma}^{+}(V_i))) = d_{ij}^{-}(\pi)$. If π is out-equitable, we define the *front divisor* Γ/π^{+} of Γ as the digraph on the vertex set $\{V_1, \dots, V_r\}$ such that the number of arcs going from V_i to V_j is $d_{ij}^{+}(\pi)$. If π is in-equitable, we then define the *rear divisor* Γ/π^{-} of Γ as the digraph on the vertex set $\{V_1, \dots, V_r\}$ such that the number of arcs going from V_i to V_j is $d_{ij}^{-}(\pi)$ [3,7,16]. A partition π which is both out-equitable and in-equitable is said to be *equitable* [27]. If A is a matrix whose lines are indexed by a set X, then for any $X_1, X_2 \subseteq X$, $A(X_1, X_2)$ stands for the submatrix of A formed by deleting those rows not indexed by elements of X_1 and those columns not indexed by elements of X_2.

We proceed to introduce three simple lemmas which, basically, can be found in [7,13,22]. The first two indicate the relationship between the adjacency matrices of a digraph and its front/rear divisor while the third one exposes that of a digraph and its one-sided cover.

Lemma 2.6. *Let Γ be a digraph and $\pi : V(\Gamma) = \cup_{i=1}^{r} V_i$ a partition.*

(i) *π is out-equitable if and only if for each $i, j \in [r]$, there exists an integer $d_{ij}^{+}(\pi)$ such that $\mathbf{A}(\Gamma)(V_i, V_j)$ has constant row sum $d_{ij}^{+}(\pi)$.*

(ii) *π is in-equitable if and only if for each $i, j \in [r]$, there exists an integer $d_{ij}^{-}(\pi)$ such that $\mathbf{A}(\Gamma)(V_i, V_j)$ has constant column sum $d_{ij}^{-}(\pi)$.*

Lemma 2.7. *Let Γ be a digraph and $\pi : V(\Gamma) = \cup_{i=1}^{r} V_r$ a partition.*

(i) *If there is a matrix B such that*

$$M_{p_\pi} B = \mathbf{A}(\Gamma) M_{p_\pi}, \tag{2.7}$$

then π is out-equitable; while if there is a matrix B such that

$$B M_{p_\pi}^{\top} = M_{p_\pi}^{\top} \mathbf{A}(\Gamma), \tag{2.8}$$

then π is in-equitable.

(ii) *If π is out-equitable, then $B = \mathbf{A}(\Gamma/\pi^{+})$ is the unique solution to Eq. (2.7) while if π is in-equitable, then $B = \mathbf{A}(\Gamma/\pi^{-})$ is the unique solution to Eq. (2.8).*

(iii) *π is out-equitable (in-equitable) if and only if Eq. (2.7) (Eq. (2.8)) holds.*

Proof. Each column of M_{p_π} corresponds to the characteristic vector of a cell of π and so each column of $M_{p_\pi} B$ takes constant value on each cell of π. Applying Eq. (2.7) then yields that for any v and w in the same cell of π the vth row and the wth row of $\mathbf{A}(\Gamma) M_{p_\pi}$ are equal. Clearly, this implies that π is out-equitable. This proves the first part of claim (i) and the other part can be proved similarly.

To prove the second reading, we first note that Lemma 2.1 says that M_{p_π} is of full column rank and so both Eqs. (2.7) and (2.8) have a unique solution. Hence it remains to check the following equalities:

$$M_{p_\pi}\mathbf{A}(\Gamma/\pi^+) = \mathbf{A}(\Gamma)M_{p_\pi}, \quad \text{if } \pi \text{ is out-equitable;} \tag{2.9}$$
$$\mathbf{A}(\Gamma/\pi^-)M_{p_\pi}^\top = M_{p_\pi}^\top\mathbf{A}(\Gamma), \quad \text{if } \pi \text{ is in-equitable.} \tag{2.10}$$

But it is easy to see that the (v, V_j)-terms of both sides of Eq. (2.9) are $d_{ij}^+(\pi)$ for $v \in V_i$, while the (V_i, w)-terms of both sides of Eq. (2.10) are $d_{ij}^-(\pi)$ for $w \in V_j$, $i, j \in [r]$. This is the result.

The third claim follows from the preceding two ones. □

Lemma 2.8. *Let Γ and Σ be two digraphs and $\phi : \Gamma \to \Sigma$ a digraph homomorphism. If ϕ is right covering, then*

$$\mathbf{A}(\Gamma)M_{\phi_0} = M_{\phi_0}\mathbf{A}(\Sigma); \tag{2.11}$$

while if ϕ is left covering, then

$$M_{\phi_0}^\top\mathbf{A}(\Gamma) = \mathbf{A}(\Sigma)M_{\phi_0}^\top.$$

Proof. Assume that ϕ is a right covering projection. We find that

$$\begin{aligned}\mathbf{A}(\Gamma)M_{\phi_0} &= M_{\mathtt{i}_\Gamma}^\top M_{\mathtt{t}_\Gamma} M_{\phi_0} \quad \text{(by Lemma 2.3)}\\ &= M_{\mathtt{i}_\Gamma}^\top M_{\phi_1} M_{\mathtt{t}_\Sigma} \quad \text{(by Eq. (2.4))}\\ &= M_{\phi_0} M_{\mathtt{i}_\Sigma}^\top M_{\mathtt{t}_\Sigma} \quad \text{(by Eq. (2.5))}\\ &= M_{\phi_0}\mathbf{A}(\Sigma) \quad \text{(by Lemma 2.3)}.\end{aligned}$$

The proof goes through in a dual way when ϕ is left covering. □

We have established some matrix equations to characterize various geometrical objects. In the following, for several classes of digraph homomorphisms, we will obtain some more matrix representations of corresponding geometrical objects, from which we are able to gain some knowledge on the corresponding characteristic polynomials.

3. One-sided cover

In this section, we will first prove that there is a one-sided covering from Γ to Σ if and only if Σ is a one-sided divisor of Γ. Then we discuss the relationship among the characteristic polynomials of a digraph, its front divisor and its rear divisor.

3.1. Divisor and cover

We recall a result in the folklore.

Lemma 3.1. *If π is an out-equitable partition of a digraph Γ, then there is a right covering projection ϕ from Γ to its front divisor Γ/π^+ such that the fiber partition $\pi(\phi_0)$ induced by ϕ_0 coincides with π. Conversely, if ϕ is a right covering projection from a digraph Γ to a digraph Σ, then $\pi(\phi_0)$ is an out-equitable partition of Γ and $\Sigma = \Gamma/\pi(\phi_0)^+$.*

Proof. The first claim is proved by constructing the right covering projection ϕ. Suppose the partition π is $V(\Gamma) = \cup_{i=1}^r V_i$. Let $\phi_0 = p_\pi$. To construct ϕ_1, since $E(\Gamma) = \cup_{\substack{v\in V(\Gamma)\\ j\in[r]}} (E_\Gamma^+(v) \cap$

$(\cup_{u\in V_j} E^-_\Gamma(u)))$, we only need to specify the map of ϕ_1 restricted to each $E^+_\Gamma(v)\cap(\cup_{u\in V_i} E^-_\Gamma(u))$, which is of size $d^+_{ij}(\pi)$ for $v\in V_i$. This is done by requiring ϕ_1 to induce a bijection from these $d^+_{ij}(\pi)$ arcs to the $d^+_{ij}(\pi)$ arcs of Γ/π^+ going from V_i to V_j. It is easy to check that $\phi=(\phi_0,\phi_1)$ is the required projection.

We now shift our attention to the second reading which comprises two assertions. We first show that $\pi(\phi_0)$ is out-equitable. For any $v,w\in V(\Sigma)$ and any $x\in\phi_0^{-1}(v)$ let $X=E^+_\Gamma(x)\cap E^-_\Gamma(\phi_0^{-1}(w))$ and $Y=E^+_\Sigma(v)\cap E^-_\Sigma(w)$. Since $\phi_1(X)\subseteq Y$, $\phi_1^{-1}(Y)\subseteq X$ and ϕ_1 induces a bijection from $E^+_\Gamma(x)$ onto $E^+_\Sigma(v)$, it follows that $\#X=\#Y$. Hence $\#X$ is independent of the choice of $x\in\phi_0^{-1}(v)$ and therefore, $\pi(\phi_0)$ is out-equitable. Now it remains to prove $\Sigma=\Gamma/\pi(\phi_0)^+$. In virtue of Lemma 2.7, it follows from the fact $\pi(\phi_0)$ is out-equitable that

$$\mathbf{A}(\Gamma)M_{p_{\pi(\phi_0)}}=M_{p_{\pi(\phi_0)}}\mathbf{A}(\Gamma/\pi(\phi_0)^+). \tag{3.1}$$

But Lemma 2.2 says $M_{p_{\pi(\phi_0)}}=M_{\phi_0}$. This along with Eqs. (2.11) and (3.1) yields that $M_{\phi_0}\mathbf{A}(\Gamma/\pi(\phi_0)^+)=M_{\phi_0}\mathbf{A}(\Sigma)$. Note that ϕ_0 is onto and hence Lemma 2.1 guarantees that M_{ϕ_0} is of full column rank. This allows us to conclude $\mathbf{A}(\Gamma/\pi(\phi_0)^+)=\mathbf{A}(\Sigma)$, as required. $\square$

3.2. Eigenvalues and eigenvectors

The next result has appeared in many places [5,7,13,16] and we list it here just for completeness.

Theorem 3.2. *If there is a right covering projection ϕ from Γ to Σ, then $\chi(\Sigma;\lambda)|\chi(\Gamma;\lambda)$.*

Proof. Since ϕ_0 is an onto map, Lemma 2.1 asserts that M_{ϕ_0} is of full column rank and thus M_{ϕ_0} represents an isomorphism from $\mathbb{C}^{V(\Sigma)}$ to $L=M_{\phi_0}\mathbb{C}^{V(\Sigma)}\leqslant\mathbb{C}^{V(\Gamma)}$. By Lemma 2.8, L is invariant under the left multiplication of $\mathbf{A}(\Gamma)$. Furthermore, the action of $\mathbf{A}(\Gamma)$ restricted on L is similar to the action of $\mathbf{A}(\Sigma)$ on $\mathbb{C}^{V(\Sigma)}$; see Fig. 2.

The above argument implies that the characteristic polynomial of $\mathbf{A}(\Sigma)$ equals that of the transformation of $\mathbf{A}(\Gamma)$ restricted to L and hence divides the characteristic polynomial of the transformation of $\mathbf{A}(\Gamma)$ on the whole space $\mathbb{C}^{V(\Gamma)}$, namely $\chi(\mathbf{A}(\Gamma);\lambda)$. $\square$

Let us proceed with a review of some classic results on lifting/projecting eigenvectors and eigenvalues [7, p. 121].

Suppose there is a right covering projection ϕ from Γ to Σ. Let $\eta\in\mathbb{C}^{V(\Sigma)}$ and $\zeta\in\mathbb{C}^{V(\Gamma)}$ such that $\zeta=M_{\phi_0}\eta$. Since M_{ϕ_0} is of full column rank (Lemma 2.1(ii)), we know that

$$\zeta\neq 0\Leftrightarrow\eta\neq 0.$$

$$\begin{tikzcd}
\mathbb{C}^{V(\Sigma)} \arrow[r, "M_{\phi_0}"] \arrow[d, "\mathbf{A}(\Sigma)"'] & L \arrow[d, "\mathbf{A}(\Gamma)"] \\
\mathbb{C}^{V(\Sigma)} \arrow[r, "M_{\phi_0}"] & L
\end{tikzcd}$$

Fig. 2. $\mathbf{A}(\Sigma)$ and $\mathbf{A}(\Gamma)$ generate similar transformations.

For any $\lambda \in \mathbb{C}$, by the fact that M_{ϕ_0} is of full column rank and that $\mathbf{A}(\Gamma)M_{\phi_0} = M_{\phi_0}\mathbf{A}(\Sigma)$ (Lemma 2.8), we can write down

$$\mathbf{A}(\Gamma)\zeta = \lambda\zeta \Leftrightarrow M_{\phi_0}(\mathbf{A}(\Sigma)\eta) = M_{\phi_0}(\lambda\eta) \Leftrightarrow \mathbf{A}(\Sigma)\eta = \lambda\eta.$$

This implies that each right eigenvector η of $\mathbf{A}(\Sigma)$ corresponding to an eigenvalue λ can be lifted to an eigenvector $\zeta = M_{\phi_0}\eta$ of $\mathbf{A}(\Gamma)$ corresponding to eigenvalue λ; conversely, for each right eigenvector ζ of $\mathbf{A}(\Gamma)$ corresponding to eigenvalue λ that lies in the column space of M_{ϕ_0}, i.e., ζ is constant on each fiber of ϕ_0, we can project it down to a right eigenvector $\eta = (M_{\phi_0}^{\top}M_{\phi_0})^{-1}M_{\phi_0}^{\top}\zeta$ of $\mathbf{A}(\Sigma)$ corresponding to eigenvalue λ.

Theorem 3.3. *Suppose there is a right covering projection ϕ from Γ to Σ and λ is an eigenvalue of Γ. Then λ is outside of the spectrum of Σ if and only if there is no right eigenvector of $\mathbf{A}(\Gamma)$ corresponding to λ which is constant on each fiber of ϕ_0.*

Proof. This follows from our discussion above on the lifting/projecting of right eigenvectors. □

We now turn to left eigenvectors. Still assume that ϕ is a right covering projection from Γ to Σ. Then, for any $\lambda \in \mathbb{C}$,

$$x\mathbf{A}(\Gamma) = \lambda x \Rightarrow x\mathbf{A}(\Gamma)M_{\phi_0} = \lambda x M_{\phi_0} \Leftrightarrow xM_{\phi_0}\mathbf{A}(\Sigma) = \lambda x M_{\phi_0}.$$

This means that each left eigenvector x of $\mathbf{A}(\Gamma)$ satisfying $xM_{\phi_0} \neq 0$ can be projected to a left eigenvector xM_{ϕ_0} of $\mathbf{A}(\Sigma)$ possessing the same eigenvalue. Recall that a *main eigenvalue* [8, p. 25] of a graph G is an eigenvalue of $\mathbf{A}(G)$ paired with an eigenvector x fulfilling $xJ \neq 0$. Now suppose that ϕ is a covering projection from the graph G to a graph H. Note that $xJ \neq 0$ implies $xM_{\phi_0} \neq 0$. This observation immediately leads to:

Corollary 3.4 [6,8, Theorem 2.4.5]. *The spectrum of any divisor of a graph G includes all those main eigenvalues of G.*

As with the lift of left eigenvectors, we can only prove the following:

Theorem 3.5. *Suppose there is a right covering projection ϕ from Γ to Σ and that $\mathbf{A}(\Gamma)$ is diagonalizable. Then $\mathbf{A}(\Sigma)$ is diagonalizable and each left eigenvector y of $\mathbf{A}(\Sigma)$ can be lifted to a left eigenvector x of $\mathbf{A}(\Gamma)$ with the same eigenvalue such that $y = xM_{\phi_0}$.*

Proof. Our assumption implies that we can take a set of linearly independent left eigenvectors of $\mathbf{A}(\Gamma)$, say $x_1, \ldots, x_n$, where $n = \#V(\Gamma)$. By Lemma 2.1(ii), M_{ϕ_0} is of full column rank. But each matrix has the same row rank and column rank. This enables us assert that $\{x_iM_{\phi_0} : i \in [n]\}$ must contain m linearly independent vectors, say $x_iM_{\phi_0}, i \in [m]$, where $m = \#V(\Sigma)$. In virtue of our preceding remarks on projecting left eigenvectors, these m vectors consist of a set of m linearly independent left eigenvectors of $\mathbf{A}(\Sigma)$, implying $\mathbf{A}(\Sigma)$ is diagonable, and the eigenvalue of $\mathbf{A}(\Sigma)$ paired with $x_iM_{\phi_0}$ equals the eigenvalue of $\mathbf{A}(\Gamma)$ paired with x_i for each $i \in [m]$. This ends the proof. □

Before leaving this subsection, we point out that what was done above for right covering projections carries over easily to left covering projections and so the details are omitted. We also mention that some interesting questions involving lifting/projecting eigenvectors can be found in [11].

3.3. Left vs. right

Lemma 3.6. *Let Γ be a digraph and $\pi : V(\Gamma) = \cup_{i=1}^{r} V_i$ an equitable partition. Then*

$$\chi(\Gamma/\pi^+, \lambda) = \chi(\Gamma/\pi^-, \lambda).$$

Proof. Let $M = M_{p_\pi}$. Then we have $\mathbf{A}(\Gamma/\pi^-)M^\top M = M^\top \mathbf{A}(\Gamma)M = M^\top M\mathbf{A}(\Gamma/\pi^+)$, where the first equality is due to Eq. (2.10) and the second comes from Eq. (2.9). Recall that Lemma 2.1 says that $M^\top M$ is invertible. This allows us to conclude that $\mathbf{A}(\Gamma/\pi^-)$ and $\mathbf{A}(\Gamma/\pi^+)$ are similar and hence have the same characteristic polynomial. □

Theorem 3.7. *Suppose that $\phi : \Gamma \to \Sigma$ is a left covering projection and $\varphi : \Gamma \to \Upsilon$ a right covering projection. Then $\chi(\Sigma, \lambda) = \chi(\Upsilon, \lambda)$ provided $\pi(\phi_0) = \pi(\varphi_0)$.*

Proof. It follows from $\pi(\phi_0) = \pi(\varphi_0)$ that $\pi = \pi(\phi_0)$ is equitable. Then Lemma 3.1 and its counterpart for left covering projection tell us that $\Sigma = \Gamma/\pi^-$ and $\Upsilon = \Gamma/\pi^+$. By now, the rest comes from Lemma 3.6. □

Observe that $\Gamma/\pi^+ = \Gamma/\pi^-$ when $d_{ij}^-(\pi) = d_{ij}^+(\pi)$ hold for all i, j, especially in the case that $\#V_i$ has a constant value for all i. But, in general, we should not expect such a result, even when Γ is symmetric.

Example 3.8. Let $\Gamma = K_3$, the complete digraph on 3 vertices without loops. Let $V(\Gamma) = \{v_1, v_2, v_3\}$, $V_1 = \{v_1\}$, $V_2 = \{v_2, v_3\}$ and $\pi : V(\Gamma) = V_1 \cup V_2$. Then π is an equitable partition of $V(\Gamma)$. It is obvious that $\mathbf{A}(\Gamma/\pi^+) = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{A}(\Gamma/\pi^-) = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$, which shows $\Gamma/\pi^+ \neq \Gamma/\pi^-$.

Question 3.9. For a given digraph Γ, can we characterize those π for which Γ/π^+ is isomorphic to Γ/π^-? Note that the relevant concept in symbolic dynamics is *finite equivalence*; c.f. [22, §8.3].

4. Cover

In this section we will provide a typical construction of a k-fold cover for a given digraph using voltage assignments. This will enable us get a standard form of the adjacency matrix of a k-fold cover.

4.1. Cover = uniform cover

A digraph Γ is (*weakly*) *connected* if for any two different vertices u and v of Γ, there is a sequence of arcs $e_0, e_1, \ldots, e_\ell$ such that for $i \in [\ell]$, e_i and e_{i-1} have at least one common endpoint and u and v are endpoints of e_0 and e_ℓ, respectively. In the sequel, we present a result in the folklore, which is a discrete analogue of the corresponding fact for covering space in topology.

Theorem 4.1. *Let Γ and Σ be two digraphs. If Σ is connected and $\phi : \Gamma \to \Sigma$ is a covering projection, then ϕ is a k-fold covering projection for some integer k.*

Proof. Since Σ is connected, our task reduces to showing that for any $e \in E(\Sigma)$ with $\mathtt{i}_\Sigma(e) = u$ and $\mathtt{t}_\Sigma(e) = v$ it holds $\#\phi_0^{-1}(u) = \#\phi_0^{-1}(v)$. Let $\phi_1^{-1}(e) = \{e_1, \ldots, e_k\}$. Since ϕ is a covering projection, on the one hand, these k arcs have different heads and different tails; on the other hand, each vertex of $\phi_0^{-1}(u)$ appears as the tail of an arc in $\phi_1^{-1}(e)$ and each vertex of $\phi_0^{-1}(v)$ appears as the head of an arc in $\phi_1^{-1}(e)$. This demonstrates that both $\phi_0^{-1}(u)$ and $\phi_0^{-1}(v)$ have cardinality k and so we are done. □

4.2. Permutation voltage digraph representation

Following largely [15], we describe in this section a way of constructing a k-fold cover for a given digraph. Note that a wide generalization in categorical terms appears in [3], where constructions of one-sided covers, called fibrations therein, are also discussed.

A *permutation voltage digraph* is a triple (Σ, S_k, α), where Σ is a digraph, S_k the symmetric group on $[k]$ and α a map which assigns to each arc $e \in E(\Sigma)$ a *permutation voltage* $\alpha(e) \in S_k$. Given a permutation voltage digraph (Σ, S_k, α), we can construct a new digraph $\Sigma \times^\alpha [k]$ by putting $V(\Sigma \times^\alpha [k]) = V(\Sigma) \times [k] = \{u_i : u \in V(\Sigma), i \in [k]\}$, $E(\Sigma \times^\alpha [k]) = E(\Sigma) \times [k] = \{e_i : e \in E(\Sigma), i \in [k]\}$, and letting the tail and head operators on $\Sigma \times^\alpha [k]$ be defined by

$$\mathtt{i}_{\Sigma \times^\alpha [k]}(e_i) = \mathtt{i}_\Sigma(e)_i, \quad \mathtt{t}_{\Sigma \times^\alpha [k]}(e_i) = \mathtt{t}_\Sigma(e)_{\alpha(e)(i)} \tag{4.1}$$

for all $i \in [k]$ and $e \in E(\Sigma)$. When Σ has a symmetric involution β, we define a voltage assignment α from $E(\Sigma)$ to S_k to be *symmetric* (with respect to β) provided for each arc $e \in E(\Sigma)$ we have $\alpha(\beta(e))^{-1} = \alpha(e)$. It is worthy noting that for a symmetric permutation voltage assignment α, the resulting digraph $\Sigma \times^\alpha [k]$ remains to be symmetric. The *natural projection* ϕ^α from $\Sigma \times^\alpha [k]$ to Σ is defined by $\phi_0^\alpha(u_i) = u$ and $\phi_1^\alpha(e_i) = e$.

Lemma 4.2. *For any digraph Σ, $\Sigma \times^\alpha [k]$ is a k-fold cover of Σ.*

Proof. Let ϕ^α be the natural projection from $\Sigma \times^\alpha [k]$ to Σ. We aim to show that ϕ^α is a k-fold covering projection. First, it is easy to check that $M(\phi_0^\alpha) = \{k\}$. Next, we see from Eq. (4.1) that ϕ^α is a homomorphism. Now consider an arbitrary vertex $u \in V(\Sigma)$ and an arbitrary $u_i \in (\phi_0^\alpha)^{-1}(u)$. By the definition of $\Sigma \times^\alpha [k]$, for each $e \in E_\Sigma^+(u)$ there exists a unique arc $e_i \in E^+_{\Sigma \times^\alpha [k]}(u_i)$ in the fiber over e and for each $f \in E_\Sigma^-(u)$ there exists a unique $f_{\alpha(f)^{-1}(i)} \in E^-_{\Sigma \times^\alpha [k]}(u_i)$ in the fiber over f. This means that ϕ^α is both a right covering projection and a left covering projection, which is the result. □

As a corollary of Lemma 4.2, we give a part answer to the question raised at the end of Section 3.

Corollary 4.3. *Let ϕ be the natural covering projection from $\Sigma \times^\alpha [k]$ to Σ. It holds*

$$\Sigma \times^\alpha [k]/\pi(\phi_0)^+ = \Sigma \times^\alpha [k]/\pi(\phi_0)^- = \Sigma.$$

Proof. From Lemma 4.2, ϕ is a both right and left covering projection. Then by Lemma 3.1, we have $\Sigma = \Sigma \times^\alpha [k]/\pi(\phi_0)^+$ and $\Sigma = \Sigma \times^\alpha [k]/\pi(\phi_0)^-$, as desired. □

$$\begin{tikzcd} \Sigma \times^{\alpha} [k] \arrow[rr, "\theta"] \arrow[dr, "\phi^{\alpha}"'] & & \Gamma \arrow[dl, "\phi"] \\ & \Sigma & \end{tikzcd}$$

Fig. 3. $\phi^{\alpha} = \phi \circ \theta$.

Going the other way, as illustrated in Fig. 3, we have

Lemma 4.4. *Let Γ and Σ be two digraphs and $\phi : \Gamma \to \Sigma$ a k-fold covering projection. Then there is a permutation voltage assignment $\alpha : E(\Sigma) \to S_k$ such that $\Sigma \times^{\alpha} [k]$ is isomorphic to Γ by an isomorphism θ and the natural covering projection ϕ^{α} from $\Sigma \times^{\alpha} [k]$ to Σ satisfies $\phi^{\alpha} = \phi \circ \theta$.*

Proof. Assume that ϕ is the covering projection from Γ to Σ. For any vertex $u \in V(\Sigma)$, label the k vertices in $\phi_0^{-1}(u)$ arbitrarily as $u_1, \ldots, u_k$. Recall that ϕ_1 induces a bijection from $E_{\Gamma}^{+}(u_i)$ to $E_{\Sigma}^{+}(u)$ for each u_i in the fiber over u. This implies that for any $e \in E_{\Sigma}^{+}(u)$ the arcs in $\phi_1^{-1}(e)$ originate at different vertices in the fiber over u. Therefore we can label the only arc in $\phi_1^{-1}(e)$ originating at u_i by e_i.

Let $e \in E(\Sigma)$ and $\mathtt{t}_{\Sigma}(e) = v$. Because ϕ is a left covering projection, each $e_i \in \phi_1^{-1}(e)$ terminates at different $v_j \in \phi_0^{-1}(v)$. Thus we can take the permutation $\alpha(e) \in S_k$ such that $\alpha(e)(i) = j$ whenever $\mathtt{t}_{\Gamma}(e_i) = v_j$.

It is straightforward to check that $\Sigma \times^{\alpha} [k]$ is the digraph obtained from Γ by giving each element of Γ a new labelling defined as above. This means there is an isomorphism θ from $\Sigma \times^{\alpha} [k]$ to Γ. Denote the vertices and the arcs in $\Sigma \times^{\alpha} [k]$ by (u, i) and (e, i), $u \in V(\Sigma)$, $e \in E(\Sigma)$, $i \in [k]$, respectively. Then by the definition of the new labelling, we have $\phi_0(\theta_0(u, i)) = u = \phi_0^{\alpha}(u_i)$ and $\phi_1(\theta_1(e, i)) = e = \phi_1^{\alpha}(e_i)$ for all $u \in V(\Sigma)$ and $e \in E(\Sigma)$. Thus we are done. $\square$

For any digraph Γ and any $E_1 \subseteq E(\Gamma)$, we denote by $\langle E_1 \rangle_{\Gamma}$ the digraph with $V(\langle E_1 \rangle_{\Gamma}) = V(\Gamma)$ and $E(\langle E_1 \rangle_{\Gamma}) = E_1$. The *transpose* of a digraph Γ, denoted $\overleftarrow{\Gamma}$, is the digraph obtained from Γ by reversing all arcs. If Γ and Σ are two digraphs on the same vertex set V, we define the *union* of Γ and Σ to be the digraph $(V, E(\Gamma) \cup E(\Sigma))$, where $E(\Gamma)$ and $E(\Sigma)$ are viewed as disjoint sets.

The next result resembles the classical result of Gross and Tucker [14,15]. The difference is that we are considering a digraph homomorphism from a symmetric digraph to another symmetric digraph which is not necessarily compatible with any involutions of the digraphs.

Theorem 4.5. *Let Γ and Σ be two symmetric digraphs and β a symmetric involution of Σ. If no loops of Σ are fixed by β and Γ is a k-fold cover of Σ, then there exists a permutation voltage assignment $\alpha : E(\Sigma) \to S_k$ which is symmetric with respect to β such that $\Sigma \times^{\alpha} [k] = \Gamma$.*

Proof. Assume that ϕ is a k-fold covering projection from Γ to Σ.

For any $u, v \in V(\Sigma)$, we denote by $E_{u,v}$ the set of arcs of Σ which connect u to v. As no loops are fixed by the involution β, each $E_{u,u}$ is divided into two parts, $L_{u,1}$ and $L_{u,2}$, such that $L_{u,2} = \beta(L_{u,1})$. Let $L_1 = \cup_{u \in V(\Sigma)} L_{u,1}$ and $L_2 = \cup_{u \in V(\Sigma)} L_{u,2}$. With any fixed linear ordering $<$ of $V(\Sigma)$, we define $E_1 = \cup_{u<v} E_{u,v} \cup L_1$ and $E_2 = \beta(E_1) = \cup_{u>v} E_{u,v} \cup L_2$. Then we have $E(\Sigma) = E_1 \cup E_2$. Correspondingly, it holds $E(\Gamma) = E_1' \cup E_2'$, where $E_1' = \phi_1^{-1}(E_1)$ and $E_2' = \phi_1^{-1}(E_2)$. Observe that $\Sigma_1 = \overleftarrow{\Sigma_2}$ and $\Gamma_1 = \overleftarrow{\Gamma_2}$, where $\Sigma_i = \langle E_i \rangle_{\Sigma}$ and $\Gamma_i = \langle E_i' \rangle_{\Gamma}$, $i = 1, 2$.

Note that ϕ induces a k-fold covering from Γ_1 to Σ_1. Therefore, Lemma 4.4 applies to give a voltage assignment $\alpha_1 : E_1 \to S_k$ such that

$$\Sigma_1 \times^{\alpha_1} [k] = \Gamma_1. \tag{4.2}$$

Define $\alpha_2 = \alpha_1^{-1} : E_2 \to S_k$ by $\alpha_1^{-1}(e) = \alpha_1(\beta(e))^{-1}$ for each $e \in E_2$. We now arrive at a symmetric voltage assignment α of Σ with respect to β by requiring

$$\alpha(e) = \begin{cases} \alpha_1(e), & \text{if } e \in E_1, \\ \alpha_2(e), & \text{if } e \in E_2. \end{cases}$$

It remains to check that for this symmetric voltage assignment α (with respect to β), we have $\Sigma \times^{\alpha} [k] = \Gamma$. However, it clearly holds

$$\Sigma = \Sigma_1 \cup \Sigma_2 \quad \text{and} \quad \Gamma = \Gamma_1 \cup \Gamma_2. \tag{4.3}$$

Furthermore, making use of Eq. (4.2), we obtain

$$\Sigma_2 \times^{\alpha_2} [k] = \overleftarrow{\Sigma_1} \times^{\alpha_1^{-1}} [k] = \overleftarrow{\Sigma_1 \times^{\alpha_1} [k]} = \overleftarrow{\Gamma_1} = \Gamma_2. \tag{4.4}$$

At this moment, we can see that the result follows from the combination of Eqs. (4.2), (4.3) and (4.4). $\square$

We conclude this subsection by pointing out that it is vital for the validity of Theorem 4.5 that the given symmetric involution has no fixed loops.

Example 4.6. As depicted in Fig. 4, Γ is a 3-regular symmetric digraphs (each undirected edge corresponding to a pair of arcs with opposite directions) and Σ is a symmetric digraphs consisting of one vertex and 3 loops. By Hall's Marriage theorem, we know that Γ is a 22-fold cover of Σ. Assume that there is a permutation voltage assignment $\alpha : E(\Sigma) \to S_{22}$ which is symmetric with respect to a symmetric involution β of Σ such that $\Sigma \times^{\alpha} [k] = \Gamma$. Note that β must fix a loop of Σ. Clearly, this loop is lifted to be a perfect matching in Γ. It is easy to check that Γ does not possess any perfect matching, yielding a contradiction.

4.3. Adjacency matrix representation

We continue to search for a suitable matrix representation for a k-fold cover. For this purpose, we have to prepare some more notations. Let Γ and Σ be two digraphs. The *tensor product* $\Gamma \otimes \Sigma$

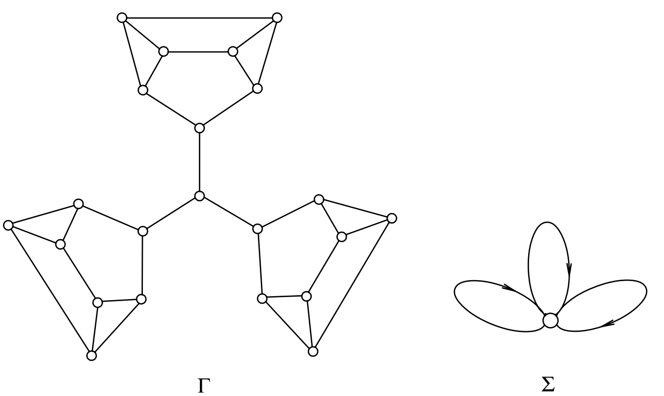

Fig. 4. There is no symmetric assignment $\alpha : E(\Sigma) \to S_{22}$ such that $\Sigma \times^{\alpha} [22] = \Gamma$.

of them is defined to be the digraph $(V(\Gamma) \times V(\Sigma),\ E(\Gamma) \times E(\Sigma))$ whose head and tail operators are given by $\mathtt{i}_{\Gamma\otimes\Sigma}(e_1, e_2) = (\mathtt{i}_\Gamma(e_1), \mathtt{i}_\Sigma(e_2))$ and $\mathtt{t}_{\Gamma\otimes\Sigma}(e_1, e_2) = (\mathtt{t}_\Gamma(e_1), \mathtt{t}_\Sigma(e_2))$, respectively. It is obvious that

$$\mathbf{A}(\Gamma \otimes \Sigma) = \mathbf{A}(\Gamma) \otimes \mathbf{A}(\Sigma) \quad \text{and} \quad \mathbf{A}(\Gamma \cup \Sigma) = \mathbf{A}(\Gamma) + \mathbf{A}(\Sigma). \tag{4.5}$$

Let Γ be a k-fold cover of a digraph Σ. By Lemma 4.4, $\Gamma = \Sigma \times^\alpha [k]$ for some permutation voltage assignment $\alpha : E(\Sigma) \to S_k$. Put $G = \langle\{\alpha(e) : e \in E(\Sigma)\}\rangle$, the subgroup of S_k generated by $\{\alpha(e) : e \in E(\Sigma)\}$. For each $g \in G$, let P_g be the $k \times k$ matrix such that

$$P_g(i, j) = \delta_{g(i),j}, \tag{4.6}$$

where δ is the Kronecker Delta. For $\alpha : E(\Sigma) \to S_k$, let $\Sigma_{\alpha,g}$ be the digraph with $V(\Sigma_{\alpha,g}) = V(\Sigma)$ and $E(\Sigma_{\alpha,g}) = \{e \in E(\Sigma) : \alpha(e) = g\}$. It is a simple observation that

$$\Sigma = \cup_{g\in G}\Sigma_{\alpha,g}. \tag{4.7}$$

With the aid of Eqs. (4.5), (4.6) and (4.7), we are well prepared to assert.

Lemma 4.7. *If* $\Gamma = \Sigma \times^\alpha [k]$, *then* $\mathbf{A}(\Gamma) = \sum_{g\in G} P_g \otimes \mathbf{A}(\Sigma_{\alpha,g})$.

5. Branched cover with branch index one

We discuss in this section the representations of several objects related to branched covers with branch index 1. Since a cover is just a branched cover with branch set being the empty set, it is natural that we will follow the same line of study as in Section 4.

5.1. Permutation voltage digraph representation

We begin with a construction of a branched cover with branch index 1 of a given digraph Σ. Let $V_1 \subseteq V(\Sigma)$, $V_2 = V(\Sigma)\backslash V_1$ and $E_{ij} = \{e \in E(\Sigma) : \mathtt{i}_\Sigma(e) \in V_i,\ \mathtt{t}_\Sigma(e) \in V_j\}$, $i, j = 1, 2$. For any permutation voltage assignment $\alpha : E(\langle V_2\rangle_\Sigma) \to S_k$, we construct the digraph $\Sigma \times^{\alpha,V_1} [k]$ with respect to Σ, V_1 and α such that $V(\Sigma \times^{\alpha,V_1} [k]) = (V_2 \times [k]) \cup V_1$ and $E(\Sigma \times^{\alpha,V_1} [k]) = (\cup_{i,j=1,2,ij\neq 11} E_{ij} \times [k]) \cup E_{11}$. It is a simple observation about $\Gamma = \Sigma \times^{\alpha,V_1} [k]$ that for each $e_i \in E_{22} \times [k]$, it holds $\mathtt{i}_\Gamma(e_i) = \mathtt{i}_\Sigma(e)_i$ and $\mathtt{t}_\Gamma(e_i) = \mathtt{t}_\Sigma(e)_{\alpha(e)(i)}$; for each $e_i \in E_{12} \times [k]$, $\mathtt{i}_\Gamma(e_i) = \mathtt{i}_\Sigma(e)$ and $\mathtt{t}_\Gamma(e_i) = \mathtt{t}_\Sigma(e)_i$; and for each $e_i \in E_{21} \times [k]$, $\mathtt{i}_\Gamma(e_i) = \mathtt{i}_\Sigma(e)_i$ and $\mathtt{t}_\Gamma(e_i) = \mathtt{t}_\Sigma(e)$.

Lemma 5.1. *Let Σ be a digraph, $V_1 \subseteq V(\Sigma)$ and α a permutation voltage assignment from $E(\langle V_2\rangle_\Sigma)$ to S_k. Then the digraph $\Sigma \times^{\alpha,V_1} [k]$ is a k-fold branched cover of Σ with branch set V_1 and branch index* 1.

Proof. Immediate from the definition of $\Sigma \times^{\alpha,V_1} [k]$. □

Lemma 5.2. *Let $\phi : \Gamma \to \Sigma$ be a k-fold branched covering projection with branch index* 1 *and branch set V_1. Then there exists a permutation voltage assignment $\alpha : E(\langle V_2\rangle_\Sigma) \to S_k$ such that $\Sigma \times^{\alpha,V_1} [k] = \Gamma$, where $V_2 = V(\Sigma)\backslash V_1$.*

Proof. We prove the result by establishing an isomorphism ψ from $\Sigma \times^{\alpha,V_1} [k]$ to Γ. First note that it follows readily from the definition of $\Sigma \times^{\alpha,V_1} [k]$ that $\langle\phi_0^{-1}(V_2)\rangle_\Gamma$ is a k-fold cover of

$\langle V_2\rangle_\Sigma$. Thus, Lemma 4.4 says that there is a permutation voltage assignment $\alpha : E(\langle V_2\rangle_\Sigma) \to S_k$ such that $\langle V_2\rangle_\Sigma \times^\alpha [k] = \langle \phi_0^{-1}(V_2)\rangle_\Gamma$. Moreover, it is easy to check that $\langle V_1\rangle_{\Sigma\times^{\alpha,V_1}[k]} = \langle V_1\rangle_\Sigma = \langle \phi_0^{-1}(V_1)\rangle_\Gamma$. At this stage we can require the homomorphism ψ induces the asserted isomorphism from $\langle V_2\rangle_\Sigma \times^\alpha [k]$ to $\langle \phi_0^{-1}(V_2)\rangle_\Gamma$ and induces the other asserted isomorphism from $\langle V_1\rangle_{\Sigma\times^{\alpha,V_1}[k]}$ to $\langle \phi_0^{-1}(V_1)\rangle_\Gamma$. This then defines ψ_0 completely as well as ψ_1 incompletely.

Let $E_{ij} = \{e \in E(\Sigma) : \mathtt{i}_\Sigma(e) \in V_i, \mathtt{t}_\Sigma(e) \in V_j\}$, $i, j = 1, 2$. To finish the construction of the required isomorphism ψ, we still need to specify the mapping ψ_1 when restricted on $(E_{12} \cup E_{21}) \times [k]$. However, due to the structure of a k-fold branched cover, there is a unique arc $\tilde{e}_i$ in $\phi_1^{-1}(e)$ such that it holds

$$\begin{cases} \mathtt{t}_\Gamma(\tilde{e}_i) = \psi_0(\mathtt{t}_\Sigma(e)_i), & \text{if } e \in E_{12},\\ \mathtt{i}_\Gamma(\tilde{e}_i) = \psi_0(\mathtt{i}_\Sigma(e)_i), & \text{if } e \in E_{21}. \end{cases}$$

Accordingly, for each $e \in E_{12} \cup E_{21}$ and $i \in [k]$, we let ψ_1 map $e_i \in E(\Sigma \times^{\alpha,V_1} [k])$ to $\tilde{e}_i \in E(\Gamma)$. This then completes the construction of the pair of maps $\psi = (\phi_0, \phi_1)$.

It is not hard to verify that ψ is a required isomorphism and hence the theorem is proved. □

5.2. *Adjacency matrix representation*

We now deduce a block form of the adjacency matrix of a branched cover of the base digraph with branch index 1.

Let $V_1 = \{v^1, \ldots, v^{n_1}\}$ and $V_2 = V(\Sigma) \setminus V_1 = \{u^1, \ldots, u^{n_2}\}$. Then we can write out

$$\mathbf{A}(\Sigma) = \begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22} \end{pmatrix}, \tag{5.1}$$

where $A_{ij} = \mathbf{A}(\Sigma)(V_i, V_j),\ i, j = 1, 2$.

By Lemma 5.2, if Γ is a k-fold branched cover of Σ with a branch index 1, then $\Gamma = \Sigma \times^{\alpha,V_1} [k]$ for some permutation voltage assignment α and some $V_1 \subseteq V(\Sigma)$. For each $g \in S_k$, put $A_{22}(\alpha; g) = \mathbf{A}((\langle V_2\rangle_\Sigma)_{\alpha,g})$. Enumerate the vertices of Γ as $v^1, \ldots, v^{n_1}; u_1^1, \ldots, u_1^{n_2}; \ldots; u_k^1, \ldots, u_k^{n_2}$ in that order, where $V_1 = \{v^1, \ldots, v^{n_1}\}$ and $V(\Sigma) \setminus V_1 = \{u^1, \ldots, u^{n_2}\}$. Let ϕ^{α,V_1} be the *natural branched projection* from Γ to Σ such that $\phi_0^{\alpha,V_1} : v \mapsto v,\ u_i \mapsto u$ for all $v \in V_1$ and $u \in V(\Sigma) \setminus V_1$ and $\phi_1^{\alpha,V_1} : e \mapsto e,\ f_i \mapsto f$ for all $e \in E(\langle V_1\rangle_\Gamma)$ and $f_i \in E(\Gamma) \setminus E(\langle V_1\rangle_\Gamma)$. Following the notation above and those in Section 4 we are ready to state:

Lemma 5.3. $\mathbf{A}(\Gamma) = \begin{pmatrix} A_{11} & J_k \otimes A_{12}\\ J_k^\top \otimes A_{21} & \sum_{g\in G} P_g \otimes A_{22}(\alpha; g) \end{pmatrix}.$

Proof. First, Lemma 4.4 tells us that $\mathbf{A}(\Gamma)(V_2 \times [k], V_2 \times [k]) = \mathbf{A}(\langle V_2 \times [k]\rangle_\Gamma) = \mathbf{A}(\langle V_2\rangle_\Sigma \times^\alpha [k])$, which in turn is just $\sum_{g\in G} P_g \otimes A_{22}(\alpha; g)$ according to Lemma 4.7. It is also immediate that $\mathbf{A}(\Gamma)(V_1, V_1) = \mathbf{A}(\langle V_1\rangle_\Gamma) = \mathbf{A}(\langle V_1\rangle_\Sigma) = A_{11}$, taking into account that the natural branched projection from Γ to Σ restricted on $\langle V_1\rangle_\Gamma$ is an isomorphism.

It remains to have a look at those $(V_1, V_2 \times \{j\})$-blocks and $(V_2 \times \{j\}), V_1)$-blocks, $j \in [k]$. We only consider the former ones below as the latter ones can be treated similarly. From the definition of $\Sigma \times^{\alpha,V_1} [k]$, for each $v \in V_1,\ u \in V_2$ and $e \in E_\Sigma^+(v) \cap E_\Sigma^-(u)$, the k arcs in $e \times [k] \subseteq E_\Gamma^+(v) \cap E_\Gamma^-(u \times [k])$ have different tails. Henceforth, we have $\#(E_\Sigma^+(v) \cap E_\Sigma^-(u)) = \#(E_\Gamma^+(v) \cap E_\Gamma^-(u_j))$ for any $v \in V_1,\ u \in V_2$ and $j \in [k]$. This is nothing but $\mathbf{A}(\Sigma)(v, u) =$

$\mathbf{A}(\Gamma)(v, u_j)$. Consequently, we arrive at $\mathbf{A}(\Gamma)(V_1, V_2 \times \{j\}) = \mathbf{A}(\Sigma)(V_1, V_2) = A_{12}$ for all $j \in [k]$ and hence the proof is complete. $\square$

Note that a k-fold branched cover of a digraph usually fails to be a cover of it. In light of Theorem 3.2, we establish here an adjacency matrix representation for the front divisor and rear divisor of a branched cover.

Lemma 5.4. *Let Γ be a branched cover of Σ such that $\Gamma = \Sigma \times^{\alpha, V_1} [k]$ for some $V_1 \subseteq V(\Sigma)$ and permutation voltage assignment $\alpha : E(\langle V_2 \times [k]\rangle_\Sigma)$, where $V_2 = V(\Sigma) \setminus V_1$. Let $\phi = \phi^{\alpha, V_1}$, the natural branched projection from Γ to Σ. Then*

$$\mathbf{A}(\Gamma/\pi(\phi)^+) = \begin{pmatrix} A_{11} & kA_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \textit{and} \quad \mathbf{A}(\Gamma/\pi(\phi)^-) = \begin{pmatrix} A_{11} & A_{12} \\ kA_{21} & A_{22} \end{pmatrix}, \tag{5.2}$$

where $A_{ij},\ i, j = 1, 2$ are the same as in Eq. (5.1).

Proof. We just prove the first equation. The proof for the second one is similar.

Recall that $V(\Gamma/\pi(\phi)^+) = V(\Sigma)$. Accordingly, we have $\mathbf{A}(\Gamma/\pi(\phi)^+)(V_1, V_1) = A_{11}$, because ϕ restricted on $\langle V_1\rangle_\Gamma$ is an isomorphism, and have $\mathbf{A}(\Gamma/\pi(\phi)^+)(V_2, V_2) = A_{22}$, as a result of Corollary 4.3. To end the proof, we still need to have a look at $\mathbf{A}(\Gamma/\pi(\phi)^+)(V_1, V_2)$ and $\mathbf{A}(\Gamma/\pi(\phi)^+)(V_2, V_1)$.

By the definition of a k-fold cover, ϕ_1 induces bijections from $E^+_\Gamma(u_j)$ to $E^+_\Sigma(u)$ and from $E^-_\Gamma(u_j)$ to $E^-_\Sigma(u)$ for each $u_j \in V_2 \times [k]$. Hence for any $v \in V_1$ and $u \in V_2$,

$$\begin{aligned}\mathbf{A}(\Gamma/\pi(\phi)^+)(v, u) &= \#\{e \in E(\Gamma) : \mathtt{i}_\Gamma(e) = v,\ \mathtt{t}_\Gamma(e) \in \phi_0^{-1}(u)\} \\ &= k\#\{e \in E(\Sigma) : \mathtt{i}_\Sigma(e) = v,\ \mathtt{t}_\Sigma(e) = u\},\end{aligned}$$

which implies that $\mathbf{A}(\Gamma/\pi(\phi)^+)(V_1, V_2) = kA_{12}$. On the other hand, by the definition of the out-equitable partition, it holds

$$\begin{aligned}\mathbf{A}(\Gamma/\pi(\phi)^+)(u, v) &= \#\{e \in E(\Gamma) : \mathtt{i}_\Gamma(e) = u_i,\ \mathtt{t}_\Gamma(e) = v\} \\ &= \#\{e \in E(\Sigma) : \mathtt{i}_\Sigma(e) = u,\ \mathtt{t}_\Sigma(e) = v\}\end{aligned}$$

for any $v \in V_1$, $u \in V_2$ and any $u_i \in \phi_0^{-1}(u)$. This then yields that $\mathbf{A}(\Gamma/\pi(\phi)^+)(V_2, V_1) = A_{21}$, completing the proof. $\square$

A direct consequence of Lemma 5.4 is:

Corollary 5.5. *We follow the notation used in Lemma 5.4. It occurs $\Gamma/\pi(\phi)^+ = \Gamma/\pi(\phi)^-$ in case that $A_{11} = A_{22}$ and $A_{21} = A_{12}$.*

6. Decomposition formula for characteristic polynomial

Our main result in this final section is a decomposition formula for the characteristic polynomial of a branched cover with branch index 1. To achieve it, we will first prepare a lemma on group representations; while to illustrate the use of the decomposition formula, we present an example at the end of this paper.

6.1. Preliminary facts on group representation

Before going further, let us recall some basic facts about linear representation of finite groups. Let H be a finite group acting on a finite set X and let W be a vector space over the field of complex numbers having a basis $\{e_x : x \in X\}$ indexed by the elements of X. The *permutation representation* ρ of H associated with X represents each $h \in H$ by the matrix $\rho(h)$ satisfying $\rho(h)(e_x) = e_{h(x)}$ for all $h \in H$ [29, 1.2 (c)]. Note that $\rho(h)$ can be viewed as a linear map on W in a natural way. Suppose that there are m_1 orbits of H on X, say $\mathscr{O}_1, \dots, \mathscr{O}_{m_1}$. Then W turns out to be a direct sum of the invariant subspaces of ρ, $W = W_1 \oplus \cdots \oplus W_{m_1}$, where W_i has the basis $\{e_x : x \in \mathscr{O}_i\}$, $i \in [m_1]$. We write ρ^{W_i} for the subrepresentation of ρ restricted to the subspace W_i. It thus follows

$$\rho = \rho^{W_1} \oplus \cdots \oplus \rho^{W_{m_1}}. \tag{6.1}$$

Moreover, we mention that the permutation representation ρ is just a group homomorphism from H to S_X, i.e., the action of H on X is a lifting of the action of some subgroup of S_X on X [17, Chap. 17]. Consequently, we can assume that $\rho_1 = 1, \rho_2, \dots, \rho_s$ form a complete set of inequivalent irreducible representations of H obtained as liftings of those of S_X, the symmetric group on X, and write

$$\begin{aligned}\rho &= (m_{11} \circ 1 \oplus \cdots \oplus m_{1s} \circ \rho_s) \oplus \cdots \oplus (m_{m_1 1} \circ 1 \oplus \cdots \oplus m_{m_1 s} \circ \rho_s)\\ &= \left(\sum_{i=1}^{m_1} m_{i1}\right) \circ 1 \oplus \cdots \oplus \left(\sum_{i=1}^{m_1} m_{is}\right) \circ \rho_s,\end{aligned} \tag{6.2}$$

where

$$\rho^{W_i} = m_{i1} \circ 1 \oplus \cdots \oplus m_{is} \circ \rho_s, \ i \in [m_1],$$

and $t \circ A$ means $\underbrace{A \oplus \cdots \oplus A}_{t}$. Observe that $m_{ij} = 0$ if ρ_i is not any lifting of an irreducible representation of $S_{\mathscr{O}_j} = S_{\#\mathscr{O}_j}$. On the one hand, Cauchy-Frobenius Lemma [26] tells us that the average number of fixed points of elements of H is equal to the number of orbits of it. On the other hand, by the orthogonality relations of characters, we conclude that the average number of fixed points of elements of H is nothing but $\sum_{i=1}^{m_1} m_{i1}$. As H acts transitively on each orbit, the previous argument leads to:

Lemma 6.1 [29, Exercise 2.6(a)]. $m_{i1} = 1$ *for* $i \in [m_1]$. *Consequently*, $\sum_{i=1}^{m_1} m_{i1}$ *equals to* m_1, *the number of orbits of* H *acting on* X.

Letting $m_j = \sum_{i=1}^{m_1} m_{ij}, \ j = 2, \dots, s$, Lemma 6.1 enables us rewrite Eq. (6.2) as

$$\rho = m_1 \circ \rho_1 \oplus m_2 \circ \rho_2 \oplus \cdots \oplus m_s \circ \rho_s. \tag{6.3}$$

In matrix form, this means that there exists a nonsingular matrix U such that for all $h \in H$,

$$U^{-1}\rho(h)U = m_1 \circ \rho_1(h) \oplus m_2 \circ \rho_2(h) \oplus \cdots \oplus m_s \circ \rho_s(h). \tag{6.4}$$

A basic result in group representation theory is that we can require all $\rho_i(h)$ to be unitary matrices and we will make this assumption below.

For the purpose of establishing our decomposition formula in the next subsection, we will be concerned with the possibility of taking the matrix U to be of some special form. We start from the case that the given group action is transitive, for which we follow the proof of [9, Lemma 4.2].

Lemma 6.2. *If the permutation representation ρ corresponds to a transitive group action of H on a set X of size k, then the matrix U in Eq.* (6.4) *can be chosen to satisfy*

$$J_k U = \left(\sqrt{k}\, 0\ \cdots\ 0\right) \quad \text{and} \quad U^{-1} J_k^{\top} = \left(\sqrt{k}\, 0\ \cdots\ 0\right)^{\top}. \tag{6.5}$$

Proof. Assume that X has k elements and the ith line of $\rho(h)$, $h \in H$, corresponds to $x_i \in X$.

Write $Q(h) = \oplus_{i=1}^{s} m_i \circ \rho_i(h)$ for $h \in H$. Then Eq. (6.4) becomes

$$\rho(h)U = UQ(h), \quad \forall h \in H. \tag{6.6}$$

The form of $Q(h)$ and the fact that $\rho_1 = 1$ implies that $(UQ(h))(1,1) = U(1,1)$ for all $h \in H$. Moreover, recall that $\rho(h)(1,j) = 1$ if $x_j = h(x_1)$ and 0 otherwise and thus we derive $(\rho(h)U)(1,1) = U(j,1)$ where $x_j = h(x_1)$. We can now derive from Eq. (6.6) that $U(j,1) = U(1,1)$ if there is $h \in H$ which sends x_1 to x_j. Since the action of H on X is transitive, this tells us $U(j,1) = U(1,1)$ for all $j \in [k]$. Considering that U is nonsingular, we know that there is a complex number $c \neq 0$ such that the first column of U is $\frac{c}{\sqrt{k}}(11\cdots 1)^{\top}$.

Observe that for any $h \in H$ we have assumed that $Q(h)$ is a unitary matrix and it is clear from definition that $\rho(h)$ is a permutation matrix and hence a unitary matrix as well. By dint of this and Eq. (6.6), for any $h \in H$ we obtain

$$U^*UQ(h) = U^*\rho(h)U = U^*(\rho(h)^{-1})^*U = (\rho(h)^{-1}U)^*U = (UQ(h)^{-1})^*U = Q(h)U^*U.$$

This says that U^*U commutes with $Q(h)$ for all $h \in H$. Since we know already that $m_1 = 1$, $\rho_1 = 1$ and ρ_1 is inequivalent to ρ_i, $i \geqslant 2$, we then deduce from Schur's lemma that for any $i > 1$

$$(U^*U)(1,i) = (U^*U)(i,1) = 0. \tag{6.7}$$

Along with what we obtained in the preceding paragraph, Eq. (6.7) then yields $JU = \left(c\sqrt{k}\, 0\ \cdots\ 0\right)$.

We can repeat the above process for U^{-1} to find that $U^{-1}J^{\top} = \left(c^{-1}\sqrt{k}\, 0\ \cdots\ 0\right)^{\top}$.

Because Eq. (6.4) is still valid after substituting $c^{-1}U$ for U, choose $c^{-1}U$ as the new U if necessary and then we are finished. □

Keeping the hypothesis and notation used before Lemma 6.2 in this section, we deduce from Lemma 6.2 the result for a general group action.

Lemma 6.3. *If ρ has a decomposition as given in Eq.* (6.3), *then the matrix U in Eq.* (6.4) *can be chosen to satisfy*

$$J_k U = \left(\sqrt{k_1}\, \sqrt{k_2}\ \cdots\ \sqrt{k_{m_1}}\, 0\ \cdots\ 0\right) \quad \text{and} \quad U^{-1} J_k^{\top} = \left(\sqrt{k_1}\, \sqrt{k_2}\ \cdots\ \sqrt{k_{m_1}}\, 0\ \cdots\ 0\right)^{\top},$$

where $k = \#X$ and $k_i = \dim W_i$, $i \in [m_1]$.

Proof. By Lemma 6.2, for each $i \in [m_1]$, there is a nonsingular matrix U_i such that for all $h \in H$

$$U_i^{-1}\rho^{W_i}(h)U_i = 1 \circ \rho_1(h) \oplus \cdots \oplus m_{is} \circ \rho_s(h). \tag{6.8}$$

Let $U_0 = U_1 \oplus \cdots \oplus U_{m_1}$. Then we have

$$U_0^{-1}\rho(h)U_0 = \oplus_{i=1}^{m_1}(1 \oplus m_{i2} \circ \rho_2(h) \oplus \cdots \oplus m_{is} \circ \rho_s(h)),$$

$$J_k U_0 = \left(\underbrace{\sqrt{k_1}\, 0 \cdots 0}_{k_1} \underbrace{\sqrt{k_2}\, 0 \cdots 0}_{k_2} \cdots \underbrace{\sqrt{k_{m_1}} 0 \cdots 0}_{k_{m_1}} \right)$$

and

$$U_0^{-1} J_k^\top = \left(\underbrace{\sqrt{k_1}\, 0 \cdots 0}_{k_1}\ \underbrace{\sqrt{k_2}\, 0 \cdots 0}_{k_2} \cdots \underbrace{\sqrt{k_{m_1}}\, 0 \cdots 0}_{k_{m_1}} \right)^\top.$$

Set T to be the matrix obtained from I_k by an interchange of its $(i+1)$th column and its $(k_1 + \cdots + k_i + 1)$th column for $i \in [m_1]$. The matrix $U = U_0 T$ is what we want. $\square$

6.2. Main result

We are now equipped with all the tools to calculate the characteristic polynomial of a k-fold branched cover $\Gamma = \Sigma \times^{\alpha, V_1} [k]$ with branch index 1. Take $H_\alpha = \langle \{\alpha(e) : e \in E(\Sigma)\} \rangle \leqslant S_k$ and call it the *voltage group* of α. Let ρ be the permutation representation of H_α associated with $[k]$ and $\rho_1 = 1, \rho_2, \ldots, \rho_s$ all the inequivalent irreducible representations of H_α. Assume that $\rho = m_1 \circ \rho_1 \oplus m_2 \circ \rho_2 \oplus \cdots \oplus m_s \circ \rho_s$. Here comes our main result, which somehow highlights the role of the base set and of the voltage group action, and separates clearly the contribution of each irreducible component of the representation.

Theorem 6.4. *Let $\Gamma = \Sigma \times^{\alpha, V_1} [k]$. Then*

$$\chi(\Gamma; \lambda) = \chi(\Gamma/\pi(\phi_0)^+; \lambda)[\det(\lambda I - A_{22})]^{m_1 - 1} \times \prod_{i=2}^{s} \left[\det \left(\lambda I - \sum_{g \in G} \rho_i(g) \otimes A_{22}(\alpha; g) \right) \right]^{m_i},$$

where $\phi = \phi^{\alpha, V_1}$, $A_{22}(\alpha; g) = \mathbf{A}((\langle V_2 \rangle_\Sigma)_{\alpha, g})$ and $V_2 = V(\Sigma) \backslash V_1$ (Note that Lemma 3.6 says that $\chi(\Gamma/\pi(\phi_0)^+; \lambda) = \chi(\Gamma/\pi(\phi_0)^-; \lambda)$).

Proof. Assume that the sizes of the m_1 orbits of H on $[k]$ are $k_1, \ldots, k_{m_1}$, respectively. Let $n_i = \#V_i, i = 1, 2$, $T = \begin{pmatrix} I_{n_1} & \\ & U \otimes I_{n_2} \end{pmatrix}$, where U is as in Lemma 6.3, and

$$D = \begin{pmatrix} A_{11} & \sqrt{k_1} A_{12} & \sqrt{k_2} A_{12} & \cdots & \sqrt{k_{m_1}} A_{12} \\ \sqrt{k_1} A_{21} & A_{22} & 0 & \cdots & 0 \\ \sqrt{k_2} A_{21} & 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{k_{m_1}} A_{21} & 0 & 0 & \cdots & A_{22} \end{pmatrix}.$$

Note that the matrix P_g defined by Eq. (4.6) is just $\rho(g)$. Thus, the combination of Lemmas 5.3 and 6.3 asserts

$$T^{-1}\mathbf{A}(\Gamma)T = D \oplus \left[\oplus_{i=2}^{s} m_i \circ \left(\sum_{g\in G} \rho_i(g) \otimes A_{22}(\alpha; g) \right) \right].$$

Put

$$Z = \begin{pmatrix} I_{n_1} & & & & \\ & \sqrt{k_1} I_{n_2} & & & \\ & & \sqrt{k_2} I_{n_2} & & \\ & & & \ddots & \\ & & & & \sqrt{k_{m_1}} I_{n_2} \end{pmatrix} \begin{pmatrix} I_{n_1} & & & & \\ & I_{n_2} & & & \\ & I_{n_2} & I_{n_2} & & \\ & \vdots & & \ddots & \\ & I_{n_2} & & & I_{n_2} \end{pmatrix}.$$

By noting that $k = \sum_{i=1}^{m_1} k_i$, a simple calculation shows

$$Z^{-1}DZ = \begin{pmatrix} A_{11} & kA_{12} & k_2 A_{12} & \cdots & k_{m_1} A_{12} \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ 0 & 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{22} \end{pmatrix}.$$

But the 2×2 block matrix on the upper left corner of $Z^{-1}DZ$ is just the matrix $\mathbf{A}(\Gamma/\pi(\phi_0)^+)$ given in Eq. (5.2). Therefore the result follows. $\square$

Suppose we have known the data about a faithful permutation representation (those m_i and ρ_i), what are the implications of Theorem 6.4? Especially, let us ask:

Question 6.5. Fix a group H. For any digraph Γ how can we tell whether or not it holds $\Gamma = \Sigma \times^{\alpha, V_1} [k]$ and $H_\alpha = H$ for some Σ, V_1 and α?

Question 6.6. If $\Sigma \times^{\alpha_1, V_1} [k] = \Sigma \times^{\alpha_2, V_1} [k]$, what is the relationship between H_{α_1} and H_{α_2}?

We also put here:

Question 6.7. Can we obtain any analogue of Theorem 6.4 for a branched cover with general branch index?

We remark that, besides Question 6.7, another further natural direction for possible generalizations of Theorem 6.4 is to consider homomorphisms whose multiplicity set can contain more than two elements. Based on the work in this paper, we have been able to define the so-called ramified covers and deduce the corresponding decomposition results [10].

6.3. An example

We close this note with an application of Theorem 6.4. Let Σ be the digraph depicted on the right of Fig. 5. We list its vertices as $v_1, v_2, v_3, w_1, w_2, w_3$ in that order and get accordingly

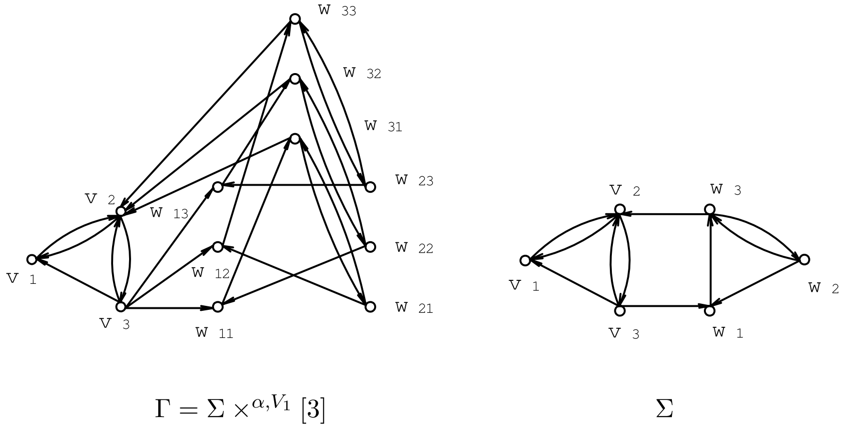

Fig. 5. Branched cover.

$$\mathbf{A}(\Sigma) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{w_1, w_2, w_3\}$. Furthermore, let $\alpha : E(\langle V_2 \rangle_\Sigma) \to S_3$ be the permutation voltage assignment such that $\alpha(w_2, w_1) = (12)$, $\alpha(w_1, w_3) = (23)$, $\alpha(w_2, w_3) = \alpha(w_3, w_2) = 1$. This then gives rise to the digraph $\Gamma = \Sigma \times^{\alpha, V_1} [3]$ which is pictured on the left of Fig. 5.

We now try to compute the characteristic polynomial of Γ.

Since

$$\mathbf{A}(\Gamma/\pi(\phi_0^{\alpha, V_1})^+) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

it follows that

$$\chi(\Gamma/\pi(\phi_0^{\alpha, V_1})^+; \lambda) = \det \begin{pmatrix} \lambda & -1 & 0 & 0 & 0 & 0 \\ -1 & \lambda & -1 & 0 & 0 & 0 \\ -1 & -1 & \lambda & -3 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & -1 & \lambda & -1 \\ 0 & -1 & 0 & 0 & -1 & \lambda \end{pmatrix}$$
$$= \lambda^6 - \lambda^4 - 5\lambda^2 - 3\lambda - 1.$$

Simple calculation shows $\langle \{\alpha(e) : e \in E(\langle V_2 \rangle_\Sigma)\} \rangle = \langle \{(12), (23)\} \rangle = S_3$. Note that S_3 has three irreducible representations, the unit representation $\rho_1 = 1$ with degree $f_1 = 1$, the sign representation ρ_2 with degree $f_2 = 1$, and the representation ρ_3 with degrees $f_3 = 2$ given by

$$\rho_3(1) = I_2, \quad \rho_3(123) = \begin{pmatrix} \eta & 0 \\ 0 & \eta^2 \end{pmatrix}, \quad \rho_3(132) = \begin{pmatrix} \eta^2 & 0 \\ 0 & \eta \end{pmatrix},$$
$$\rho_3((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_3((23)) = \begin{pmatrix} 0 & \eta \\ \eta^2 & 0 \end{pmatrix}, \quad \rho_3((13)) = \begin{pmatrix} 0 & \eta^2 \\ \eta & 0 \end{pmatrix},$$

where $\eta = \exp \frac{2\pi\sqrt{-1}}{3} = \frac{-1+\sqrt{-3}}{2}$. Let ρ be the permutation representation of S_3 associated with the permutation voltage assignment described previously. It is an easy verification that $\rho = 1 \oplus \rho_3$.

We proceed to determine $\mathbf{A}_{22}(\alpha, h),\ h \in S_3$. They are

$$\mathbf{A}_{22}(\alpha, 1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{A}_{22}(\alpha, (12)) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\mathbf{A}_{22}(\alpha, (23)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{A}_{22}(\alpha, (13)) = \mathbf{A}_{22}(\alpha, (123)) = \mathbf{A}_{22}(\alpha, (132)),$$

the last three ones all being the 3×3 zero matrix.

Finally, by utilizing all the above preparative results and appealing to Theorem 6.4, we see that

$$\begin{aligned}
\chi(\Gamma; \lambda) &= \chi(\Gamma/\pi(\phi_0^{\alpha, V_1})^+; \lambda) \det \left(\lambda I_6 - \sum_{h \in S_3} \rho_3(h) \otimes \mathbf{A}_{22}(\alpha, h) \right) \\
&= (\lambda^6 - \lambda^4 - 5\lambda^2 - 3\lambda - 1) \det \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & -\eta \\ 0 & \lambda & -1 & -1 & 0 & 0 \\ 0 & -1 & \lambda & 0 & 0 & 0 \\ 0 & 0 & -\eta^2 & \lambda & 0 & 0 \\ -1 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & -1 & \lambda \end{pmatrix} \\
&= (\lambda^6 - \lambda^4 - 5\lambda^2 - 3\lambda - 1)(\lambda^6 - 2\lambda^4 + \lambda^3 + \lambda^2 - \lambda + 1).
\end{aligned}$$

References

[1] R.L. Adler, B. Marcus, Topological entropy and equivalence of dynamical systems, Mem. Amer. Math. Soc. 219 iv+84 (1979).

[2] N. Biggs, Algebraic Graph Theory, second ed., Cambridge University Press, London, 1993.

[3] P. Boldi, S. Vigna, Fibrations of graphs, Discrete Math. 243 (2002) 21–66.

[4] Y. Chae, J.H. Kwak, J. Lee, Characteristic polynomials of some graph bundles, J. Korean Math. Soc. 30 (1993) 229–249.

[5] A. Chan, C.D. Godsil, Symmetry and eigenvectors, in: G. Hahn, G. Sabidussi (Eds.), Graph Symmetry—Algebraic Methods and Applications, Kluwer Academic Publishers, Dordrecht, 1997, pp. 75–106.

[6] D. Cvetković, The main part of the spectrum, divisors and switching of graphs, Publ. Inst. Math. (Beograd) 23 (1978) 31–38.

[7] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs: Theory and Applications, third revised and enlarged ed., Johann Ambrosius Barth Verlag, Heidelberg–Leipzig, 1995.

[8] D. Cvetković, P. Rowlinson, S. Simić, Eigenspaces of Graphs, Cambridge University Press, Cambridge, 1997.

[9] A. Deng, Y. Wu, Characteristic polynomials of digraphs having a semi-free action, Linear Algebra Appl. 408 (2005) 189–206.
[10] A. Deng, I. Sato, Y. Wu, Characteristic polynomials of ramified uniform covering digraphs, Eur. J. Combin. (2006), doi: 10.1016/j.ejc.2006.04.001.
[11] A. Dress, D. Stevanović, A note on a theorem of Horst Sachs, Ann. Comb. 8 (2004) 487–497.
[12] R. Feng, J.H. Kwak, J. Lee, Characteristic polynomials of graph coverings, Bull. Austral. Math. Soc. 69 (2004) 133–136.
[13] C. Godsil, G. Royle, Algebraic Graph Theory, Springer-Verlag, New York, 2001.
[14] J.L. Gross, T.W. Tucker, Generating all graph coverings by permutation voltage assignments, Discrete Math. 18 (1977) 273–283.
[15] J.L. Gross, T.W. Tucker, Topological Graph Theory, Wiley-Interscience, New York, 1987.
[16] P. Híc, R. Nedela, S. Pavlíková, Front divisors of trees, Acta Math. Univ. Comenianae 1 (1992) 69–84.
[17] G. James, M. Liebeck, Representations and Characters of Groups, second ed., Cambridge University Press, 2001.
[18] B.P. Kitchens, Symbolic Dynamics: One-sided, Two-Sided and Countable State Markov Shifts, Springer, 1998.
[19] D.E. Knuth, Partitioned tensor products and their spectra, J. Algebraic Combin. 6 (1997) 259–267.
[20] J.H. Kwak, J. Lee, Characteristic polynomials of some graph bundles II, Linear and Multilinear Algebra 32 (1992) 61–73.
[21] J. Lee, H.K. Kim, Characteristic polynomials of graphs having a semi-free action, Linear Algebra Appl. 307 (2000) 35–46.
[22] D. Lind, B. Marcus, An Introduction to Symbolic Dynamics and Coding, Cambridge University Press, 1995.
[23] H. Mizuno, I. Sato, Characteristic polynomials of some graph coverings, Discrete Math. 142 (1995) 295–298.
[24] H. Mizuno, I. Sato, Characteristic polynomials of some covers of symmetric digraphs, Ars Combin. 45 (1997) 3–12.
[25] M. Nasu, Constant-to-one and onto global maps of homomorphisms between strongly connected graphs, Ergodic Theory Dynam. Syst. 3 (1983) 387–413.
[26] P.M. Neumann, A lemma that is not Burnside's, Math. Scientist 4 (1979) 133–141.
[27] A.J. Schwenk, Computing the characteristic polynomial of a graph, Lect. Notes Math. 406 (1974) 153–172.
[28] H. Sachs, Simultane Überlangerung gegebener Graphen, Magyar Tud. Akad. Mat. Kutató Int. Közl. 9 (1965) 415–427.
[29] J.P. Serre, Linear Representations of Finite Group, Springer-Verlag, New York, 1977.
[30] R.F. Williams, Classification of subshifts of finite type, Ann. Math. 98 (1973) 120–153, errata, ibid. 99 (1974) 380–381.
[31] Y. Wu, A. Deng, Hoffman polynomials of irreducible matrices and digraphs, Linear Algebra Appl. 414 (2006) 138–171.