On Moore Bipartite Digraphs

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Abstract: In the context of the degree/diameter problem for directed graphs, it is known that the number of vertices of a strongly connected bipartite digraph satisfies a Moore-like bound in terms of its diameter k and the maximum out-degrees (d_1, d_2) of its partite sets of vertices. It has been proved that, when $d_1 d_2 > 1$, the digraphs attaining such a bound, called Moore bipartite digraphs, only exist when $2 \leq k \leq 4$. This paper deals with

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the problem of their enumeration. In this context, using the theory of circulant matrices and the so-called De Bruijn near-factorizations of cyclic groups, we present some new constructions of Moore bipartite digraphs of diameter three and composite out-degrees. By applying the iterated line digraph technique, such constructions also provide new families of dense bipartite digraphs with arbitrary diameter. Moreover, we show that the line digraph structure is inherent in any Moore bipartite digraph G of diameter $k = 4$, which means that $G = L\,G'$, where G' is a Moore bipartite digraph of diameter $k = 3$. J Graph Theory 43: 171–187, 2003

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1. INTRODUCTION

For its implications in the design of interconnection networks, it is interesting to find families of dense digraphs, that is to say, strongly connected digraphs with a relatively large number of vertices in comparison with the largest order allowed by their maximum out-degree and diameter. In the case of bipartite digraphs with maximum out-degrees (d_1, d_2), not necessarily equal in both partite sets of vertices (V_1, V_2), the order N is bounded above by the *Moore-like bound*

$$N \le M(d_1, d_2, k) = \begin{cases} 2(1 + d_1 d_2 + \cdots + (d_1 d_2)^m) & \text{if } k = 2m + 1, \\ (d_1 + d_2)(1 + d_1 d_2 + \cdots + (d_1 d_2)^{m-1}) & \text{if } k = 2m, \end{cases} \tag{1}$$

where k stands for the diameter.

To derive this bound, we simply consider that, from any vertex v, all vertices in the same (respectively, different) partite set of v must be reached in at most $k - 1$ steps, if k is odd (respectively, even); see [9]. A bipartite digraph $G = (V, E)$, $V = V_1 \cup V_2$, with maximum out-degrees (d_1, d_2), diameter k, and (optimal) order $N = M(d_1, d_2, k)$ is called a *Moore bipartite digraph*. In fact, such a digraph must be *semi-out-regular* with out-degrees (d_1, d_2) (or (d_1, d_2)-*outregular* for short); that is, $d^+(v) = d_\alpha$ for any $v \in V_\alpha$ (throughout the paper, $\alpha \in \{1, 2\}$). The problem of the Moore bipartite digraphs existence has already been solved in the regular case ($d_1 = d_2$) by Fiol and Yebra [9], and their results can be easily extended to the general case, as was noted by Gómez, Padró, and Perennes [12] in the context of generalized cycles. Thus, it is known that the bound $M(d_1, d_2, k)$ is only attained when $2 \le k \le 4$, if $d_1 d_2 > 1$. The non-existence of a Moore bipartite digraph $G = (V_1 \cup V_2, E)$ with diameter $k > 4$, out-degrees (d_1, d_2), $d_1 d_2 > 1$, and order $N = n_1 + n_2$,

$$n_\alpha := |V_\alpha| = \begin{cases} 1 + d_1 d_2 + \cdots + (d_1 d_2)^m & \text{if } k = 2m + 1, \\ d_{\bar{\alpha}}(1 + d_1 d_2 + \cdots + (d_1 d_2)^{m-1}) & \text{if } k = 2m, \end{cases} \tag{2}$$

(where $\bar{1} = 2$ and $\bar{2} = 1$), has been deduced by applying spectral techniques to the equation that its adjacency matrix

$$A = \mathrm{bip}\,(A_1, A_2) := \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix} \tag{3}$$

must satisfy. Depending on the parity of k, such an equation is

$$I + A + \cdots + A^{2m+1} = \begin{pmatrix} J_n & d_1 J_n \\ d_2 J_n & J_n \end{pmatrix} \quad \text{if } k = 2m+1 \tag{4}$$

(in this case $n := n_1 = n_2$; see (2)), and

$$A + A^2 + \cdots + A^{2m} = \begin{pmatrix} d_1 J_{n_1} & J_{n_1,n_2} \\ J_{n_2,n_1} & d_2 J_{n_2} \end{pmatrix} \quad \text{if } k = 2m, \tag{5}$$

where J denotes the all-one matrix with the appropriate dimensions. In matricial terms, these equations formulate two essential properties of G. Namely, the uniqueness of a walk of length $\leq k-1$ between each pair of vertices in the same or different partite set of $V(G)$, depending upon the parity of k; and the semi-out-regularity of the digraph. Using the expression (3) for A, such equations are equivalent to $A_\alpha J = d_\alpha J$ (in which case we say that A_α is d_α-*outregular*) and

$$\left.\begin{aligned} I_n + A_1A_2 + \cdots + (A_1A_2)^m &= J_n \\ I_n + A_2A_1 + \cdots + (A_2A_1)^m &= J_n \end{aligned}\right\} \quad \text{if } k = 2m+1, \tag{6}$$

$$\left.\begin{aligned} A_1(I_{n_2} + A_2A_1 + \cdots + (A_2A_1)^{m-1}) &= J_{n_1,n_2} \\ A_2(I_{n_1} + A_1A_2 + \cdots + (A_1A_2)^{m-1}) &= J_{n_2,n_1} \end{aligned}\right\} \quad \text{if } k = 2m, \tag{7}$$

respectively; see [9,12].

Nevertheless, the enumeration of Moore bipartite digraphs remains unsettled, apart from the trivial cases $k = 2$, and $d_1 = d_2 = 1$ (k odd), which have unique solutions: the complete bipartite digraph K_{d_1,d_2} and the directed cycle of length $k+1$, respectively. For diameter $k = 3$, only one construction was known. Namely, the family of Moore bipartite digraphs $BD(d_1, d_2, n)$, where $n := n_\alpha = c1 + d_1d_2$, introduced in [12] as a natural extension of the regular Moore bipartite digraphs $BD(d, n)$ presented in [9]; see also [2]. A digraph $BD(d_1, d_2, n)$ has set of vertices $V = V_1 \cup V_2$, where $V_\alpha = \{\alpha\} \times \mathbb{Z}_n$, and the vertices $(1, i)$ and $(2, i)$ are, respectively adjacent to the vertices of the sets

$$N^+(1, i) = \{(2, d_1 i + t);\ t = 0, 1, \ldots, d_1 - 1\}, \tag{8}$$

$$N^+(2, i) = \{(1, -d_2(i+1) + t);\ t = 0, 1, \ldots, d_2 - 1\}. \tag{9}$$

(From now on, and unless otherwise stated, arithmetic must be understood in the considered ring, e.g., $\mathbb{Z}_n$.) Then, *De Bruijn type adjacencies* (8), from V_1 to V_2, are represented by a binary submatrix A_1 in which each row (excluding the first one) is the previous row moved to the right d_1 places, with wraparound; that is, A_1 is a d_1-*circulant matrix*. Notice that A_1 is uniquely determined by its "shifting parameter" d_1 and its first row $(a_0, a_1, \ldots, a_{n-1}) = (1, \overset{d_1}{\ldots}, 1, 0, \ldots, 0)$. This can be represented by the polynomial

$$\theta_{A_1}(x) := \sum_{i=0}^{n-1} a_i x^i = 1 + x + \cdots + x^{d_1 - 1},$$

which is called the *Hall polynomial* of A_1. Analogously, *Kautz type adjacencies* (9), from V_2 to V_1, are described by a $(-d_2)$-circulant binary submatrix A_2 with Hall polynomial $\theta_{A_2}(x) = x^{n-d_2} + x^{n-d_2+1} + \cdots + x^{n-1}$, since $N^+(2,0) = \{(1, -d_2 + t);\ t = 0, 1, \ldots, d_2 - 1\}$. This fact suggests seeking other solutions to (6), in the case $m = 1$ ($k = 3$), using the theory of circulant matrices. In this context, the key idea is to use the known isomorphism between the subring of circulant matrices of order n with rational coefficients and the quotient ring $\mathbb{Q}[x]/(x^n - 1)$, which assigns to each matrix its Hall polynomial; see e.g. [5]. Thus, in Section 2, we reformulate such a search in polynomial and additive terms, which allows us to construct new Moore bipartite digraphs of diameter three and composite out-degrees by using the so-called De Bruijn near-factorizations of cyclic groups. From such optimal digraphs, and using the iterated line digraph technique introduced in [7,8], we can obtain new families of dense bipartite digraphs with arbitrary diameter.

We point out that, from the results in [7,8], if G is a Moore bipartite digraph with diameter $k = 3$ and out-degrees (d_1, d_2), $d_1 d_2 > 1$, then its $(2r-1)$-th iterated line digraph $L^{2r-1} G$, $r \geq 1$, is a (d_1, d_2)-outregular bipartite digraph of diameter $k = 2(r+1)$ and order $N = (d_1 + d_2)((d_1 d_2)^r + (d_1 d_2)^{r-1})$. Similarly, $L^{2r} G$ has also the same out-degrees, diameter $k = 2(r+1) + 1$, and order $N = 2((d_1 d_2)^{r+1} + (d_1 d_2)^r)$. In particular, LG is a Moore bipartite digraph of diameter $k = 4$. Furthermore, $\{L^t G\}_t$ is a family of dense bipartite digraphs with fixed out-degrees (d_1, d_2) and diameter $k(t) = 3 + t$, since its order function $N(t)$ satisfies that $N(t) = (1 - \varepsilon) M(d_1, d_2, 3 + t)$, where

$$\varepsilon := \frac{M(d_1, d_2, 3+t) - N(t)}{M(d_1, d_2, 3+t)} = \begin{cases} \dfrac{(d_1 d_2)^{r-1} - 1}{(d_1 d_2)^{r+1} - 1} & \text{if } t = 2r - 1, \\[2ex] \dfrac{(d_1 d_2)^{r} - 1}{(d_1 d_2)^{r+2} - 1} & \text{if } t = 2r, \end{cases}$$

is always smaller than $1/(d_1 d_2)^2$.

Finally, in Section 3, we show that the line digraph structure appears as a characteristic and extremal property of the class of Moore bipartite digraphs of diameter four. That is, any such digraph is the line digraph of a Moore bipartite digraph of diameter three. As a result, the enumeration of Moore bipartite digraphs has been reduced to the case of diameter three.

2. MOORE BIPARTITE DIGRAPHS OF DIAMETER THREE

From (6), the problem of enumeration of (d_1, d_2)-outregular Moore bipartite digraphs of diameter $k = 3$ leads us to the search of all binary matrices A_1, A_2 of order $n = 1 + d_1 d_2$ satisfying that

$$A_\alpha J_n = d_\alpha J_n \quad \text{and} \quad A_\alpha A_{\bar{\alpha}} = J_n - I_n. \tag{10}$$

Next, we present a "characterization" of these solutions, from which the semi-inregularity of their associated digraphs is derived.

Lemma 2.1. *Let $A_1 \in \mathcal{M}_n(0,1)$ be a d_1-outregular matrix of order $n = 1 + d_1 d_2$. Then, there is a (d_1, d_2)-outregular Moore bipartite digraph $G = (V_1 \cup V_2, E)$ of diameter $k = 3$ such that A_1 represents the adjacencies of G from V_1 to V_2, if and only if, A_1 is nonsingular and d_1-regular (that is, $J_n A_1 = A_1 J_n = d_1 J_n$) such that $A_2 = A_1^{-1}(J_n - I_n)$ is a (d_2-regular) binary matrix. In this case, A_2 represents the adjacencies of G from V_2 to V_1.*

Proof. Assume first that G is a (d_1, d_2)-outregular Moore bipartite digraph of diameter $k = 3$, such that A_1 is a submatrix of its adjacency matrix $A = \mathrm{bip}\,(A_1, A_2)$. Then, since $A_1 A_2 = J_n - I_n$ and $\det(J_n - I_n) = (-1)^{n-1}\,(n-1) \neq 0$, we deduce that A_1 is nonsingular and $A_2 = A_1^{-1}(J_n - I_n) \in \mathcal{M}_n(0,1)$. Moreover, using the identity $A_2 A_1 = J_n - I_n$, we obtain that $(J_n - I_n)A_1^{-1} = A_2 = A_1^{-1}(J_n - I_n)$, which implies that $J_n A_1 = A_1 J_n = d_1 J_n$ and A_1 is d_1-regular.

Conversely, if $A_1 \in \mathcal{M}_n(0,1)$ is a d_1-regular non-singular matrix such that $A_2 = A_1^{-1}(J_n - I_n)$ is also a $(0,1)$-matrix, then $A_1 A_2 = J_n - I_n = A_2 A_1$, since $A_1^{-1} J_n = J_n A_1^{-1}$. Moreover, from $A_1 J_n = d_1 J_n$ and $n = 1 + d_1 d_2$, we get $A_1^{-1} J_n = \frac{1}{d_1} J_n = \frac{d_2}{n-1} J_n$, whence

$$A_2 J_n = A_1^{-1}(J_n - I_n) J_n = (n-1) A_1^{-1} J_n = d_2 J_n.$$

Consequently, the associated bipartite digraph with binary matrix $A = \mathrm{bip}\ (A_1, A_2)$ is a Moore bipartite digraph of diameter $k = 3$. ∎

The following property is a direct consequence of Lemma 1.1, taking into account that the column sums of the submatrix A_α represent the in-degrees of the vertices in $V_{\bar{\alpha}}$.

Corollary 2.1. *Let G be a (d_1, d_2)-outregular Moore bipartite digraph of diameter $k = 3$. Then, G is (d_2, d_1)-inregular. In particular, G is regular if $d_1 = d_2$.*

From the above, given the out-degrees (d_1, d_2), we are interested in finding non-singular d_1-regular matrices $A_1 \in \mathcal{M}_n(0,1)$, where $n = 1 + d_1 d_2$, such that $A_1^{-1}(J_n - I_n)$ is also a $(0,1)$-matrix. In this work, we confine our search to the class of g-circulant matrices. A well-known property of these matrices states that if A_α, $\alpha = 1, 2$, are g_α-circulant matrices of order n, then $A_1 A_2$ is a $(g_1 g_2)$-circulant matrix and its Hall polynomial is

$$\theta_{A_1 A_2}(x) = \theta_{A_1}(x^{g_2})\theta_{A_2}(x),$$

where the polynomial arithmetic is modulo $x^n - 1$; see [1]. Moreover, if A_α is non-singular and has order n, then g_α has an inverse in $\mathbb{Z}_n$ ($g_\alpha \in \mathbb{Z}_n^*$) and A_α^{-1} is a (g_α^{-1})-circulant matrix; see [5]. Thus, for instance, since the corresponding A_1 matrix of the digraph $BD(d_1, d_2, n)$ is a non-singular d_1-circulant matrix, with Hall polynomial $\theta_{A_1}(x) = 1 + x + \cdots + x^{d_1 - 1}$, it follows that A_1^{-1} is (d_1^{-1})-circulant, with $d_1^{-1} = -d_2$, and has Hall polynomial $\theta_{A_1^{-1}}(x)$ such that $\theta_{A_1}(x^{n-d_2})$ $\theta_{A_1^{-1}}(x) = 1$. Then, taking into account that $J_n - I_n$ is 1-circulant, we have that $A_2 = A_1^{-1}(J_n - I_n)$ is a $(-d_2)$-circulant matrix with Hall polynomial

$$\begin{aligned}
\theta_{A_2}(x) &= \theta_{A_1^{-1}}(x)\theta_{J_n - I_n}(x) \\
&= \left(\frac{1}{d_1}\sum_{i=0}^{n-d_2-1} x^i + \frac{1-d_1}{d_1}\sum_{i=n-d_2}^{n-1} x^i\right)\sum_{i=1}^{n-1} x^i \\
&= x^{n-d_2} + x^{n-d_2+1} + \cdots + x^{n-1}.
\end{aligned}$$

We point out that if A_1 is a g-circulant nonsingular matrix of order n, then there is a permutation matrix P such that $A_1' := PA_1$ is a circulant matrix with the same Hall polynomial as A_1; see again [5]. Moreover, $A_2 = A_1^{-1}(J_n - I_n)$ is a (g^{-1})-circulant matrix and $A_2' := (A_1')^{-1}(J_n - I_n)$ is a circulant matrix such that

$$A_2' = A_1^{-1}P^\top(J_n - I_n) = A_1^{-1}(J_n - I_n)P^\top = A_2 P^\top,$$

where we have used that $P^{-1} = P^\top$ and $PJ_n = J_n P$ since P is a permutation matrix. Therefore, $A = \mathrm{bip}\,(A_1, A_2)$ and $A' := \mathrm{bip}\,(A_1', A_2') = \mathrm{bip}\,(PA_1, A_2 P^\top)$ are the adjacency matrices of isomorphic digraphs since

$$A' = \begin{pmatrix} 0 & A_1' \\ A_2' & 0 \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}^\top. \tag{11}$$

In the particular case of the digraph $BD(d_1, d_2, n)$, the permutation of the rows of A_1 (which is d_1-circulant) giving a circulant matrix A_1' is represented by a

$(-d_2)$-circulant matrix P (recall that $d_1^{-1} = -d_2$) with Hall polynomial $\theta_P(x) = 1$, and we get $A_1' = PA_1$. Then, $A_2' = A_2P^\top$ is also a circulant matrix with Hall polynomial

$$\begin{aligned}\theta_{A_2'}(x) &= \theta_{A_2}(x^{d_1})\theta_{P^\top}(x) = x^{(n-d_2)d_1} + x^{(n-d_2+1)d_1} + \cdots + x^{(n-1)d_1}\\ &= x + x^{1+d_1} + \cdots + x^{1+(d_2-1)d_1}.\end{aligned}$$

Hence, we can restrict our attention to the case of (ordinary) circulant matrices. In the following result, we reformulate the conditions on such matrices in two ways: First, in additive terms, by showing the presence of a near-factorization of the cyclic group $\mathbb{Z}_n$; and second in terms of the coefficients of their Hall polynomials.

Proposition 2.1. *For some given positive integers d_1, d_2, let $A_1 \in \mathcal{M}_n(0,1)$ be a circulant matrix with Hall polynomial $\theta_{A_1}(x) = x^{i_1} + x^{i_2} + \cdots + x^{i_{d_1}}$, where $0 \le i_1 < i_2 < \cdots < i_{d_1} < n = 1 + d_1d_2$, and let $S_1 = \{i_1, i_2, \ldots, i_{d_1}\}$. Then, the following statements are equivalent:*

(i) *There is a $(0,1)$-matrix A_2 such that $A = bip\,(A_1, A_2)$ is the adjacency matrix of a (d_1, d_2)-outregular Moore bipartite digraph of diameter $k = 3$;*

(ii) *There exists a subset $S_2 \subseteq \mathbb{Z}_n$ of cardinal d_2 such that*

$$S_1 + S_2 = \mathbb{Z}_n \backslash \{0\}; \tag{12}$$

(iii) *The polynomials $\theta_{A_1}(x)$ and $x^n - 1$ are relatively prime and $\theta_{A_1}(x)^{-1}$ is a polynomial with d_2 coefficients equal to $(1-d_1)/d_1$ and $n - d_2$ coefficients equal to $1/d_1$.*

Proof. First, we notice that A_1 is d_1-regular, since $A_1 \in \mathcal{M}_n(0,1)$ is circulant and $\theta_{A_1}(1) = d_1$. Thus, from Lemma 1.1, and recalling that the inverse of a non-singular matrix is also circulant, we can reformulate (i) by saying that there exists a circulant matrix $A_2 \in \mathcal{M}_n(0,1)$ such that $A_1A_2 = J_n - I_n$ or, equivalently, there is a polynomial $\theta_{A_2}(x) = x^{j_1} + x^{j_2} + \cdots + x^{j_{d_2}}$, with $0 \le j_1 < j_2 < \cdots < j_{d_2} < n$, satisfying that

$$\theta_{A_1}(x)\theta_{A_2}(x) = \sum_{(i,j)\in S_1\times S_2} x^{i+j} = x + x^2 + \cdots + x^{n-1},$$

where $S_2 = \{j_1, j_2, \ldots, j_{d_2}\} \subseteq \mathbb{Z}_n$. In additive terms, such a polynomial relation can be expressed as

$$S_1 + S_2 = \{i + j;\ (i,j) \in S_1 \times S_2\} = \mathbb{Z}_n \backslash \{0\}$$

and, consequently, (i) $\Longleftrightarrow$ (ii).

Besides, since a circulant matrix A_1 is nonsingular if, and only if, there exists the inverse polynomial of $\theta_{A_1}(x)$ in $\mathbb{Q}[x]/(x^n-1)$, it turns out that (i) is also equivalent to saying that $\theta_{A_1}(x)$ and x^n-1 are relatively prime and the polynomial $\theta_{A_1}(x)^{-1}=\theta_{A_1^{-1}}(x)=\sum_{i=0}^{n-1}a_ix^i$, with $\theta_{A_1^{-1}}(1)=\sum_{i=0}^{n-1}a_i=1/d_1$, satisfies

$$\begin{aligned}\theta_{A_2}(x)=\theta_{A_1^{-1}}(x)\theta_{J_n-I_n}(x)&=\left(\sum_{i=0}^{n-1}a_ix^i\right)\left(\sum_{i=1}^{n-1}x^i\right)\\&=(a_1+a_2+\cdots+a_{n-1})+\cdots+(a_0+a_1+\cdots+a_{n-2})x^{n-1}\\&=\left(\frac{1}{d_1}-a_0\right)+\left(\frac{1}{d_1}-a_1\right)x+\cdots+\left(\frac{1}{d_1}-a_{n-1}\right)x^{n-1},\end{aligned}$$

which means that d_2 coefficients of $\theta_{A_1^{-1}}(x)$ are equal to $(1-d_1)/d_1$ and the remaining ones are equal to $1/d_1$, since $A_2=A_1^{-1}(J_n-I_n)\in\mathcal{M}_n(0,1)$ and $\theta_{A_2}(1)=d_2$. ■

From now on, we will use the additive version (ii) since it turns out to be more convenient from the theoretical point of view, whereas the polynomial version (iii) seems to be more appropriate for a computational exploration. Thus, we want to find solutions to (12); that is, pairs (S_1,S_2) of subsets of $\mathbb{Z}_n$, $n=1+d_1d_2$, with cardinality $|S_\alpha|=d_\alpha$, such that all the sums $i+j$, where $(i,j)\in S_1\times S_2$, are distinct and non-null. In the context of group theory, this is a special type of "(*t*-)*near-factorization*" of $\mathbb{Z}_n$, where it is only required that $|S_1'+S_2'|=|S_1'||S_2'|$ or, equivalently, $S_1'+S_2'=\mathbb{Z}_n\backslash\{t\}$ for some $t\in\mathbb{Z}_n$. Of course, every t-near-factorization provides us with a solution to our problem or 0-near-factorization, by simply taking $S_1=-t+S_1'$ and $S_2=S_2'$.

The pair of integer sets constituting a near-factorization are also known in the literature as "number systems" (see e.g., Grinstead [13]) because of their similarity to the standard number systems extensively studied. As we shall see later, an easy way of obtaining such systems is from the so-called "degenerate British Number Systems" (introduced by De Bruijn [4] in the context of the non-negative integers). Grinstead [13] realized that, if (S_1,S_2) is a number system, then (S_1+s,S_2+t) with $s,t\in\mathbb{Z}_n$, (hS_1,hS_2+t) with $h\in\mathbb{Z}_n^*$, and $(-S_1,S_2)$ also are number systems, and all of them are considered isomorphic to (S_1,S_2). From the above, one can prove that every isomorphic class contains a near-factorization (S_1,S_2); satisfying $S_\alpha=-S_\alpha$, which is called *symmetric* (see Caen et al. [6]). In the case of our 0-near-factorizations, however, this is not true because not all the above transformations fix the zero. Before concentrating in such a case, let us state the following general result, which was proved by Bacsó et al. in [3]:

Lemma 2.2. *Let (S_1,S_2) be a near-factorization of $\mathbb{Z}_n$ with $|S_1|\geq 2$. Then,*

$$\gcd(S_1\cup\{n\})=1. \tag{13}$$

In our context, let us denote by $\mathcal{S}$ the set of all number systems corresponding to 0-near-factorizations

$$\mathcal{S} := \{(S_1, S_2) \in \mathcal{P}_{d_1}(\mathbb{Z}_n) \times \mathcal{P}_{d_2}(\mathbb{Z}_n);\ S_1 + S_2 = \mathbb{Z}_n \backslash \{0\}\},$$

and by $BD(S_1, S_2)$ the Moore bipartite digraph associated with $(S_1, S_2) \in \mathcal{S}$, which means that the submatrices $A_1 := A(S_1)$ and $A_2 := A(S_2)$ of its adjacency matrix A are circulant with Hall polynomials $\theta_{A_1}(x) = \sum_{i \in S_1} x^i$ and $\theta_{A_2}(x) = \sum_{j \in S_2} x^j$, respectively. In graphical terms, the adjacencies of the digraph $BD(S_1, S_2)$, with set of vertices $V = \{1, 2\} \times \mathbb{Z}_n$, are $(\alpha, i) \to (\bar{\alpha}, i + S_\alpha)$. Thus, using this notation

$$BD(d_1, d_2, n) = BD(\{0, 1, \ldots, d_1 - 1\}, \{1, 1 + d_1, \ldots, 1 + (d_2 - 1)d_1\}).$$

Now the set $\mathcal{S}$ is closed under some "translations and homothetic transformations," in the sense that if $(S_1, S_2) \in \mathcal{S}$, then $(t + S_1, -t + S_2) \in \mathcal{S}$, for each $t \in \mathbb{Z}_n$ and $(h \cdot S_1, h \cdot S_2) \in \mathcal{S}$ if $h \in \mathbb{Z}_n^*$. In the following lemma, we show that, in order to construct non-isomorphic digraphs, it is necessary to consider the quotient set of $\mathcal{S}$ modulo such transformations.

Lemma 2.3. *Let (S_1, S_2) be a 0-near-factorization of $\mathbb{Z}_n$, where $|S_\alpha| = d_\alpha$ and $n = 1 + d_1 d_2$. Let $BD(S_1, S_2)$ be its associated (d_1, d_2)-outregular Moore bipartite digraph of diameter three. Then,*

(i) *$BD(t + S_1, -t + S_2)$ is isomorphic to $BD(S_1, S_2)$, for each $t \in \mathbb{Z}_n$;*
(ii) *$BD(h \cdot S_1, h \cdot S_2)$ is isomorphic to $BD(S_1, S_2)$, for each $h \in \mathbb{Z}_n^*$;*
(iii) *$BD(S_1, S_2)$ is isomorphic to $BD(d_1, d_2, n)$ if, for some α, S_α is an arithmetic progression with common difference $h \in \mathbb{Z}_n^*$ modulo n.*

Proof. Let $A_1^{(t)} := A(t + S_1)$ and $A_2^{(-t)} := A(-t + S_2)$ be the circulant submatrices associated with $(t + S_1, -t + S_2) \in \mathcal{S}$. Then,

$$A_1^{(t)} = PA_1 \quad \text{and} \quad A_2^{(-t)} = A_2 P^\top,$$

where P is a permutation circulant matrix (with $\theta_P(x) = x^t$). Thus, taking into account (13), we deduce that all Moore bipartite digraphs $BD(t + S_1, -t + S_2)$ are isomorphic. Analogously, if we denote by $A_\alpha^{[h]} := A(h \cdot S_\alpha)$ the circulant submatrices associated with $(h \cdot S_1, h \cdot S_2) \in \mathcal{S}$, then

$$\begin{pmatrix} 0 & A_1^{[h]} \\ A_2^{[h]} & 0 \end{pmatrix} = \begin{pmatrix} 0 & P^\top A_1 P \\ P^\top A_2 P & 0 \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}^\top \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix},$$

where P is a permutation h-circulant matrix ($\theta_P(x) = 1$), since $h \in \mathbb{Z}_n^*$. Therefore, $BD(h \cdot S_1, h \cdot S_2)$ and $BD(S_1, S_2)$ are isomorphic.

Finally, if S_1 is an arithmetic progression with common difference $h \in \mathbb{Z}_n^*$ modulo n; that is, $S_1 = t + h\{0, 1, \ldots, d_1 - 1\}$, then $S_2 = -t + h\{1, 1 + d_1, \cdots, 1 + (d_2 - 1)d_1\}$ and, consequently, from (i) and (ii), we conclude that $BD(S_1, S_2) \cong BD(d_1, d_2, n)$. ■

Notice that, in graphical terms, the isomorphism from the vertex set of $BD(S_1, S_2)$ to the vertex set of $BD(t + S_1, -t + S_2)$ is just $(1, i) \mapsto (1, i - t)$ and $(2, i) \mapsto (2, i)$. Correspondingly, the isomorphism from $BD(S_1, S_2)$ to $BD(h \cdot S_1, h \cdot S_2)$ is $(\alpha, i) \mapsto (\alpha, hi)$.

Intuitively, we can imagine a pair $(S_1, S_2) \in \mathcal{S}$ as a kind of tiling of $\mathbb{Z}_n \backslash \{0\}$, where S_1 represents the set of initial positions of the 1×1 tiles and S_2 provides the translation steps such that $\{t + S_1\}_{t \in S_2}$ completely fills $\mathbb{Z}_n \backslash \{0\}$. When we employ the usual integer arithmetic, this approach suggests the construction of a solution to $S_1 + S_2 = \{1, \ldots, d_1 d_2\}$, where $d_\alpha = \rho_\alpha \tau_\alpha$, by composing a (simpler) solution to $R_1 + R_2 = \{1, \ldots, \rho_1 \rho_2\}$ with one to $T_1 + T_2 = \{1, \ldots, \tau_1 \tau_2\}$. The idea is to "blow up" each unit tile centered at $R_1 + R_2$ to transform it into all the tiles of $T_1 + T_2$. In fact, this leads to a special type of what is called a *degenerate British Number System* [13], or *De Bruijn near-factorization* of the cyclic group $\mathbb{Z}_n$ [3]. The second name is because such factorizations are the discrete version of a construction of De Bruijn, which gives all the possible number systems for the nonnegative integers (see [4] for more details).

As we will see later, the interest of our De Bruijn (0-)near-factorizations is that, contrarily to the case when they are used to construct graphs [13,3], they sometimes give non-isomorphic bipartite digraphs. We recall that in the following result we use standard integer arithmetic.

Lemma 2.4. *Let (R_1, R_2) and (T_1, T_2) be two pairs of subsets of non-negative integers such that $R_1 + R_2 = \{1, \ldots, \rho_1 \rho_2\}$ and $T_1 + T_2 = \{1, \ldots, \tau_1 \tau_2\}$, with $|R_\alpha| = \rho_\alpha$ and $|T_\alpha| = \tau_\alpha$. If $0 \in T_1$, then $(R_1 + \rho_1 \rho_2 T_1, R_2 + \rho_1 \rho_2 (T_2 - 1))$ is a solution to $S_1 + S_2 = \{1, \ldots, d_1 d_2\}$, where $|S_\alpha| = d_\alpha = \rho_\alpha \tau_\alpha$.*

Proof. Notice that each of the elements $\{0, 1\}$ belongs to exactly one of the integer subsets T_α, since $1 \in T_1 + T_2$ and $|T_1 \cap T_2| \leq 1$, which means that $\min(\mathrm{T}_2 - 1) = 0$, if $0 \in T_1$. Clearly,

$$\begin{aligned} S_1 + S_2 &= (R_1 + \rho_1 \rho_2 T_1) + (R_2 + \rho_1 \rho_2 (T_2 - 1)) \\ &= R_1 + R_2 + \rho_1 \rho_2 (T_1 + T_2 - 1) \\ &= \{1, \ldots, \rho_1 \rho_2\} + \{0, \ldots, (\tau_1 \tau_2 - 1)\rho_1 \rho_2\} \\ &= \{1, \ldots, d_1 d_2\}, \end{aligned}$$

where $|S_\alpha| = |R_\alpha||T_\alpha|$. ■

Thus, for instance, in the regular even case $d_1 = d_2 = 2\rho$, we can compose the pair (R_1, R_2),

$$R_1 = \{0, 1, \ldots, \rho - 1\} \quad \text{and} \quad R_2 = \{1, 1 + \rho, \ldots, 1 + (\rho - 1)\rho\},$$

with the unique solution to $T_1 + T_2 = \mathbb{Z}_5 \backslash \{0\}$ (modulo transformations of the type $T \mapsto t + hT$),

$$T_1 = \{0, 1\} \quad \text{and} \quad T_2 = \{1, 3\},$$

in two different ways:

$$\begin{aligned}
\text{(M1)} \qquad S_1 &= R_1 + \rho^2 T_1 \\
&= \{0, 1, \ldots, \rho - 1, \rho^2, \rho^2 + 1, \ldots, \rho^2 + \rho - 1\} \\
S_2 &= R_2 + \rho^2 (T_2 - 1) \\
&= \{1, 1 + \rho, \ldots, \rho^2 - \rho + 1, 2\rho^2 + 1, \ldots, 3\rho^2 - \rho + 1\} \\
\text{(M2)} \qquad S_1 &= R_2 + \rho^2 T_1 \\
&= \{1, 1 + \rho, \ldots, \rho^2 - \rho + 1, \rho^2 + 1, \ldots, 2\rho^2 - \rho + 1\} \\
S_2 &= R_1 + \rho^2 (T_2 - 1) \\
&= \{0, 1, \ldots, \rho - 1, 2\rho^2, 2\rho^2 + 1, \ldots, 2\rho^2 + \rho - 1\}.
\end{aligned}$$

While the Moore bipartite digraph obtained from (M2) is isomorphic to $BD(2\rho, 1 + 4\rho^2)$, since the corresponding set $S_1 = 1 + \rho\{0, 1, \ldots, 2\rho - 1\}$ is an arithmetic progression with common difference $\rho \in \mathbb{Z}^*_{1+4\rho^2}$, the associated digraph with the pair defined by (M1) is a new construction, as we will show in the next theorem that generalizes the previous ideas. Before this, let us give a particular example: For $d = 4$ ($\rho = 2$), we get

$$\begin{aligned}
(M1) \qquad & S_1 = \{0, 1, 4, 5\} \quad S_2 = \{1, 3, 9, 11\}, \\
(M2) \qquad & S_1 = \{1, 3, 5, 7\} \quad S_2 = \{0, 1, 8, 9\},
\end{aligned}$$

from which we construct two nonisomorphic regular Moore bipartite digraphs of diameter $k = 3$ and degree $d = 4$, which turn out to be unique if we just consider digraphs of the type $BD(S_1, S_2)$.

Theorem 2.1. *Let $d_1 \geq d_2$ be positive integers. Given a factorization,*

$$d_1 = \rho_1 \tau_1 \text{ and } d_2 = \rho_2 \tau_2, \text{ where } \rho_\alpha \geq \tau_\alpha \geq 1 \ (\alpha = 1, 2),$$

let us consider the pair (S_1, S_2) of subsets of $\mathbb{Z}_n$, where $n = 1 + d_1 d_2$, defined by

$$S_1 = \cup_{i=0}^{\tau_1 - 1}(R_1 + i\rho_1\rho_2) \quad \textit{and} \quad S_2 = \cup_{j=0}^{\tau_2 - 1}(R_2 + j\rho_1\rho_2\tau_1), \tag{14}$$

where $R_1 = \{0, 1, \ldots, \rho_1 - 1\}$ and $R_2 = \{1, 1 + \rho_1, \ldots, 1 + (\rho_2 - 1)\rho_1\}$. Then, the Moore bipartite digraph $BD(S_1, S_2)$ is not isomorphic to $BD(d_1, d_2, n)$, if and only if, $\tau_\alpha > 1$.

Proof. If we take the integer subsets,

$$T_1 = \{0, 1, \ldots, \tau_1 - 1\} \quad \text{and} \quad T_2 = \{1, 1 + \tau_1, \ldots, 1 + (\tau_2 - 1)\tau_1\},$$

then $S_1 = R_1 + \rho_1\rho_2 T_1$ and $S_2 = R_2 + \rho_1\rho_2(T_2 - 1)$. So, applying Lemma 2.4, we have that $S_1 + S_2 = \{1, \ldots, d_1 d_2\}$, where $|S_\alpha| = d_\alpha$.

We know that if $G = (V_1 \cup V_2, E)$ is a Moore bipartite digraph of diameter three, then given a vertex $v_1 \in V_1$ there is a unique $v_1 \rightarrow w$ walk of length two, for each $w \in V_1 \backslash \{0\}$. Moreover, since $d^-(v_2) = d_1$, for each $v_2 \in V_2$, it follows that there are exactly $d_1(d_1 - 1)$ arcs incident from vertices of $N_2^+(v_1) = V_1 \setminus \{v_1\}$ (at distance two from v_1) to vertices of $N^+(v_1)$. We will show that these arcs are distributed in a different manner when we consider $G = BD(d_1, d_2, n)$ and $G = BD(S_1, S_2)$, if $\tau_\alpha > 1$. First, we point out that their automorphisms group act transitively on each partite set V_α, since the associated submatrix A_α is circulant. Thus, let us take the vertex $v_1 = 0$ of V_1 and let us compute the maximum number of vertices of $N^+(0)$ that are adjacent from a given vertex of $N_2^+(0) = \{1, 2, \ldots, n - 1\}$. In the case of the digraph $BD(d_1, d_2, n)$, such a value is

$$\max\{|N^+(w) \cap N^+(0)|;\ w \in N_2^+(0)\} = d_1 - 1,$$

which is attained by the vertices 1 and $n - 1$, since $N^+(0) = \{0, 1, \ldots, d_1 - 1\}$ and A_1 is circulant. Nevertheless, in the case of the digraph $BD(S_1, S_2)$ such a maximum is $d_1 - \min\{\rho_1, \tau_1\} = d_1 - \tau_1 < d_1 - 1$, since there is a partition of $N^+(0)$ into $\tau_1 > 1$ non-contiguous segments $R_1 + i\rho_1\rho_2 = \{i\rho_1\rho_2, \ldots, i\rho_1\rho_2 + \rho_1 - 1\}$ of cardinality ρ_1, where the greatest integer of $N^+(0)$ is $\rho_1 - 1 + \rho_1\rho_2(\tau_1 - 1) < \rho_1\rho_2\tau_1 \leq (n - 1)/2$ (since $\tau_2 > 1$). Hence, there does not exist an isomorphism between both digraphs that identify the vertices $v = 0$ of the partite set V_1 of each digraph, which implies that they are not isomorphic, if $d_1 > d_2$. This also holds in the regular case $d := d_1 = d_2$, since $BD(d, 1 + d^2)$ is vertex-transitive; see [9].

Notice that in the extremal case $\tau_1 = 1$ or $\tau_2 = 1$, the digraph $BD(S_1, S_2)$ is isomorphic to $BD(d_1, d_2, n)$, since S_1 (respectively, S_2) represents an arithmetic progression modulo $n = 1 + d_1 d_2$ with common difference $s = 1$ (respectively, $s = \rho_1 \in \mathbb{Z}_n^*$); see Lemma 2.3. ■

As a consequence, if we have two distinct factorizations of the composite degrees $d_\alpha = \rho_\alpha \tau_\alpha = \rho'_\alpha \tau'_\alpha$, where $\rho_\alpha \geq \tau_\alpha > 1$ and $\rho'_\alpha \geq \tau'_\alpha > 1$, such that

$$\tau_1 \neq \tau'_1, \text{ if } d_1 > d_2, \quad \text{and} \quad \{\tau_1, \tau_2\} \neq \{\tau'_1, \tau'_2\}, \text{ if } d_1 = d_2,$$

then $BD(S_1, S_2)$ and $BD(S'_1, S'_2)$ are not isomorphic. Hence, the previous theorem provides new constructions of Moore bipartite digraphs with composite out-degrees (d_1, d_2), where $d_1 \geq d_2 > 1$.

Notice that if $d_2 = 1$, there is a unique solution to our enumeration problem, namely $BD(d_1, 1, 1 + d_1)$. In the cases $d_1 = 2, 3$, we have also showed that the only possible Moore bipartite digraphs are the $BD(d_1, d_2, d_1 d_2 + 1)$. Here we only discuss the simple case $d_1 = 2$: By Lemma 2.3 (i) we can assume that $S_1 = \{0, s\}$, so that the isomorphism $BD(S_1, S_2) \cong BD(2, d_2, 2d_2 + 1)$ is a direct consequence of Lemma 2.2 and Lemma 2.3 (iii).

3. MOORE BIPARTITE DIGRAPHS OF DIAMETER FOUR

As we have already mentioned, the line digraph LG of a Moore bipartite digraph G with diameter three and out-degrees (d_1, d_2), where $d_1 d_2 > 1$, is a Moore bipartite digraph of diameter four. Next, we will prove that the converse result is also true and, therefore, the problem of the enumeration of Moore bipartite digraphs can be reduced to the case of diameter three. In fact, it is shown that the rank of the adjacency matrix A of a (d_1, d_2)-outregular Moore bipartite digraph of diameter four attains the minimum possible value, $\operatorname{rank} A = 2(1 + d_1 d_2)$, whence the line digraph structure follows. This approach has already been used in some other $(0,1)$-matrix equations; see [17,10].

First, we will see that all Moore bipartite digraphs with a given diameter $k \in \{3, 4\}$ and fixed out-degrees (d_1, d_2) are cospectral.

Lemma 3.1. *Let G be a (d_1, d_2)-outregular Moore bipartite digraph of diameter $k > 2$. Then, the spectrum of G is*

$$\begin{pmatrix} \sqrt{d_1 d_2} & i & -i & -\sqrt{d_1 d_2} \\ 1 & d_1 d_2 & d_1 d_2 & 1 \end{pmatrix} \quad \text{if } k = 3,$$

$$\begin{pmatrix} \sqrt{d_1 d_2} & i & 0 & -i & -\sqrt{d_1 d_2} \\ 1 & d_1 d_2 & (d_1 + d_2 - 2)(1 + d_1 d_2) & d_1 d_2 & 1 \end{pmatrix} \quad \text{if } k = 4,$$

(where $i = \sqrt{-1}$).

Proof. Let $A = \mathrm{bip}\,(A_1, A_2)$ be the adjacency matrix of $G = (V_1 \cup V_2, E)$ and let $n_\alpha = |V_\alpha|$. From the out-regularity of A_α (that can also be expressed as $A_\alpha \boldsymbol{j} = d_\alpha \boldsymbol{j}$, where $\boldsymbol{j}$ is the all-one vector), it turns out that

$$(\sqrt{d_1}\boldsymbol{j}, \sqrt{d_2}\boldsymbol{j}) := (\sqrt{d_1}, \overset{n_1}{\ldots}, \sqrt{d_1}, \sqrt{d_2}, \overset{n_2}{\ldots}, \sqrt{d_2})$$

is a positive eigenvector of A associated to the eigenvalue $\sqrt{d_1 d_2}$. Since A is irreducible (G is strongly connected) and using Perron-Frobenius theorem, we know that $\sqrt{d_1 d_2}$ has multiplicity equal to 1. Moreover, since G is bipartite, the characteristic polynomial of G, $\phi(G,x)$, satisfies that $\phi(G,-x) = (-1)^{n_1+n_2}\phi(G,x)$, which implies that $-\sqrt{d_1 d_2}$ is also a simple eigenvalue of G; see e.g. [11,16].

Furthermore, from the matrix equation (4) of the case $k = 3$,

$$I + A + A^2 + A^3 = \begin{pmatrix} J_n & d_1 J_n \\ d_2 J_n & J_n \end{pmatrix}, \tag{15}$$

where $n := n_\alpha = 1 + d_1 d_2$, and taking into account that 0 is an eigenvalue of the right-hand matrix of (17) with multiplicity equal to $2n-2$, we deduce that the remaining $2n-2$ eigenvalues of A must be roots of the equation $1 + x + x^2 + x^3 = (x+1)(x^2+1) = 0$. Hence, since $\phi(G,x) = \phi(G,-x)$, we have that $\pm i$ are eigenvalues of G with the same multiplicity $n - 1 = d_1 d_2$.

Analogously, from the corresponding matrix equation (5) for $k = 4$

$$A + A^2 + A^3 + A^4 = \begin{pmatrix} d_1 J_{n_1} & J_{n_1,n_2} \\ J_{n_2,n_1} & d_2 J_{n_2} \end{pmatrix},$$

where $n_\alpha = d_{\bar{\alpha}}(1 + d_1 d_2)$, we obtain that $\pm i$ and 0 are eigenvalues of A with multiplicity $m_A(i) = m_A(-i)$ and $m_A(0) = n_1 + n_2 - 2 - 2m_A(i)$, respectively. Therefore, using the condition $\operatorname{tr} A^2 = 2d_1 d_2 - 2m_A(i) = 0$, we conclude that $m_A(i) = d_1 d_2$ and $m_A(0) = (d_1 + d_2 - 2)(1 + d_1 d_2)$. ■

Theorem 3.1. *Every Moore bipartite digraph of diameter four is the line digraph of a Moore bipartite digraph of diameter three.*

Proof. Let $A = \mathrm{bip}\,(A_1, A_2)$ be the adjacency matrix of a (d_1, d_2)-outregular Moore bipartite digraph $G = (V_1 \cup V_2, E)$ of diameter $k = 4$. From (7), we know that

$$A_1(I_{n_2} + A_2 A_1) = J_{n_1,n_2} \quad \text{and} \quad A_2(I_{n_1} + A_1 A_2) = J_{n_2,n_1}, \tag{16}$$

where $n_\alpha = |V_\alpha| = d_{\bar{\alpha}}(1 + d_1 d_2)$. Thus,

$$B := A + A^3 = \begin{pmatrix} 0 & A_1(I_{n_2} + A_2 A_1) \\ A_2(I_{n_1} + A_1 A_2) & 0 \end{pmatrix} = \begin{pmatrix} 0 & J_{n_1,n_2} \\ J_{n_2,n_1} & 0 \end{pmatrix},$$

which corresponds to the adjacency matrix of the complete bipartite digraph K_{n_1,n_2}. Then, since

$$(B + \sqrt{n_1 n_2} I_{n_1+n_2})B = \begin{pmatrix} n_2 J_{n_1} & \sqrt{n_1 n_2} J_{n_1,n_2} \\ \sqrt{n_1 n_2} J_{n_2,n_1} & n_1 J_{n_2} \end{pmatrix}$$

and taking into account that $A_\alpha J = d_\alpha J$, we deduce that

$$(A - \sqrt{d_1 d_2} I_{n_1+n_2})(A + A^3 + \sqrt{n_1 n_2} I_{n_1+n_2})(A + A^3) = 0,$$

which implies that the minimum polynomial of A, $\mu_A(x)$, divides

$$q(x) = (x - \sqrt{d_1 d_2})(x + \sqrt{d_1 d_2})(x^2 - \sqrt{d_1 d_2} x + 1 + d_1 d_2)(x^2 + 1)x.$$

But, since $\mu_A(x)$ also divides the characteristic polynomial of G, which is

$$\phi(G, x) = (x^2 - d_1 d_2)(x^2 + 1)^{d_1 d_2} x^{n_1 + n_2 - 2(1 + d_1 d_2)}$$

(by Lemma 3.1), we conclude that

$$\mu_A(x) = (x^2 - d_1 d_2)(x^2 + 1)x.$$

Hence,

$$\operatorname{rank} A = n_1 + n_2 - \dim \ker A = n_1 + n_2 - m_A(0) = 2(1 + d_1 d_2).$$

Taking into account that $\operatorname{rank} A = \operatorname{rank} A_1 + \operatorname{rank} A_2$ and

$$\min\{\operatorname{rank} A_1, \operatorname{rank} A_2\} \geq \min\{n_2/d_1, n_1/d_2\} = 1 + d_1 d_2,$$

since $A_\alpha J = d_\alpha J$ and each column of A_α is non-null, it turns out that

$$\operatorname{rank} A_1 = \operatorname{rank} A_2 = 1 + d_1 d_2.$$

This extremal property implies that the rows of A are mutually orthogonal or identical, which is equivalent, by Heuchenne's condition [15], to say that G is a line digraph. Therefore, we conclude that $G = L G'$, where G' is a Moore bipartite digraph with diameter three and out-degrees (d_1, d_2). ∎

As a consequence of the previous theorem and Corollary 2.1, we deduce that any Moore bipartite digraph $G = (V_1 \cup V_2, E)$ of diameter four is *semidiregular*, which means that $d^+(v) = d^-(v) = d_\alpha, \forall v \in V_\alpha$, since each vertex v of $G = LG'$ represents an arc (u, w), say, of G' and the corresponding degrees satisfy the relation

$$d_G^-(v) = d_{G'}^-(u) = d_{G'}^+(w) = d_G^+(v).$$

Finally, as a result of Theorem 2.1, and taking into account that the line digraph transformation is one-to-one from the class of strongly connected digraphs onto the class of line digraphs, up to isomorphisms (see [14]), we obtain new constructions of Moore bipartite digraphs of diameter four and composite degrees.

Lemma 3.2. *Let $d_1 \geq d_2$ be positive integers such that $d_1 d_2 > 1$. Let $d_1 = \rho_1 \tau_1$ and $d_2 = \rho_2 \tau_2$, with $\rho_\alpha \geq \tau_\alpha \geq 1$, be integer factorizations. If (S_1, S_2) represents the solution to $S_1 + S_2 = \mathbb{Z}_{1+d_1 d_2} \setminus \{0\}$ defined by* (14)*, then $LBD(S_1, S_2)$ is a Moore bipartite digraph of diameter $k = 4$ and degrees (d_1, d_2), which is not isomorphic to $LBD(d_1, d_2, 1 + d_1 d_2)$, if $\tau_\alpha > 1$.*

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