Noname manuscript No.
(will be inserted by the editor)

Competition Numbers and Phylogeny Numbers

Uniform Complete Multipartite Graphs

Yaokun Wu · Yanzhen Xiong · Soesoe Zaw

Received: date / Accepted: date

Abstract Let D be a digraph. The competition graph of D is the graph sharing the same vertex set with D such that two different vertices are adjacent if and only if they have a common out-neighbor in D; the phylogeny graph of D is the competition graph of the digraph obtained from D by adding a loop at every vertex. For any graph G with n vertices, its competition number $\kappa(G)$ is the least nonnegative integer k such that G is an induced subgraph of the competition graph of an acyclic digraph with $n+k$ vertices, while its phylogeny number $\phi(G)$ is the least nonnegative integer p such that G is an induced subgraph of the phylogeny graph of an acyclic digraph with $n+p$ vertices. This paper provides new estimates of the competition numbers and phylogeny numbers of complete multipartite graphs with uniform part sizes. Accordingly, we can show that the range of the function $\phi - \kappa + 1$ is the set of all nonnegative integers. We also report results about a hypergraph version of competition number and phylogeny number.

Keywords Edge clique cover · Mutually orthogonal Latin squares

Mathematics Subject Classification (2000) 05C35

Yaokun Wu
School of Mathematical Sciences and MOE-LSC, Shanghai Jiao Tong University, Shanghai, 200240, P.R. China
E-mail: ykwu@sjtu.edu.cn

Yanzhen Xiong
School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai, 200240, P.R. China
E-mail: xyz920805@sjtu.edu.cn

Soesoe Zaw
School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai, 200240, P.R. China
E-mail: soesoezaw@sjtu.edu.cn

1 Introduction

In 1968, based on the idea that two species compete if and only if they have a common prey, Cohen [Coh68] proposed the concept of the competition graph of an acyclic digraph. He intended to apply this concept like a mathematical microscope to detect the structure of a food web [Coh78]. Indeed, he has observed that almost all competition graphs arising from actual ecosystems are interval graphs. This motivates Roberts [Rob78] to study how to recognize/characterize competition graphs. A finding of Roberts [Rob78] then is that every graph can be made into a competition graph by adding some isolated vertices and so he defined the least such number to be the competition number of the graph. Surely, a graph is a competition graph if and only if its competition number is zero. Stimulated by problems of phylogenetic tree reconstruction, Roberts and Sheng [RS98] defined the concept of the phylogeny graph of an acyclic digraph and the concept of the phylogeny number of a graph. Note that phylogeny graphs are known as moral graphs in Bayesian network theory [LCKS17]. Opsut [Ops82] found that it is an NP-complete problem to recognize competition graphs and to determine competition numbers. Analogously, Roberts and Sheng [RS98] proved that it is an NP-complete problem to calculate the phylogeny numbers. As an effort to tackle a difficult extremal combinatorics problem, many researchers have tried to determine the exact value or estimate the upper/lower bounds of the competition/phylogeny numbers of various classes of graphs. As a variant of competition graph, Sonntag and Teichert [ST04] proposed the definition of competition hypergraph, which naturally leads to the concepts of ST-competition numbers and ST-phylogeny numbers of hypergraphs. Some other interesting parameters similar to competition numbers and phylogeny numbers include double competition numbers and poset competition numbers of graphs and hypergraphs [GST16, WL10].

This paper is concerned with relationships between competition numbers and phylogeny numbers of graphs as well as ST-competition numbers and ST-phylogeny numbers of hypergraphs. We especially try to estimate these parameters for complete multipartite graphs. In § 2, we formally define all relevant concepts and describe some relevant former results as well as our main results. § 3 is devoted to proofs to our main results. We conclude the paper in § 4 with some research questions.

2 Main results

A *graph* G is a pair consisting of its vertex set $\mathrm{V}(G) \neq \emptyset$ and its edge set $\mathrm{E}(G) \subseteq \binom{\mathrm{V}(G)}{2}$. For each graph G and nonnegative integer k, let $\mathrm{I}_k(G)$ stand for the graph obtained from G by adding k isolated vertices. An *induced subgraph* of a graph G, or simply known as a subgraph of G, is a graph G' such that $\mathrm{V}(G') \subseteq \mathrm{V}(G)$ and $\mathrm{E}(G') = \mathrm{E}(G) \cap \binom{\mathrm{V}(G')}{2}$. Let us write $G' \lhd G$ to mean that G' is a subgraph of G. A *hypergraph* H comprises its vertex set $\mathrm{V}(H) \neq \emptyset$ and its hyperedge set $\mathscr{E}(H) \subseteq \binom{\mathrm{V}(H)}{\geq 2}$. For each hypergraph H and nonnegative integer

k, let $\mathrm{I}_k(H)$ stand for the hypergraph for which $\mathrm{V}\left(\mathrm{I}_k(H)\right)$ is the disjoint union of $\mathrm{V}(H)$ and a set of size k while $\mathscr{E}\left(\mathrm{I}_k(H)\right)$ equals $\mathscr{E}(H)$. The *subhypergraph* induced by a nonempty subset $A \subseteq \mathrm{V}(H)$ is the hypergraph H' with vertex set A and hyperedge set $\mathscr{E}(H') = \{e \cap A : \ e \in \mathscr{E}(H), |e \cap A| \geq 2\}$. For two hypergraphs H and H', we write $H' \lhd H$ to mean that H' is a subhypergraph of H. A digraph D is a pair consisting of its vertex set $\mathrm{V}(D) \neq \emptyset$ and its arc set $\mathrm{A}(D) \subseteq \mathrm{V}(D) \times \mathrm{V}(D)$. For each digraph D, let D° stand for the digraph with $\mathrm{V}(D^\circ) = \mathrm{V}(D)$ and $\mathrm{A}(D^\circ) = \mathrm{A}(D) \cup \{(v,v) : \ v \in \mathrm{V}(D)\}$. For any $(u,v) \in \mathrm{A}(D)$, we call u an *in-neighbor* of v in D, and call v an *out-neighbor* of u in D. For each $v \in \mathrm{V}(D)$, let $\mathrm{N}^-_D(v)$ denote the set of in-neighbors of v, i.e., $\mathrm{N}^-_D(v) := \{u \in \mathrm{V}(D) : \ (u,v) \in \mathrm{A}(D)\}$. A digraph D is *acyclic* if it contains no directed cycle. Note that a digraph D is acyclic if and only if D has an *acyclic ordering*, namely a total order $\prec$ defined on $\mathrm{V}(D)$ such that $\mathrm{A}(D) \subseteq \{(v,w) \in \mathrm{V}(D) \times \mathrm{V}(D) : \ \ v \prec w\}$. When we use $\prec$ to stand for a total order, we will say v is *less than* w in $\prec$ whenever $v \prec w$. [1]

For every digraph D, the *competition graph* of D [Coh68], denoted by $\mathcal{C}(D)$, is the graph with $\mathrm{V}\left(\mathcal{C}(D)\right) = \mathrm{V}(D)$ and with two vertices being adjacent if and only if they have at least one common out-neighbor in D. The *competition number* of a graph G, denoted by $\kappa(G)$, is the least nonnegative integer k such that $\mathrm{I}_k(G)$ becomes the competition graph of an acyclic digraph. Equivalently,

$$\kappa(G) = \min\left\{|\,\mathrm{V}(D)| - |\,\mathrm{V}(G)|\right\},$$

where D runs through all acyclic digraphs such that $G \lhd \mathcal{C}(D)$.

For every digraph D, the *phylogeny graph* of D [RS98], denoted by $\mathcal{P}(D)$, is the competition graph of D°, that is, $\mathcal{P}(D) = \mathcal{C}(D^\circ)$. The *phylogeny number* of a graph G, denoted by $\phi(G)$, is the least number p such that we can find a phylogeny graph of an acyclic digraph that has $p + |\,\mathrm{V}(G)|$ vertices and has G as an induced subgraph.

For every digraph D, the *competition hypergraph* of D [ST04], denoted by $\mathcal{CH}(D)$, is the hypergraph with vertex set $\mathrm{V}\left(\mathcal{CH}(D)\right) = \mathrm{V}(D)$ and hyperedge set

$$\mathscr{E}\left(\mathcal{CH}(D)\right) = \{e \in \binom{\mathrm{V}(D)}{\geq 2} : \ \exists v \in \mathrm{V}(D) \text{ s.t. } e = \mathrm{N}^-_D(v)\}.$$

The ST-*competition number of a hypergraph H*, denoted by $\kappa_{\mathrm{ST}}(H)$, is the least nonnegative integer k such that $\mathrm{I}_k(H)$ becomes the competition hypergraph of an acyclic digraph. Equivalently, $\kappa_{\mathrm{ST}}(H)$ is the least value of $|\,\mathrm{V}(D) \setminus \mathrm{V}(H)|$ where D runs through all acyclic digraphs satisfying $H \lhd \mathcal{CH}(D)$.

For every digraph D, the ST-*phylogeny hypergraph* of D, denoted by $\mathcal{PH}(D)$, is the competition hypergraph of D°, that is, $\mathcal{PH}(D) = \mathcal{CH}(D^\circ)$. The ST-*phylogeny number of a hypergraph H*, which we write as $\phi_{\mathrm{ST}}(H)$, is the least value of $|\,\mathrm{V}(D) \setminus \mathrm{V}(H)|$ where D runs through all acyclic digraphs satisfying $H \lhd \mathcal{PH}(D)$.

---

[1] A graph as used in this paper is sometimes referred to as a simple graph, and a digraph, again as used in this paper, is sometimes referred to as a digraph without multiple arcs and with at most one loop at each vertex. Surely an acyclic digraph does not have any loops.

For any positive integers $m, n_1, \dots, n_m$, let $[m] = \{1, \dots, m\}$ and let $\mathrm{K}^{n_1,\dots,n_m}$ denote the graph with

$$\mathrm{V}(\mathrm{K}^{n_1,\dots,n_m}) = \bigcup_{i=1}^{m} \mathrm{V}_i$$

where $\mathrm{V}_i = \{v_i^j : j \in [n_i]\}$ for $i \in [m]$, and with

$$\mathrm{E}(\mathrm{K}^{n_1,\dots,n_m}) = \{v_i^j v_{i'}^{j'} : i \neq i', j \in [n_i], j' \in [n_{i'}]\}.$$

We call $\mathrm{K}^{n_1,\dots,n_m}$ a *complete multipartite graph* with m parts and part sizes $n_1, \dots, n_m$. The complete multipartite graph $\mathrm{K}^{n,\dots,n}$ with m parts and uniform part size n is named a *uniform complete multipartite graph* and denoted by K_m^n.

Theorem 1 *(i) (Kim-Park-Sano [PKS08, Theorem 7])* $\kappa(\mathrm{K}_m^2) = 2$ *for* $m \geq 2$.
(ii) (Kim-Park-Sano [PKS08, Theorem 8]) $\kappa(\mathrm{K}_m^3) = 4$ *for* $m \geq 3$.
(iii) (Kim-Sano [KS08, Theorem 1]) $\kappa(\mathrm{K}_3^n) = n^2 - 3n + 4$ *for* $n \geq 2$.

We mention that the proof of Theorem 1 by Kim-Park-Sano [PKS08, Theorem 7, Theorem 8] and Kim-Sano [KS08, Theorem 1] relies on very intricate constructions. By adapting their main constructions a bit, we yield the following.

Theorem 2 *(i)* $\phi(\mathrm{K}_m^2) = 1 = \kappa(\mathrm{K}_m^2) - 1$ *for* $m \geq 2$.
(ii) $\phi(\mathrm{K}_m^3) = 3 = \kappa(\mathrm{K}_m^3) - 1$ *for* $m \geq 3$.
(iii) $\phi(\mathrm{K}_3^n) = n^2 - 3n + 3 = \kappa(\mathrm{K}_3^n) - 1$ *for* $n \geq 2$.

For a graph G, a *clique* of G is a subset of $\mathrm{V}(G)$ in which every two distinct elements are adjacent in G. A clique of G is called *maximal* if it is not properly contained in any clique of G. The *clique hypergraph* of G, denoted by $\mathcal{K}(G)$, is the hypergraph with vertex set $\mathrm{V}(G)$ and with the set of all maximal cliques of G which are of size at least two as its hyperedge set. For $G = \mathrm{K}^n$, namely the graph with n vertices and no edges, it is easy to see that $\phi_{\mathrm{ST}}\left(\mathcal{K}(G)\right) = \kappa_{\mathrm{ST}}\left(\mathcal{K}(G)\right) = 0$. When G is a complete multipartite graph with more than one parts, some more effort leads to the next result.

Theorem 3 *Let* $m, n_1, \dots, n_m$ *be positive integers and let* $H = \mathcal{K}(\mathrm{K}^{n_1,\dots,n_m})$. *If* $m \geq 2$, *then* $\phi_{\mathrm{ST}}(H) + 1 = \kappa_{\mathrm{ST}}(H) = \prod_{\ell=1}^{m} n_\ell - \sum_{\ell=1}^{m} n_\ell + m$.

For any positive integer n, a *Latin square of order* n is an $n \times n$ matrix with entries using n distinct elements such that none of the elements appear twice in any row or column of the matrix. For convenience, we shall assume the n distinct elements to be $1, \dots, n$. Let L denote a Latin square of order n and $L(i, j)$ denote the entry in row i and column j of L. For two Latin squares L_1 and L_2 of order n, we say that L_1 and L_2 are *orthogonal* if all ordered pairs from corresponding entries of L_1 and L_2 are different, i.e. for every pair $(i^*, j^*) \in [n] \times [n]$, there is a unique pair $(i, j) \in [n] \times [n]$ such that $L_1(i, j) = i^*$

and $L_2(i,j) = j^*$. A number of Latin squares of the same order form a set of *mutually orthogonal Latin squares*, often abbreviated in the literature to MOLS, if any two of them are orthogonal. The largest size of a set of MOLS of order n is denoted by $\mathcal{L}(n)$. It is known that $\mathcal{L}(n) \leq n-1$ [KD15, Theorem 5.1.2] with equality if and only if there exists a finite projective plane of order n [KD15, p. 166]. Note that a finite projective plane of order n does exist when n is a prime power.

Let G be a graph. A *vertex clique cover* of G is a set of cliques of G such that every vertex of G is contained in some clique in this set. An *edge clique cover* of G is a set of cliques of G such that every edge of G is contained in some clique in this set. The *edge clique cover number* of G, denoted by $\theta_e(G)$, is the minimum size of an edge clique cover of G. A *minimum edge clique cover* of G is an edge clique cover of G of size $\theta_e(G)$.

Theorem 4 *(Chang-Li [LC12, Theorem 17]) Let m and n be positive integers such that $2 \leq m \leq \mathcal{L}(n)+2$. Then for every minimum edge clique cover $\mathscr{S}$ of K_m^n, there exists a digraph D satisfying the following two properties:*

(i) $\mathscr{S} = \{\mathrm{N}_D^-(v) : \ v \in \mathrm{V}(D), |\,\mathrm{N}_D^-(v)| \geq 1\}$*;*
(ii) $\mathcal{C}(D) = \mathrm{I}_{n^2-2n+2}(\mathrm{K}_m^n)$*.*

Especially, $\kappa(\mathrm{K}_m^n) \leq n^2 - 2n + 2$*.*

Remark 1 For every connected and triangle-free graph G, one can check easily that $\phi(G) = \kappa(G) - 1 = |\,\mathrm{E}(G)| - |\,\mathrm{V}(G)| + 1$ [Rob78, Theorem 2] [RS98, Theorem 12]. It follows that $\phi(\mathrm{K}_2^n) + 1 = \kappa(\mathrm{K}_2^n) = n^2 - 2n + 2$ and so the bound in Theorem 4 is tight.

Theorem 5 *(Kim-Park-Sano [KPS12, Theorem 3]) Let n be an integer such that $3 \leq n \leq \mathcal{L}(n)+2$. Then for every integer m which is not smaller than n, we have $\kappa(\mathrm{K}_m^n) \leq n^2 - n + 1$.*

For any integer $n \geq 2$, Zhang [Zha63] showed that every set of MOLS of order n and size $n-2$ can be uniquely extended to a set of MOLS of order n and size $n-1$. When $n \neq 4$, Shrikhande [Shr61] proved that every set of MOLS of order n and size $n-3$ can be uniquely extended to a set of MOLS of order n and size $n-1$. Surely, the result of Zhang implies that for each integer $n \geq 2$, $n \leq \mathcal{L}(n)+2$ if and only if $\mathcal{L}(n) = n-1$. This then illustrates that the following result is an extension of Theorem 5.

Theorem 6 *Let n be a positive integer such that $\mathcal{L}(n) = n-1$. Then for every integer m greater than 1, it holds $\kappa(\mathrm{K}_m^n) \leq n^2 - 2n + 2$.*

For two sets S and T, their disjoint union is designated by $S \sqcup T$. The *disjoint union* of two graphs G_1 and G_2, denoted by $G_1 \sqcup G_2$, is the graph with vertex set $\mathrm{V}(G_1) \sqcup \mathrm{V}(G_2)$ and edge set $\mathrm{E}(G_1) \sqcup \mathrm{E}(G_2)$. Similarly, we can define the disjoint union of two digraphs or two hypergraphs. Roberts and Sheng [RS98, Lemma 6] derived the following additivity property of the phylogeny number.

Theorem 7 *(Roberts-Sheng [RS98, Lemma 6]) For any two graphs G_1 and G_2, it holds $\phi(G_1 \sqcup G_2) = \phi(G_1) + \phi(G_2)$.*

Let us establish below a counterpart of Theorem 7.

Theorem 8 *(i) For any two hypergraphs H_1 and H_2, it holds $\phi_{\mathrm{ST}}(H_1 \sqcup H_2) = \phi_{\mathrm{ST}}(H_1) + \phi_{\mathrm{ST}}(H_2)$.*
(ii) For any two graphs G_1 and G_2, it holds $\kappa(G_1 \sqcup G_2) \le \kappa(G_1) + \kappa(G_2)$, with equality if and only if $\kappa(G_1) = \kappa(G_2) = 0$.
(iii) For any two hypergraphs H_1 and H_2, it holds $\kappa_{\mathrm{ST}}(H_1 \sqcup H_2) \le \kappa_{\mathrm{ST}}(H_1) + \kappa_{\mathrm{ST}}(H_2)$, with equality if and only if $\kappa_{\mathrm{ST}}(H_1) = \kappa_{\mathrm{ST}}(H_2) = 0$.

Subsequently, we can determine the range of the functions $\phi - \kappa + 1$ and $\phi_{\mathrm{ST}} - \kappa_{\mathrm{ST}} + 1$.

Theorem 9 *(i) For any integer k, there exists a graph G such that $\phi(G) - \kappa(G) + 1 = k$ if and only if $k \ge 0$.*
(ii) For any integer k, there exists a hypergraph H such that $\phi_{\mathrm{ST}}(H) - \kappa_{\mathrm{ST}}(H) + 1 = k$ if and only if $k \ge 0$.

3 Proofs

The starting point of almost all research on competition numbers and phylogeny numbers is the following easy observation.

Lemma 1 *Let D be a digraph, G a graph, and H a hypergraph. Then, the following hold:*

(i) $G \lhd \mathcal{C}(D)$ if and only if $\{\mathrm{N}_D^-(v) \cap \mathrm{V}(G) : v \in \mathrm{V}(D)\}$ is an edge clique cover of G.
(ii) $G \lhd \mathcal{P}(D)$ if and only if $\{(\mathrm{N}_D^-(v) \cup \{v\}) \cap \mathrm{V}(G) : v \in \mathrm{V}(D)\}$ is an edge clique cover of G.
(iii) $H \lhd \mathcal{CH}(D)$ if and only if $\{\mathrm{N}_D^-(v) \cap \mathrm{V}(H) : v \in \mathrm{V}(D), |\mathrm{N}_D^-(v) \cap \mathrm{V}(H)| \ge 2\} = \mathscr{E}(H)$.
(iv) $H \lhd \mathcal{PH}(D)$ if and only if $\{(\mathrm{N}_D^-(v) \cup \{v\}) \cap \mathrm{V}(H) : v \in \mathrm{V}(D), |(\mathrm{N}_D^-(v) \cup \{v\}) \cap \mathrm{V}(H)| \ge 2\} = \mathscr{E}(H)$.

Lemma 2 *(i) For every graph G, it holds $\phi(G) - \kappa(G) + 1 \ge 0$.*
(ii) For every hypergraph H, it holds $\phi_{\mathrm{ST}}(H) - \kappa_{\mathrm{ST}}(H) + 1 \ge 0$.

Proof (i). Let D_1 be an acyclic digraph such that $G \lhd \mathcal{P}(D_1)$ and $|\mathrm{V}(D_1)| - |\mathrm{V}(G)| = \phi(G)$. Let $\prec$ be an acyclic ordering of D_1 and we enumerate the vertices in D_1 with at least one in-neighbor as $v_1, \ldots, v_\ell$, where $v_1 \prec \cdots \prec v_\ell$. Define $S_i = (N_{D_1}^-(v_i) \cup \{v_i\}) \cap \mathrm{V}(G)$ for $i \in [\ell]$. By Lemma 1 (ii), $\{S_i\}_{i=1}^{\ell}$ is an edge clique cover of G.

Let D_2 be the digraph with $\mathrm{V}(D_2) = \mathrm{V}(D_1) \sqcup \{u\}$ and $\mathrm{A}(D_2) = \{(v, u) : v \in S_\ell\} \cup \left(\bigcup_{i=1}^{\ell-1} \{(v, v_{i+1}) : v \in S_i\}\right)$. It is clear that D_2 is acyclic. By

Lemma 1 (i) and the fact that $\{S_i\}_{i=1}^{\ell}$ is an edge clique cover of G, $G \lhd \mathcal{C}(D_2)$ holds. This then implies $\kappa(G) \leq |\,\mathrm{V}(D_2)| - |\,\mathrm{V}(G)| = \phi(G) + 1$, as required.

(ii). Let D_1 be an acyclic digraph such that $H \lhd \mathcal{PH}(D_1)$ and $|\,\mathrm{V}(D_1)| - |\,\mathrm{V}(H)| = \phi_{\mathrm{ST}}(H)$. Assume that $v_1, \ldots, v_\ell$ are all the vertices in D_1 such that $|\big(N^-_{D_1}(v_i) \cup \{v_i\}\big) \cap \mathrm{V}(H)| \geq 2$ for $i \in [\ell]$. Let $\prec$ be an acyclic ordering of D_1 and we may suppose $v_1 \prec \cdots \prec v_\ell$. Let $S_i = \big(N^-_{D_1}(v_i) \cup \{v_i\}\big) \cap \mathrm{V}(H)$ for $i \in [\ell]$. By Lemma 1 (iv), $\{S_i : \ i \in [\ell]\} = \mathscr{E}(H)$.

Let D_2 be the digraph with $\mathrm{V}(D_2) = \mathrm{V}(D_1) \sqcup \{u\}$ and $\mathrm{A}(D_2) = \{(v, u) : v \in S_\ell\} \cup \big(\bigcup_{i=1}^{\ell-1}\{(v, v_{i+1}) : \ v \in S_i\}\big)$. It is apparent that D_2 is acyclic. By Lemma 1 (iii) and the fact that $\{S_i : \ i \in [\ell]\} = \mathscr{E}(H)$, we obtain $H \lhd \mathcal{CH}(D_2)$. Therefore, $\kappa_{\mathrm{ST}}(H) \leq |\,\mathrm{V}(D_2)| - |\,\mathrm{V}(G)| = \phi_{\mathrm{ST}}(H) + 1$, finishing the proof. $\square$

Lemma 3 *(Kim-Sano [KS08, Lemma 4]) For every positive integer $n \geq 4$, the complete tripartite graph K_3^n possesses an edge clique cover $\{S_\ell\}_{\ell=1}^{n^2}$ such that*

$$\begin{cases} S_1 = \{v_1^1, v_2^1, v_3^1\}, & S_2 = \{v_1^2, v_2^n, v_3^1\}, \quad S_3 = \{v_1^1, v_2^n, v_3^n\}, \\ S_4 = \{v_1^n, v_2^1, v_3^n\}, & S_5 = \{v_1^n, v_2^2, v_3^1\}, \quad S_6 = \{v_1^1, v_2^2, v_3^2\}, \\ S_7 = \{v_1^{n-1}, v_2^2, v_3^n\}, & S_8 = \{v_1^2, v_2^{n-1}, v_3^n\}, \quad S_9 = \{v_1^1, v_2^{n-1}, v_3^{n-1}\}, \\ & \cdots \\ S_{3s+4} = \{v_1^{n-s}, v_2^2, v_3^{n-s+1}\}, & S_{3s+5} = \{v_1^2, v_2^{n-s}, v_3^{n-s+1}\}, \quad S_{3s+6} = \{v_1^1, v_2^{n-s}, v_3^{n-s}\}, \\ & \cdots \\ S_{3n-5} = \{v_1^3, v_2^2, v_3^4\}, & S_{3n-4} = \{v_1^2, v_2^3, v_3^4\}, \quad S_{3n-3} = \{v_1^1, v_2^3, v_3^3\}. \end{cases} \tag{1}$$

Proof (Proof of Theorem 2) (i). It follows from Theorem 1 (i) that $\kappa(\mathrm{K}_m^2) = 2$ for $m \geq 2$. Consequently, by Lemma 2 (i), our task is to show that $\phi(\mathrm{K}_m^2) \leq 1$ for $m \geq 2$.

When $m = 2$, let D be the digraph with $\mathrm{V}(D) = \mathrm{V}(\mathrm{K}_2^2) \sqcup \{u\}$ and $\mathrm{A}(D) = \{(v_1^1, v_2^1), (v_2^1, v_1^2), (v_1^2, v_2^2), (v_2^2, u), (v_1^1, u)\}$. It is clear that D is acyclic and $\mathrm{K}_2^2 \lhd \mathcal{P}(D)$. This gives $\phi(\mathrm{K}_2^2) \leq 1$.

When $m \geq 3$, we consider the following edge clique cover of K_m^2:

$$\begin{cases} S_1 = \{v_1^2, v_2^1, v_3^1, \ldots, v_{m-1}^1, v_m^1\}, \\ S_2 = \{v_1^1, v_2^2, v_3^1, \ldots, v_{m-1}^1, v_m^1\}, \\ \cdots \\ S_m = \{v_1^1, v_2^1, v_3^1, \ldots, v_{m-1}^1, v_m^2\}, \\ S_{m+1} = \{v_1^2, v_2^2, v_3^2, \ldots, v_{m-1}^2, v_m^2\}. \end{cases} \tag{2}$$

Let D be the digraph with $\mathrm{V}(D) = \mathrm{V}(\mathrm{K}_m^2) \sqcup \{u\}$ and

$$\mathrm{A}(D) = \{(v, u) : \ v \in S_{m+1}\} \cup \Big(\bigcup_{i=1}^{m} \big\{(v, v_i^2) : \ v \in S_i \setminus \{v_i^2\}\big\}\Big).$$

Observe that the digraph D is acyclic with an acyclic ordering $\prec$ such that

$$v_1^1 \prec v_2^1 \prec \cdots \prec v_m^1 \prec v_1^2 \prec v_2^2 \prec \cdots \prec v_m^2 \prec u.$$

Note that $\{(\mathrm{N}_D^-(v)\cup\{v\})\cap \mathrm{V}(\mathrm{K}_m^2):\ v\in \mathrm{V}(D)\}$ is the edge clique cover of K_m^2 as demonstrated in (2). By Lemma 1 (ii), $\mathrm{K}_m^2 \lhd \mathcal{P}(D)$ holds, which then implies that $\phi(\mathrm{K}_m^2)\le 1$ when $m\ge 3$, as expected.

(ii). In view of Theorem 1 (ii), $\kappa(\mathrm{K}_m^3)=4$. By Lemma 2 (i), it remains to show that $\phi(\mathrm{K}_m^3)\le 3$.

For $m\ge 3$, there exists a unique positive integer t and a unique integer $r\in\{0,1,2\}$ such that $m=3t+r$. For each $i\in[t]$, let

$$\begin{cases} S(v^1_{3(i-1)+1}) = \{v^1_{3(i-1)+1}, v^2_{3(i-1)+2}, v^2_{3(i-1)+3}\},\\ S(v^1_{3(i-1)+2}) = \{v^2_{3(i-1)+1}, v^1_{3(i-1)+2}, v^2_{3(i-1)+3}\},\\ S(v^1_{3(i-1)+3}) = \{v^2_{3(i-1)+1}, v^2_{3(i-1)+2}, v^1_{3(i-1)+3}\},\\ S(v^2_{3(i-1)+1}) = \{v^2_{3(i-1)+1}, v^3_{3(i-1)+2}, v^3_{3(i-1)+3}\},\\ S(v^2_{3(i-1)+2}) = \{v^3_{3(i-1)+1}, v^2_{3(i-1)+2}, v^3_{3(i-1)+3}\},\\ S(v^2_{3(i-1)+3}) = \{v^3_{3(i-1)+1}, v^3_{3(i-1)+2}, v^2_{3(i-1)+3}\},\\ S(v^3_{3(i-1)+1}) = \{v^3_{3(i-1)+1}, v^1_{3(i-1)+2}, v^1_{3(i-1)+3}\},\\ S(v^3_{3(i-1)+2}) = \{v^1_{3(i-1)+1}, v^3_{3(i-1)+2}, v^1_{3(i-1)+3}\},\\ S(v^3_{3(i-1)+3}) = \{v^1_{3(i-1)+1}, v^1_{3(i-1)+2}, v^3_{3(i-1)+3}\}. \end{cases}$$

When $r=0$, we put

$$\begin{aligned} S(v^1_{3t+1}) &= S(v^2_{3t+1}) = S(v^3_{3t+1}) = S(v^1_{3t+2}) = S(v^2_{3t+2})\\ = S(v^3_{3t+2}) &= S(v^1_{3t+3}) = S(v^2_{3t+3}) = S(v^3_{3t+3}) = \emptyset. \end{aligned}$$

When $r=1$, we put

$$\begin{aligned} S(v^1_{3t+1}) = S(v^3_{3t+2}) = S(v^3_{3t+3}) = \{v^1_{3t+1}\},\\ S(v^2_{3t+1}) = S(v^1_{3t+2}) = S(v^1_{3t+3}) = \{v^2_{3t+1}\},\\ S(v^3_{3t+1}) = S(v^2_{3t+2}) = S(v^2_{3t+3}) = \{v^3_{3t+1}\}. \end{aligned}$$

When $r=2$, we put

$$\begin{aligned} S(v^1_{3t+1}) = \{v^1_{3t+1}, v^2_{3t+2}\}, S(v^2_{3t+1}) = \{v^2_{3t+1}, v^3_{3t+2}\}, S(v^3_{3t+1}) = \{v^3_{3t+1}, v^1_{3t+2}\},\\ S(v^1_{3t+2}) = \{v^2_{3t+1}, v^1_{3t+2}\}, S(v^2_{3t+2}) = \{v^3_{3t+1}, v^2_{3t+2}\}, S(v^3_{3t+2}) = \{v^1_{3t+1}, v^3_{3t+2}\},\\ S(v^1_{3t+3}) = \{v^2_{3t+1}, v^2_{3t+2}\}, S(v^2_{3t+3}) = \{v^3_{3t+1}, v^3_{3t+2}\}, S(v^3_{3t+3}) = \{v^1_{3t+1}, v^1_{3t+2}\}. \end{aligned}$$

We demonstrate $9t+9$ cliques of K_m^3 as below:

$$\begin{cases} C_{1,i} = S(v^1_{3(i-1)+1}) \cup \big(\bigcup_{\ell=i+1}^{t+1} S(v^1_{3(\ell-1)+1})\big),\\ C_{2,i} = S(v^1_{3(i-1)+2}) \cup \big(\bigcup_{\ell=i+1}^{t+1} S(v^2_{3(\ell-1)+1})\big),\\ C_{3,i} = S(v^1_{3(i-1)+3}) \cup \big(\bigcup_{\ell=i+1}^{t+1} S(v^3_{3(\ell-1)+1})\big),\\ C_{4,i} = S(v^2_{3(i-1)+1}) \cup \big(\bigcup_{\ell=i+1}^{t+1} S(v^1_{3(\ell-1)+1})\big),\\ C_{5,i} = S(v^2_{3(i-1)+2}) \cup \big(\bigcup_{\ell=i+1}^{t+1} S(v^2_{3(\ell-1)+1})\big),\\ C_{6,i} = S(v^2_{3(i-1)+3}) \cup \big(\bigcup_{\ell=i+1}^{t+1} S(v^3_{3(\ell-1)+1})\big),\\ C_{7,i} = \mathrm{V}(\mathrm{K}_m^3) \cap \Big(\{v^3_{3(i-2)+1}\} \cup S(v^3_{3(i-1)+1}) \cup \big(\bigcup_{\ell=i+1}^{t+1} S(v^1_{3(\ell-1)+1})\big)\Big),\\ C_{8,i} = \mathrm{V}(\mathrm{K}_m^3) \cap \Big(\{v^3_{3(i-2)+2}\} \cup S(v^3_{3(i-1)+2}) \cup \big(\bigcup_{\ell=i+1}^{t+1} S(v^2_{3(\ell-1)+1})\big)\Big),\\ C_{9,i} = \mathrm{V}(\mathrm{K}_m^3) \cap \Big(\{v^3_{3(i-2)+3}\} \cup S(v^3_{3(i-1)+3}) \cup \big(\bigcup_{\ell=i+1}^{t+1} S(v^3_{3(\ell-1)+1})\big)\Big), \end{cases} \tag{3}$$

where $i \in [t+1]$. We claim that these $9t+9$ cliques form an edge clique cover of K_m^3. Take $v_{i_1}^{j_1} v_{i_2}^{j_2} \in \mathrm{E}(\mathrm{K}_m^3)$ with $i_1 < i_2$. We need to prove that $v_{i_1}^{j_1} v_{i_2}^{j_2}$ is contained in one of the cliques as listed in (3). Assume that $i_1 = 3t_1 + r_1, i_2 = 3t_2 + r_2$ where $t_1, t_2 \in \{0, 1, \ldots, t\}$ and $r_1, r_2 \in [3]$. If $t_1 = t_2$, it is clear that $v_{i_1}^{j_1} v_{i_2}^{j_2} \subseteq S(v_{3t_1+r}^{\ell})$ for some $\ell, r \in [3]$, and $v_{i_1}^{j_1} v_{i_2}^{j_2}$ is contained in the clique $C_{3(\ell-1)+r,t_1+1}$. We next turn to the case of $t_1 < t_2$. We can find that $v_{i_1}^{j_1} \in S(v_{3t_1+1}^{s_1}) \cap S(v_{3t_1+2}^{s_2}) \cap S(v_{3t_1+3}^{s_3})$ for some $s_1, s_2, s_3 \in [3]$. Note that $v_{i_2}^{j_2} \in S(v_{3t_2+1}^{\ell})$ for some $\ell \in [3]$. Therefore, $v_{i_1}^{j_1} v_{i_2}^{j_2}$ is contained in the clique $C_{3(s_\ell-1)+\ell,t_1+1}$.

Define D to be the digraph with $\mathrm{V}(D) = \mathrm{V}(\mathrm{K}_m^3) \sqcup \{v_0^3, v_{-1}^3, v_{-2}^3\}$ and $\mathrm{A}(D) = \bigcup_{\ell=1}^{9} A_\ell$ where

$$\begin{cases}
A_1 = \bigcup_{i=1}^{t+1} \left\{(v, v_{3(i-1)+1}^1) : \ v \in C_{1,i} \setminus \{v_{3(i-1)+1}^1\}\right\}, \\
A_2 = \bigcup_{i=1}^{t+1} \left\{(v, v_{3(i-1)+2}^1) : \ v \in C_{2,i} \setminus \{v_{3(i-1)+2}^1\}\right\}, \\
A_3 = \bigcup_{i=1}^{t+1} \left\{(v, v_{3(i-1)+3}^1) : \ v \in C_{3,i} \setminus \{v_{3(i-1)+3}^1\}\right\}, \\
A_4 = \bigcup_{i=1}^{t+1} \left\{(v, v_{3(i-1)+1}^2) : \ v \in C_{4,i} \setminus \{v_{3(i-1)+1}^2\}\right\}, \\
A_5 = \bigcup_{i=1}^{t+1} \left\{(v, v_{3(i-1)+2}^2) : \ v \in C_{5,i} \setminus \{v_{3(i-1)+2}^2\}\right\}, \\
A_6 = \bigcup_{i=1}^{t+1} \left\{(v, v_{3(i-1)+3}^2) : \ v \in C_{6,i} \setminus \{v_{3(i-1)+3}^2\}\right\}, \\
A_7 = \bigcup_{i=1}^{t+1} \left\{(v, v_{3(i-2)+1}^3) : \ v \in C_{7,i} \setminus \{v_{3(i-2)+1}^3\}\right\}, \\
A_8 = \bigcup_{i=1}^{t+1} \left\{(v, v_{3(i-2)+2}^3) : \ v \in C_{8,i} \setminus \{v_{3(i-2)+2}^3\}\right\}, \\
A_9 = \bigcup_{i=1}^{t+1} \left\{(v, v_{3(i-2)+3}^3) : \ v \in C_{9,i} \setminus \{v_{3(i-2)+3}^3\}\right\}.
\end{cases}$$

The digraph D is acyclic as it has the following acyclic ordering $\prec$:

$$\begin{aligned}
&v_m^3 \prec \cdots \prec v_{3i+3}^3 \prec v_{3i+2}^3 \prec v_{3i+1}^3 \prec v_{3i+3}^2 \prec v_{3i+2}^2 \prec v_{3i+1}^2 \prec v_{3i+3}^1 \prec v_{3i+2}^1 \\
&\prec v_{3i+1}^1 \prec v_{3(i-1)+3}^3 \prec v_{3(i-1)+2}^3 \prec v_{3(i-1)+1}^3 \prec \cdots \prec v_1^1 \prec v_0^3 \prec v_{-1}^3 \prec v_{-2}^3.
\end{aligned}$$

Note that the edge clique cover of K_m^3 as shown in (3) is nothing but $\{(\mathrm{N}_D^-(v) \cup \{v\}) \cap \mathrm{V}(\mathrm{K}_m^3) : \ v \in \mathrm{V}(D)\}$. By Lemma 1 (ii), we have $\mathrm{K}_m^3 \lhd \mathcal{P}(D)$. Therefore, $\phi(\mathrm{K}_m^3) \leq 3$ for $m \geq 3$, as expected.

(iii). Thanks to (i) and (ii), we can restrict our attention to the case of $n \geq 4$. Theorem 1 (iii) says that $\kappa(\mathrm{K}_3^n) = n^2 - 3n + 4$. By Lemma 2 (i), it suffices to show $\phi(\mathrm{K}_3^n) \leq n^2 - 3n + 3$.

Let $\mathscr{S} = \{S_\ell\}_{\ell=1}^{n^2}$ be an edge clique cover of K_3^n satisfying the condition of Lemma 3, namely $S_1, \ldots, S_{3n-3}$ are specified by (1). Construct D to be the digraph with $\mathrm{V}(D) = \mathrm{V}(\mathrm{K}_3^n) \sqcup \{z_1, \ldots, z_{n^2-3n+3}\}$ and $\mathrm{A}(D) = (\bigcup_{\ell=1}^{3n-3} A_\ell) \cup$

$\left(\bigcup_{\ell=1}^{n^2-3n+3}\{(v,z_\ell):\ v\in S_{\ell+3n-3}\}\right)$ where

$$\begin{array}{llll}
A_1 = \{(v_1^1,v_3^1),(v_2^1,v_3^1)\}, & A_2 = \{(v_1^2,v_2^n),(v_3^1,v_2^n)\},\\
A_3 = \{(v_1^1,v_3^n),(v_2^n,v_3^n)\}, & A_4 = \{(v_2^1,v_1^n),(v_3^n,v_1^n)\},\\
A_5 = \{(v_1^n,v_2^2),(v_3^1,v_2^2)\}, & A_6 = \{(v_1^1,v_3^2),(v_2^2,v_3^2)\},\\
A_7 = \{(v_2^2,v_1^{n-1}),(v_3^n,v_1^{n-1})\}, & A_8 = \{(v_1^2,v_2^{n-1}),(v_3^n,v_2^{n-1})\},\\
A_9 = \{(v_1^1,v_3^{n-1}),(v_2^{n-1},v_3^{n-1})\}, & \\
 & \cdots \\
A_{3s+4} = \{(v_2^2,v_1^{n-s}),(v_3^{n-s+1},v_1^{n-s})\}, & A_{3s+5} = \{(v_1^2,v_2^{n-s}),(v_3^{n-s+1},v_2^{n-s})\},\\
A_{3s+6} = \{(v_1^1,v_3^{n-s}),(v_2^{n-s},v_3^{n-s})\}, & \\
 & \cdots \\
A_{3n-5} = \{(v_2^2,v_1^3),(v_3^4,v_1^3)\}, & A_{3n-4} = \{(v_1^2,v_2^3),(v_3^4,v_2^3)\},\\
A_{3n-3} = \{(v_1^1,v_3^3),(v_2^3,v_3^3)\}. &
\end{array}$$

The digraph D is acyclic due to the existence of the acyclic ordering $\prec$:

$$\begin{aligned}
&v_1^1 \prec v_2^1 \prec v_3^1 \prec v_1^2 \prec v_2^n \prec v_3^n \prec v_1^n \prec v_2^2 \prec v_3^2 \prec v_1^{n-1} \prec v_2^{n-1} \prec v_3^{n-1}\\
&\prec v_1^{n-2} \prec v_2^{n-2} \prec v_3^{n-2} \prec \cdots \prec v_1^3 \prec v_2^3 \prec v_3^3 \prec z_1 \prec \cdots \prec z_{n^2-3n+3}.
\end{aligned}$$

It is easy to see that $\left\{\left(\mathrm{N}_D^-(v)\cup\{v\}\right)\cap \mathrm{V}(\mathrm{K}_3^n):\ v\in \mathrm{V}(D)\right\}$ coincides with the edge clique cover $\mathscr{S}$ of K_3^n. By Lemma 1 (ii), $\mathrm{K}_3^n \lhd \mathcal{P}(D)$ holds. Therefore, $\phi(\mathrm{K}_3^n) \le n^2-3n+3$ for $n\ge 4$, finishing the proof. □

Proof (Proof of Theorem 3) We assume $n_1 \le \cdots \le n_m$ and write k for $\prod_{\ell=1}^m n_\ell + m - \sum_{\ell=1}^m n_\ell$. When $n_m = 1$, H is the clique hypergraph of the complete graph $\mathrm{K}_m^1 = \mathrm{K}_m$ and so $\phi_{\mathrm{ST}}(H)+1 = \kappa_{\mathrm{ST}}(H) = 1 = k$. In the following, we will stipulate that $n_m \ge 2$.

First of all, we prove that $\phi_{\mathrm{ST}}(H) \ge \kappa_{\mathrm{ST}}(H) - 1 \ge k-1$. Suppose D is an acyclic digraph such that $\mathcal{CH}(D) = \mathrm{I}_{\kappa_{\mathrm{ST}}(H)}(H)$. The hypergraph H has $\sum_{\ell=1}^m n_\ell$ vertices and $\prod_{\ell=1}^m n_\ell$ hyperedges of the same size m. Therefore, in D there are at least $\prod_{\ell=1}^m n_\ell$ vertices which have m in-neighbors. Meanwhile, since D is acyclic, it possesses an acyclic ordering $\prec$ and so every vertex among the smallest m elements in $\prec$ has less than m in-neighbors in D. Therefore, $\kappa_{\mathrm{ST}}(H) \ge \prod_{\ell=1}^m n_\ell + m - \sum_{\ell=1}^m n_\ell = k$. By Lemma 2 (ii), $\phi_{\mathrm{ST}}(H) \ge \kappa_{\mathrm{ST}}(H) - 1 \ge k-1$.

Secondly, we intend to show $\kappa_{\mathrm{ST}}(H) = k$. For this purpose, what remains is to get $\kappa_{\mathrm{ST}}(H) \le k$. We need to find an acyclic digraph D_1 with $\mathrm{V}(D_1) = \mathrm{V}(H) \sqcup \{z_t:\ t\in[k]\}$ such that $H \lhd \mathcal{CH}(D_1)$. Let us specify a total order $\prec$ on $\mathrm{V}(D_1)$ such that

$$\begin{aligned}
&v_1^1 \prec v_2^1 \prec \cdots \prec v_m^1 \prec v_1^2 \prec v_1^3 \prec \cdots \prec v_1^{n_1} \prec v_2^2 \prec v_2^3 \prec \cdots \prec v_2^{n_2}\\
&\prec \cdots \prec v_m^2 \prec v_m^3 \prec \cdots \prec v_m^{n_m} \prec z_1 \prec z_2 \prec \cdots \prec z_k.
\end{aligned} \tag{4}$$

In (4), when we are writing $v_1^2 \prec v_1^3 \prec \cdots \prec v_1^{n_1}$, we really refer to an empty string if $n_1 = 1$ and refer to v_1^2 when $n_1 = 2$. For every $v_i^j \in \mathrm{V}(H)$, let $\xi(v_i^j)$ be the element covering v_i^j in $\prec$, namely $\xi(v_i^j)$ is the minimum element in $\prec$ which is bigger than v_i^j. Let $\mathrm{A}(D_1) = \left\{\left(u,\xi(v_i^j)\right):\ u\in\left(\{v_1^1,\ldots,v_m^1\}\cup\{v_i^j\}\right)\setminus\right.$

$\{v_i^1\}, v_i^j \in \mathrm{V}(H) \setminus \{v_1^1, \ldots, v_{m-1}^1, v_m^{n_m}\}\} \cup \big(\bigcup_{t=1}^{k}\{(w, z_t) : w \in e_t\}\big)$, where $e_1, \ldots, e_k$ are the k hyperedges from the set

$$\mathscr{E}(H) \setminus \big\{\big(\{v_1^1, \ldots, v_m^1\} \cup \{v_i^j\}\big) \setminus \{v_i^1\} : v_i^j \in \mathrm{V}(H) \setminus \{v_1^1, \ldots, v_{m-1}^1, v_m^{n_m}\}\big\}.$$

It is easy to see $\mathrm{A}(D_1) \subseteq \{(v, w) \in \mathrm{V}(D_1) \times \mathrm{V}(D_1) : v \prec w\}$ and so $\prec$ is an acyclic ordering of D_1. Moreover, one can check that $\{\mathrm{N}^-_{D_1}(v) \cap \mathrm{V}(H) : v \in \mathrm{V}(D_1), |\mathrm{N}^-_{D_1}(v) \cap \mathrm{V}(H)| \geq 2\} = \mathscr{E}(H)$. By Lemma 1 (iii), we obtain $H \lhd \mathcal{CH}(D_1)$, as expected.

Finally, we want to prove $\phi_{\mathrm{ST}}(H) = k-1$. It now suffices to show $\phi_{\mathrm{ST}}(H) \leq k-1$. That is, we are going to construct an acyclic digraph D_2 with $\mathrm{V}(D_2) = \mathrm{V}(H) \sqcup \{\bar{z}_t : t \in [k-1]\}$ such that $H \lhd \mathcal{PH}(D_2)$. Let $\bar{e}_1, \ldots, \bar{e}_{k-1}$ be the $k-1$ hyperedges from $\mathscr{E}(H) \setminus \big\{\big(\{v_1^1, \ldots, v_m^1\} \cup \{v_i^j\}\big) \setminus \{v_i^1\} : v_i^j \in \mathrm{V}(H) \setminus \{v_1^1, \ldots, v_{m-1}^1\}\big\}$. Define $\mathrm{A}(D_2) = \big\{(u, v_i^j) : u \in \{v_1^1, \ldots, v_m^1\} \setminus \{v_i^1\}, v_i^j \in \mathrm{V}(H) \setminus \{v_1^1, \ldots, v_{m-1}^1\}\big\} \cup \big(\bigcup_{t=1}^{k-1}\{(w, \bar{z}_t) : w \in \bar{e}_t\}\big)$. We can verify that D_2 has an acyclic ordering $\prec$ such that

$$\begin{aligned} v_1^1 \prec v_2^1 \prec \cdots \prec v_m^1 \prec v_1^2 \prec v_1^3 \prec \cdots \prec v_1^{n_1} \prec v_2^2 \prec v_2^3 \prec \cdots \prec v_2^{n_2} \\ \prec \cdots \prec v_m^2 \prec v_m^3 \prec \cdots \prec v_m^{n_m} \prec z_1 \prec z_2 \prec \cdots \prec z_{k-1}, \end{aligned}$$

where this ordering should be understood as explained after (4). Note that $\big\{\big(\mathrm{N}^-_{D_2}(v) \cup \{v\}\big) \cap \mathrm{V}(H) : v \in \mathrm{V}(D_2), |\big(\mathrm{N}^-_{D_2}(v) \cup \{v\}\big) \cap \mathrm{V}(H)| \geq 2\big\} = \mathscr{E}(H)$. It follows from Lemma 1 (iv) that $H \lhd \mathcal{PH}(D_2)$, as was to be shown. □

Lemma 4 *(Park-Kim-Sano [PKS09, Theorem 2.1]) Let m and n be positive integers with $3 \leq m \leq \mathcal{L}(n)+2$ and let $L = \{L_1, \ldots, L_{m-2}\}$ be a set of MOLS of order n and size $m-2$. Then $\theta_e(\mathrm{K}_m^n) = n^2$ and an edge clique cover of K_m^n can be enumerated as*

$$\mathcal{S}_{x,y} = \{v_1^x, v_2^y, v_3^{L_1(x,y)}, v_4^{L_2(x,y)}, \ldots, v_m^{L_{m-2}(x,y)}\}, \tag{5}$$

where x and y run through all elements of $[n]$.

Lemma 5 *Let m and n be positive integers such that $3 \leq m \leq \mathcal{L}(n)+1$. There exists an edge clique cover $\mathscr{S}$ of K_m^n that satisfies the following two properties:*

(i) $|\mathscr{S}| = n^2 = \theta_e(\mathrm{K}_m^n)$;

(ii) $\mathscr{S}$ can be partitioned into $\mathcal{F}_1, \ldots, \mathcal{F}_n$ such that $\mathcal{F}_\ell$ is a vertex clique cover of K_m^n for all $\ell \in [n]$.

Proof Let $L = \{L_1, \ldots, L_{m-1}\}$ be a set of MOLS of order n and let $\mathcal{S}_{x,y}$ be defined as in (5) for $x, y \in [n]$. Let $\mathscr{S} = \{\mathcal{S}_{x,y} : x, y \in [n]\}$. By Lemma 4, $\mathscr{S}$ is an edge clique cover of K_m^n with size $|\mathscr{S}| = n^2 = \theta_e(\mathrm{K}_m^n)$.

For each $\ell \in [n]$, define $\mathcal{F}_\ell = \{\mathcal{S}_{x,y} : L_{m-1}(x, y) = \ell\}$. It is clear that $\mathcal{F}_1, \ldots, \mathcal{F}_n$ is a partition of $\mathscr{S}$. Pick $v_i^j \in \mathrm{V}(\mathrm{K}_m^n)$ and take $\ell \in [n]$. Our goal is to prove that v_i^j appears in some clique from $\mathcal{F}_\ell$.

We first address the case of $i = 1$. Since L_{m-1} is a Latin square, there exists $y \in [n]$ such that $L_{m-1}(j, y) = \ell$. This means that $\mathcal{S}_{j,y}$ is a clique from $\mathcal{F}_\ell$ which contains v_i^j.

Next assume that $i = 2$. Since L_{m-1} is a Latin square, there exists $x \in [n]$ such that $L_{m-1}(x, j) = \ell$. This means that $\mathcal{S}_{x,j}$ is a clique from $\mathcal{F}_\ell$ which contains v_i^j.

Now, we deal with the remaining case that $3 \le i \le m$. Note that L_{i-2} and L_{m-1} are orthogonal. Henceforth, there is a unique $(x, y) \in [n] \times [n]$ such that $L_{i-2}(x, y) = j$ and $L_{m-1}(x, y) = \ell$. We thus see that $\mathcal{S}_{x,y}$ is a clique from $\mathcal{F}_\ell$ which contains v_i^j. □

The statement of Lemma 5 can be found basically in [KPS12, p. 1177]. For a proof of the statement there, Kim, Park and Sano [KPS12, p. 1177] suggested that the required partition $\mathcal{F}_1, \ldots, \mathcal{F}_n$ of the edge clique cover $\{\mathcal{S}_{x,y} : x, y \in [n]\}$ of K_m^n as shown in (5) can be chosen as $\mathcal{F}_t = \{\mathcal{S}_{1,1+t}, \ldots, \mathcal{S}_{n,n+t}\}$ for $t \in [n]$, where the subscripts should be read modulo n. We point out here that this partition does not always work. Take $n = 3$ and $m = 3$. Let

$$L_1 = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \text{ and } L_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}.$$

Clearly, $\{L_1, L_2\}$ is a set of MOLS of order $n = 3$ and size $\mathcal{L}(3) = 2$. Observe that $3 \le m \le 3 = \mathcal{L}(n) + 1$. Following the construction from [KPS12, p. 1177], we have $\mathcal{F}_1 = \{\mathcal{S}_{1,2}, \mathcal{S}_{2,3}, \mathcal{S}_{3,1}\} = \big\{\{v_1^1, v_2^2, v_3^3\}, \{v_1^2, v_2^3, v_3^3\}, \{v_1^3, v_2^1, v_3^3\}\big\}$. But $\mathcal{F}_1$ is not a vertex clique cover of K_3^3 as it covers neither v_3^1 nor v_3^2.

For every positive integer n, let $\varGamma_n$ be the set of all positive integers k for which we can find a digraph D such that $\mathcal{C}(D) = \mathrm{I}_k(\mathrm{K}_n^n)$ and that

$$\mathscr{S} = \{\mathrm{N}_D^-(v) \cap \mathrm{V}(\mathrm{K}_n^n) : \ v \in \mathrm{V}(D), |\,\mathrm{N}_D^-(v) \cap \mathrm{V}(\mathrm{K}_n^n)| \ge 1\}$$

is an edge clique cover of K_n^n satisfying the ensuing properties:

(i) $|\,\mathscr{S}\,| = n^2$;
(ii) $\mathscr{S}$ can be partitioned into $\mathcal{F}_1, \ldots, \mathcal{F}_n$ such that $\mathcal{F}_\ell$ is a vertex clique cover of K_n^n for every $\ell \in [n]$.

According to Lemma 5, we know that $\varGamma_n$ is always nonempty.

Lemma 6 *(Kim-Park-Sano [KPS12, Theorem 2]) Let m, n and k be positive integers such that $n \ge 2$, $k \in \varGamma_n$ and $m \ge n$. Then it holds* $\kappa(\mathrm{K}_m^n) \le \max\{k, \max_{t \in [n-1]} \kappa(\mathrm{K}_t^n) - 1\}$.

Lemma 7 *Let n be a positive integer such that $\mathcal{L}(n) = n - 1$. Then $n^2 - 2n + 2 \in \varGamma_n$.*

Proof Taking $m = n$, an application of Theorem 4 and Lemma 5 ensures $n^2 - 2n + 2 \in \varGamma_n$.

Proof (Proof of Theorem 6) When $n = 2$ and $m \ge 2$, Theorem 1 (i) says that $\kappa(\mathrm{K}_m^2) = 2 = n^2 - 2n + 2$.

When $(n, m) = (3, 2)$, Remark 1 gives $\kappa(\mathrm{K}_m^n) = n^2 - 2n + 2$. When $n = 3$ and $m \ge 3$, Theorem 1 (ii) implies that $\kappa(\mathrm{K}_m^3) = 4 < 5 = n^2 - 2n + 2$.

We now assume $n \geq 4$ and $\mathcal{L}(n) = n-1$. By virtue of Theorem 4, Lemma 6 and Lemma 7, $\kappa(\mathrm{K}_m^n) \leq n^2-2n+2$ for $m \geq n$. Coupled with Theorem 4 again, this completes the proof. □

Proof (Proof of Theorem 8) (i). For $\ell = 1, 2$, let D_ℓ be an acyclic digraph such that $H_\ell \lhd \mathcal{PH}(D_\ell)$ and $|\mathrm{V}(D_\ell)| - |\mathrm{V}(H_\ell)| = \phi_{\mathrm{ST}}(H_\ell)$. Clearly, $(H_1 \sqcup H_2) \lhd \mathcal{PH}(D_1 \sqcup D_2)$, which then implies $\phi_{\mathrm{ST}}(H_1 \sqcup H_2) \leq \phi_{\mathrm{ST}}(H_1) + \phi_{\mathrm{ST}}(H_2)$.

Suppose D' is an acyclic digraph satisfying $(H_1 \sqcup H_2) \lhd \mathcal{PH}(D')$ and $|\mathrm{V}(D')| - |\mathrm{V}(H_1 \sqcup H_2)| = \phi_{\mathrm{ST}}(H_1 \sqcup H_2)$. For $\ell = 1, 2$, let D'_ℓ be the subgraph of D' induced by $\mathrm{V}(H_\ell) \cup \{w : (u, w) \in \mathrm{A}(D'), u \in \mathrm{V}(H_\ell)\}$. It is not hard to see that $H_\ell \lhd \mathcal{PH}(D'_\ell)$ and $\mathrm{V}(D'_1) \cap \mathrm{V}(D'_2) = \emptyset$. Therefore, $\phi_{\mathrm{ST}}(H_1 \sqcup H_2) = |\mathrm{V}(D')| - |\mathrm{V}(H_1 \sqcup H_2)| = (|\mathrm{V}(D'_1)| - |\mathrm{V}(H_1)|) + (|\mathrm{V}(D'_2)| - |\mathrm{V}(H_2)|) \geq \phi_{\mathrm{ST}}(H_1) + \phi_{\mathrm{ST}}(H_2)$, and so we are done.

(ii). Without loss of generality, we may assume that $\kappa(G_1) \geq \kappa(G_2)$. When $\kappa(G_1) = 0$, it is obvious that $\kappa(G_1 \sqcup G_2) = \kappa(G_1) + \kappa(G_2) = 0$. We now consider the case that $\kappa(G_1) \geq 1$. For each $i \in [2]$, let D_i be the acyclic digraph such that $\mathcal{C}(D_i) = \mathrm{I}_{\kappa(G_i)}(G_i)$. Since $\kappa(G_1) \geq 1$, we can find a vertex $v \in \mathrm{V}(D_1) \backslash \mathrm{V}(G_1)$. The acyclicity of D_2 allows us to obtain a vertex $u \in \mathrm{V}(D_2)$ such that $\mathrm{N}^-_{D_2}(u) = \emptyset$. Let D be the digraph obtained from $D_1 \sqcup D_2$ by deleting the vertex v and adding arcs (w, u) where w runs through $\mathrm{N}^-_{D_1}(v)$. It is not difficult to check that D is acyclic and that $G_1 \sqcup G_2$ is an induced subgraph of $\mathcal{C}(D)$. This implies that $\kappa(G_1 \sqcup G_2) \leq \kappa(G_1) + \kappa(G_2) - 1$.

(iii). We can follow the idea of proving (ii) to give a proof for this claim. □

Lemma 8 *(i) For any integers n and k such that $n \geq 2$ and $0 \leq k \leq n^2 - 3n + 4$, it holds $\phi\left(\mathrm{I}_k(\mathrm{K}_3^n)\right) - \kappa\left(\mathrm{I}_k(\mathrm{K}_3^n)\right) + 1 = k$.*

(ii) For positive integers n and m and a nonnegative integer k such that $m \geq 2$ and $0 \leq k \leq n^m - mn + m$, it holds $\phi_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(\mathrm{K}_m^n)\right)\right) - \kappa_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(\mathrm{K}_m^n)\right)\right) + 1 = k$.

Proof (i). Theorem 1 (iii) implies $\kappa\left(\mathrm{I}_k(\mathrm{K}_3^n)\right) = \kappa(\mathrm{K}_3^n) - k = n^2 - 3n + 4 - k$ when $k \leq n^2 - 3n + 4$. Meanwhile, taking into account Theorem 2 (iii) and Theorem 7, we obtain $\phi\left(\mathrm{I}_k(\mathrm{K}_3^n)\right) = \phi(\mathrm{K}_3^n) = n^2 - 3n + 3$. Therefore, it holds $\phi\left(\mathrm{I}_k(\mathrm{K}_3^n)\right) - \kappa\left(\mathrm{I}_k(\mathrm{K}_3^n)\right) + 1 = k$, as required.

(ii). We deduce from Theorem 3 that

$$\kappa_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(\mathrm{K}_m^n)\right)\right) = \kappa_{\mathrm{ST}}\left(\mathcal{K}(\mathrm{K}_m^n)\right) - k = n^m - mn + m - k.$$

Moreover, Theorem 3 and Theorem 8 (i) lead to

$$\phi_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(\mathrm{K}_m^n)\right)\right) = \phi_{\mathrm{ST}}\left(\mathcal{K}(\mathrm{K}_m^n)\right) = n^m - mn + m - 1.$$

We thus get to $\phi_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(\mathrm{K}_m^n)\right)\right) - \kappa_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(\mathrm{K}_m^n)\right)\right) + 1 = k$, finishing the proof. □

Proof (Proof of Theorem 9) (i). By Lemma 2 (i), $\phi(G) - \kappa(G) + 1 \geq 0$ holds for every graph G. Take any nonnegative integer k. There exists a positive integer n satisfying $k \leq n^2 - 3n + 4$. By Lemma 8 (i), $\phi\left(\mathrm{I}_k(\mathrm{K}_3^n)\right) - \kappa\left(\mathrm{I}_k(\mathrm{K}_3^n)\right) + 1 = k$.

(ii). Thanks to Lemma 2 (ii), $\phi_{\mathrm{ST}}(H)-\kappa_{\mathrm{ST}}(H)+1 \geq 0$ for every hypergraph H. Let k be any nonnegative integer. We can find positive integers n and m such that $m \geq 2$ and $k < n^m - mn + m$. By Lemma 8 (ii), we have $\phi_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(\mathrm{K}_m^n)\right)\right) - \kappa_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(\mathrm{K}_m^n)\right)\right) + 1 = k$, which ends the proof. □

4 Further research

Inspired by Theorem 2, we are interested in learning whether or not $\phi(G) - \kappa(G) + 1 = 0$ holds for all uniform complete multipartite graphs G. What happens if G is a strongly regular graph?

A graph G can be regarded as a hypergraph H_G with $\mathrm{V}(\mathrm{H}_G) = \mathrm{V}(G)$ and $\mathscr{E}(\mathrm{H}_G) = \mathrm{E}(G)$. How to understand the structure of those graphs G satisfying $\kappa_{\mathrm{ST}}(\mathrm{H}_G) = \kappa(G)$ or $\phi_{\mathrm{ST}}(\mathrm{H}_G) = \phi(H_G)$?

Note that the simple 'adding loops' operation is all the difference between defining competition numbers and phylogeny numbers. Theorem 9 illustrates the consequence of this small twist when considering all graphs and hypergraphs. Will this operation make bigger difference if we restrict our attention to some special classes of graphs and hypergraphs? What are the ranges of $\phi(G) - \kappa(G) + 1$ and $\phi_{\mathrm{ST}}(H) - \kappa_{\mathrm{ST}}(H) + 1$ when the graphs G and hypergraphs H are connected? Some progress on tackling this question can be found in [XZZ19].

Acknowledgements We are grateful to Peter Cameron for teaching us some useful facts on Latin squares. We thank the referee for providing a simple proof of Theorem 8 (ii). This work was supported by STCSM (17690740800) and NSFC (11671258).

References

Coh68. Joel E. Cohen. *Interval Graphs and Food Webs: A finding and a problem.* RAND Corporation, 1968. URL: http://lab.rockefeller.edu/cohenje/assets/file/014.1CohenIntervalGraphsFoodWebsRAND1968.pdf.

Coh78. Joel E. Cohen. *Food Webs and Niche Space.* Princeton University Press, 1978. URL: https://press.princeton.edu/titles/324.html.

GST16. Christian Garske, Martin Sonntag, and Hanns-Martin Teichert. Niche hypergraphs. *Discuss. Math. Graph Theory*, 36(4):819–832, 2016. doi:10.7151/dmgt.1893.

KD15. A. Donald Keedwell and József Dénes. *Latin Squares and Their Applications.* Elsevier/North-Holland, Amsterdam, second edition, 2015. With a foreword to the previous edition by Paul Erdös. URL: https://www.sciencedirect.com/book/9780444635556/latin-squares-and-their-applications.

KPS12. Suh-Ryung Kim, Boram Park, and Yoshio Sano. The competition numbers of complete multipartite graphs with many partite sets. *Discrete Appl. Math.*, 160(7-8):1176–1182, 2012. doi:10.1016/j.dam.2011.12.017.

KS08. Suh-Ryung Kim and Yoshio Sano. The competition numbers of complete tripartite graphs. *Discrete Appl. Math.*, 156(18):3522–3524, 2008. doi:10.1016/j.dam.2008.04.009.

LC12. Bo-Jr Li and Gerard J. Chang. Competition numbers of complete r-partite graphs. *Discrete Appl. Math.*, 160(15):2271–2276, 2012. doi:10.1016/j.dam.2012.05.005.

LCKS17. Seung Chul Lee, Jihoon Choi, Suh-Ryung Kim, and Yoshio Sano. On the phylogeny graphs of degree-bounded digraphs. *Discrete Appl. Math.*, 233:83–93, 2017. doi:10.1016/j.dam.2017.07.018.

Ops82. Robert J. Opsut. On the computation of the competition number of a graph. *SIAM J. Algebraic Discrete Methods*, 3(4):420–428, 1982. doi:10.1137/0603043.

PKS08. Boram Park, Suh-Ryung Kim, and Yoshio Sano. On competition numbers of complete multipartite graphs with partite sets of equal size. RIMS-1644, October 2008. URL: http://www.kurims.kyoto-u.ac.jp/preprint/file/IMS1644.pdf.

PKS09. Boram Park, Suh-Ryung Kim, and Yoshio Sano. The competition numbers of complete multipartite graphs and mutually orthogonal Latin squares. *Discrete Math.*, 309(23-24):6464–6469, 2009. doi:10.1016/j.disc.2009.06.016.

Rob78. Fred S. Roberts. Food webs, competition graphs, and the boxicity of ecological phase space. In Yousef Alavi and Don R. Lick, editors, *Theory and Applications of Graphs*, volume 642 of *Lecture Notes in Mathematics*, pages 477–490. Springer-Verlag, Berlin-New York, 1978. doi:10.1007/BFb0070404.

RS98. Fred S. Roberts and Li Sheng. Phylogeny numbers. *Discrete Appl. Math.*, 87(1-3):213–228, 1998. doi:10.1016/S0166-218X(98)00058-4.

Shr61. Sharadchandra Shankar Shrikhande. A note on mutually orthogonal Latin squares. *Sankhyā Ser. A*, 23:115–116, 1961. URL: http://repository.ias.ac.in/73430/.

ST04. Martin Sonntag and Hanns-Martin Teichert. Competition hypergraphs. *Discrete Appl. Math.*, 143(1-3):324–329, 2004. doi:10.1016/j.dam.2004.02.010.

WL10. Yaokun Wu and Junjie Lu. Dimension-2 poset competition numbers and dimension-2 poset double competition numbers. *Discrete Appl. Math.*, 158(6):706–717, 2010. doi:10.1016/j.dam.2009.12.001.

XZZ19. Yanzhen Xiong, Soesoe Zaw, and Yinfeng Zhu. Competition numbers and phylogeny numbers of connected graphs and hypergraphs. submitted, 2019. URL: http://zhuyinfeng.org/Data/Preprints/AC20190013_origin.pdf.

Zha63. Li-qian Zhang. On the maximal number of orthogonal Latin squares. I. *Shuxue Jinzhan*, 6:201–204, 1963. URL: http://www.oaj.pku.edu.cn/sxjz/CN/Y1963/V-8/I2/201.