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Competition Numbers and Phylogeny Numbers

Uniform Complete Multipartite Graphs

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Abstract Let D be a digraph. The competition graph of D is the graph sharing the same vertex set with D such that two different vertices are adjacent if and only if they have a common out-neighbor in D; the phylogeny graph of D is the competition graph of the digraph obtained from D by adding a loop at every vertex. For any graph G with n vertices, its competition number $\kappa(G)$ is the least nonnegative integer k such that G is a vertex-induced subgraph of the competition graph of an acyclic digraph with $n + k$ vertices, while its phylogeny number $\phi(G)$ is the least nonnegative integer p such that G is a vertex-induced subgraph of the phylogeny graph of an acyclic digraph with $n + p$ vertices. This paper provides new estimates of the competition numbers and phylogeny numbers of complete multipartite graphs with uniform part sizes. Accordingly, we can show that the range of the function $\phi - \kappa + 1$ is the set of all nonnegative integers. We also report results about a hypergraph version of the competition numbers and phylogeny numbers.

Keywords Edge clique cover · Mutually orthogonal Latin squares

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1 Introduction

In 1968, based on the idea that two species compete if and only if they have a common prey, Cohen [1] proposed the concept of the competition graph of an acyclic digraph. He intended to apply this concept like a mathematical microscope to detect the structure of a food web [2]. Indeed, he has observed that almost all competition graphs arising from actual ecosystems are interval graphs. This motivates Roberts [12] to study how to recognize/characterize competition graphs. A finding of Roberts [12] then is that every graph can be made into a competition graph by adding some isolated vertices and so he defined the least such number to be the competition number of the graph. Surely, a graph is a competition graph if and only if its competition number is zero. Stimulated by problems of phylogenetic tree reconstruction, Roberts and Sheng [13] defined the concept of the phylogeny graph of an acyclic digraph and the concept of the phylogeny number of a graph. Note that phylogeny graphs are known as moral graphs in Bayesian network theory [7]. Opsut [9] found in 1982 that, it is an NP-complete problem to recognize competition graphs and to determine competition numbers. Analogously, Roberts and Sheng [13] proved that it is an NP-complete problem to calculate the phylogeny numbers. As an effort to tackle a difficult extremal combinatorics problem, many researchers have tried to determine the exact value or estimate the upper/lower bounds of the competition/phylogeny numbers of various classes of graphs. As a variant of competition graph, Sonntag and Teichert [15] proposed the definition of competition hypergraph in 2004, which naturally leads to the concepts of ST-competition numbers and ST-phylogeny numbers of hypergraphs. Some other interesting parameters similar to competition numbers and phylogeny numbers include double competition numbers and poset competition numbers of graphs and hypergraphs [4, 16].

This paper is concerned with the relationship between competition numbers and phylogeny numbers of graphs as well as ST-competition numbers and ST-phylogeny numbers of hypergraphs. We especially try to estimate these parameters for complete multipartite graphs. In § 2, we formally define all relevant concepts and describe some relevant former results as well as our main results. § 3 is devoted to proofs to our main results. We conclude the paper in § 4 with some research questions.

2 Main results

A *graph* G is a pair consisting of its vertex set $\mathrm{V}(G) \neq \emptyset$ and its edge set $\mathrm{E}(G) \subseteq \binom{\mathrm{V}(G)}{2}$. For each graph G and nonnegative integer k, let $\mathrm{I}_k(G)$ stand for the graph obtained from G by adding k isolated vertices. A *vertex-induced subgraph* of a graph G, or simply known as a subgraph of G, is a graph G' such that $\mathrm{V}(G') \subseteq \mathrm{V}(G)$ and $\mathrm{E}(G') = \mathrm{E}(G) \cap \binom{\mathrm{V}(G')}{2}$. Let us write $G' \lhd G$ to mean that G' is a subgraph of G. A *hypergraph* H comprises its vertex set $\mathrm{V}(H) \neq \emptyset$ and its hyperedge set $\mathscr{E}(H) \subseteq \binom{\mathrm{V}(H)}{\geq 2}$. For each hypergraph H and nonnegative integer k, let $\mathrm{I}_k(H)$ stand for the hypergraph for which $\mathscr{E}\left(\mathrm{I}_k(H)\right)$ equals $\mathscr{E}(H)$ and $\mathrm{V}\left(\mathrm{I}_k(H)\right) \setminus \mathrm{V}(H)$ is a set of size k. The *subhypergraph* induced by a nonempty subset $A \subseteq \mathrm{V}(H)$ is the hypergraph H' with vertex set A and hyperedge set $\mathscr{E}(H') = \{e \cap A : \ e \in \mathscr{E}(H), |e \cap A| \geq 2\}$. For two hypergraphs H and H', we write $H' \lhd H$ to mean that H' is a subhypergraph of H. A digraph D is a pair consisting of its vertex set $\mathrm{V}(D) \neq \emptyset$ and its arc set $\mathrm{A}(D) \subseteq \mathrm{V}(D) \times \mathrm{V}(D)$. For each digraph D, let D° stand for the digraph with $\mathrm{V}(D^\circ) = \mathrm{V}(D)$ and $\mathrm{A}(D^\circ) = \mathrm{A}(D) \cup \{(v,v) : \ v \in \mathrm{V}(D)\}$. For any $(u,v) \in \mathrm{A}(D)$, we call u an *in-neighbor* of v in D, and call v an *out-neighbor* of u in D. For each $v \in \mathrm{V}(D)$, let $\mathrm{N}_D^-(v)$ denote the set of in-neighbors of v, i.e., $\mathrm{N}_D^-(v) := \{u \in \mathrm{V}(D) : \ (u,v) \in \mathrm{A}(D)\}$. A digraph D is *acyclic* if it contains no cycle. Note that a digraph D is acyclic if and only if D has an *acyclic ordering*, namely a total order $\prec$ defined on $\mathrm{V}(D)$ such that $\mathrm{A}(D) \subseteq \{(v,w) \in \mathrm{V}(D) \times \mathrm{V}(D) : \ \ v \prec w\}$. When we use $\prec$ to stand for a total order, we will say v is *less than* w in $\prec$ whenever $v \prec w$.

For every digraph D, the *competition graph* of D [1], denoted by $\mathcal{C}(D)$, is the graph with $\mathrm{V}\left(\mathcal{C}(D)\right) = \mathrm{V}(D)$ and with two vertices being adjacent if and only if they have at least one common out-neighbor in D. The *competition number* of a graph G, denoted by $\kappa(G)$, is the least nonnegative integer k such that $\mathrm{I}_k(G)$ becomes the competition graph of an acyclic digraph. Equivalently, $\kappa(G) = \min\left(|\mathrm{V}(D)| - |\mathrm{V}(G)|\right)$ where D runs through all acyclic digraphs such that $G \lhd \mathcal{C}(D)$.

For every digraph D, the *phylogeny graph* of D [13], denoted by $\mathcal{P}(D)$, is the competition graph of D°, that is, $\mathcal{P}(D) = \mathcal{C}(D^\circ)$. The *phylogeny number* of a graph G, denoted by $\phi(G)$, is the least number p such that we can find a phylogeny graph of an acyclic digraph that has $p + |\mathrm{V}(G)|$ vertices and has G as an induced subgraph.

For every digraph D, the *competition hypergraph* of D [15], denoted by $\mathcal{CH}(D)$, is the hypergraph with vertex set $\mathrm{V}\left(\mathcal{CH}(D)\right) = \mathrm{V}(D)$ and hyperedge set

$$\mathscr{E}\left(\mathcal{CH}(D)\right) = \{e \in \binom{\mathrm{V}(H)}{\geq 2} : \ \exists v \in \mathrm{V}(D) \text{ s.t. } e = \mathrm{N}_D^-(v)\}.$$

The ST-*competition number of a hypergraph* H, denoted by $\kappa_{\mathrm{ST}}(H)$, is the least nonnegative integer k such that $\mathrm{I}_k(H)$ becomes the competition hypergraph of an acyclic digraph. Equivalently, $\kappa_{\mathrm{ST}}(H)$ is the least value of $|\mathrm{V}(D) \setminus \mathrm{V}(H)|$ where D runs through all acyclic digraphs satisfying $H \lhd \mathcal{CH}(D)$.

For every digraph D, the ST-*phylogeny hypergraph* of D, denoted by $\mathcal{PH}(D)$, is the competition hypergraph of D°, that is, $\mathcal{PH}(D) = \mathcal{CH}(D^\circ)$. The ST-*phylogeny number of a hypergraph* H, which we write as $\phi_{\mathrm{ST}}(H)$, is the least value of $|\mathrm{V}(D) \setminus \mathrm{V}(H)|$ where D runs through all acyclic digraphs satisfying $H \lhd \mathcal{PH}(D)$.

For any positive integers $m, n_1, \ldots, n_m$, let $[m] = \{1, \ldots, m\}$ and let $K^{n_1,\ldots,n_m}$ denote the graph with

$$\mathrm{V}(K^{n_1,\ldots,n_m}) = \bigcup_{i=1}^{m} \mathrm{V}_i$$

where $\mathrm{V}_i = \{v_i^j : j \in [n_i]\}$ for $i \in [m]$, and with

$$\mathrm{E}(K^{n_1,\dots,n_m}) = \{v_i^j v_{i'}^{j'} : i \neq i', j \in [n_i], j' \in [n_{i'}]\}.$$

We call $K^{n_1,\dots,n_m}$ a *complete multipartite graph* with m parts and part sizes $n_1, \dots, n_m$. The *uniform complete multipartite graph*, denoted by K_m^n, is the complete multipartite graph $K^{n,\dots,n}$ with m parts and uniform part size n.

Theorem 1 *(i) (Kim-Park-Sano [11, Theorem 7])* $\kappa(K_m^2) = 2$ *for* $m \geq 2$*;*
(ii) (Kim-Park-Sano [11, Theorem 8]) $\kappa(K_m^3) = 4$ *for* $m \geq 3$*;*
(iii) (Kim-Sano [6, Theorem 1]) $\kappa(K_3^n) = n^2 - 3n + 4$ *for* $n \geq 2$.

We mention that the proof of Theorem 1 by Kim-Park-Sano [11, Theorem 7, Theorem 8] and Kim-Sano [6, Theorem 1] relies on very intricate constructions. By adapting their main constructions a bit, we yield the following.

Theorem 2 *(i)* $\phi(K_m^2) = 1 = \kappa(K_m^2) - 1$ *for* $m \geq 2$*;*
(ii) $\phi(K_m^3) = 3 = \kappa(K_m^3) - 1$ *for* $m \geq 3$*;*
(iii) $\phi(K_3^n) = n^2 - 3n + 3 = \kappa(K_3^n) - 1$ *for* $n \geq 2$.

For a graph G, a *clique* of G is a subset of $\mathrm{V}(G)$ such that every two vertices in this subset are adjacent. A clique of G is called *maximal* if it is not properly contained in every clique of G. The *clique hypergraph* of G, denoted by $\mathcal{K}(G)$, is the hypergraph with vertex set $\mathrm{V}(G)$ and with the set of all maximal cliques of G as its hyperedge set. For $G = K^n$, it is easy to see that $\phi_{\mathrm{ST}}\left(\mathcal{K}(G)\right) = \kappa_{\mathrm{ST}}\left(\mathcal{K}(G)\right) = 0$. When G is a complete multipartite graph with more than one parts, some more effort leads to the next result.

Theorem 3 *Let* $m, n_1, \dots, n_m$ *be positive integers and let* $H = \mathcal{K}(K^{n_1,\dots,n_m})$*. If* $m \geq 2$*, then* $\phi_{\mathrm{ST}}(H) + 1 = \kappa_{\mathrm{ST}}(H) = \prod_{\ell=1}^m n_\ell - \sum_{\ell=1}^m n_\ell + m$.

For any positive integer n, a *Latin square of order* n is an $n \times n$ matrix with entries using n distinct elements such that none of the elements appear twice in any row or column of the matrix. For convenience, we shall assume the n distinct elements to be $1, \dots, n$. Let L denote a Latin square of order n and $L(i,j)$ denote the entry in row i and column j of L. For two Latin squares L_1 and L_2 of order n, we say that L_1 and L_2 are *orthogonal* if all ordered pairs from corresponding entries of L_1 and L_2 are different, i.e. for every pair $(i^*, j^*) \in [n] \times [n]$, there is a unique pair $(i,j) \in [n] \times [n]$ such that $L_1(i,j) = i^*$ and $L_2(i,j) = j^*$. A number of Latin squares of the same order form a set of *mutually orthogonal Latin squares*, often abbreviated in the literature to MOLS, if any two of them are orthogonal. The largest size of a set of MOLS of order n is denoted by $\mathcal{L}(n)$. It is known that $\mathcal{L}(n) \leq n - 1$ [3, Theorem 5.1.2] with equality if and only if there exists a finite projective plane of order n [3, p. 166]. Note that a finite projective plane of order n does exist when n is a prime power.

Theorem 4 *(Chang-Li [8, Theorem 17]) Let* m *and* n *be positive integers such that* $2 \leq m \leq \mathcal{L}(n) + 2$*. Then for every minimum edge clique cover* $\mathscr{S}$ *of* K_m^n*, there exists a digraph* D *satisfying the following two properties:*

(i) $\mathscr{S} = \{\mathrm{N}_D^-(v):\ v \in \mathrm{V}(D), |\,\mathrm{N}_D^-(v)| \geq 1\}$;
(ii) $\mathcal{C}(D) = \mathrm{I}_{n^2-2n+2}(K_m^n)$.

Especially, $\kappa(K_m^n) \leq n^2 - 2n + 2$.

Remark 1 For every connected and triangle-free graph G, one can check easily that $\phi(G) = \kappa(G) - 1 = |\,\mathrm{E}(G)| - |\,\mathrm{V}(G)| + 1$ [12, Theorem 2] [13, Theorem 12]. It follows that $\phi(K_2^n) + 1 = \kappa(K_2^n) = n^2 - 2n + 2$ and so the bound in Theorem 4 is tight.

Theorem 5 *(Kim-Park-Sano [5, Theorem 3]) Let n be an integer such that $3 \leq n \leq \mathcal{L}(n) + 2$. Then for every integer m which is not smaller than n, we have $\kappa(K_m^n) \leq n^2 - n + 1$.*

For any integer $n \geq 2$, Zhang [18] showed that every set of MOLS of order n and size $n-2$ can be uniquely extended to a set of MOLS of order n and size $n-1$. When $n \neq 4$, Shrikhande [14] proved that every set of MOLS of order n and size $n-3$ can be uniquely extended to a set of MOLS of order n and size $n-1$. Surely, the result of Zhang implies that for each integer $n \geq 2$, $n \leq \mathcal{L}(n)+2$ if and only if $\mathcal{L}(n) = n-1$. This then illustrates that the following result is an extension of Theorem 5.

Theorem 6 *Let n be a positive integer such that $\mathcal{L}(n) = n-1$. Then for every integer m bigger than 1, it holds $\kappa(K_m^n) \leq n^2 - 2n + 2$.*

For two sets S and T, their disjoint union is designated by $S \sqcup T$. The *disjoint union* of two graphs G_1 and G_2, denoted by $G_1 \sqcup G_2$, is the graph with vertex set $\mathrm{V}(G_1) \sqcup \mathrm{V}(G_2)$ and edge set $\mathrm{E}(G_1) \sqcup \mathrm{E}(G_2)$. Similarly we can define the disjoint union of two digraphs or two hypergraphs. Roberts and Sheng [13, Lemma 6] derived the following additivity property of the phylogeny number. We establish below a counterpart of it for the ST-phylogeny number.

Theorem 7 *(Roberts-Sheng [13, Lemma 6]) For any two graphs G_1 and G_2, it holds $\phi(G_1 \sqcup G_2) = \phi(G_1) + \phi(G_2)$.*

Theorem 8 *For any two hypergraphs H_1 and H_2, it holds $\phi_{\mathrm{ST}}(H_1 \sqcup H_2) = \phi_{\mathrm{ST}}(H_1) + \phi_{\mathrm{ST}}(H_2)$.*

In next result, we determine the range of the functions $\phi - \kappa + 1$ and $\phi_{\mathrm{ST}} - \kappa_{\mathrm{ST}} + 1$.

Theorem 9 *(i) For every integer k, there exists a graph G such that $\phi(G) - \kappa(G) + 1 = k$ if and only if $k \geq 0$.*
(ii) For every integer k, there exists a hypergraph H such that $\phi_{\mathrm{ST}}(H) - \kappa_{\mathrm{ST}}(H) + 1 = k$ if and only if $k \geq 0$.

3 Proofs

Let G be a graph. A *vertex clique cover* of G is a set of cliques of G such that every vertex of G is contained in some clique in this set. An *edge clique cover* of G is a set of cliques of G such that every edge of G is contained in some clique in this set. A minimum edge clique cover of G is an edge clique cover of G with minimum size. The *edge clique cover number* of G, denoted by $\theta_e(G)$, is the size of a minimum edge clique cover of G. The next result is straightforward and is the starting point of almost all research on competition numbers and phylogeny numbers.

Lemma 1 *Let D be a digraph, G a graph, and H a hypergraph. Then, the following hold:*

(i) $G \lhd \mathcal{C}(D)$ *if and only if* $\{\mathrm{N}_D^-(v) \cap \mathrm{V}(G) :\ v \in \mathrm{V}(D)\}$ *is an edge clique cover of G;*
(ii) $G \lhd \mathcal{P}(D)$ *if and only if* $\{\big(\mathrm{N}_D^-(v) \cup \{v\}\big) \cap \mathrm{V}(G) :\ v \in \mathrm{V}(D)\}$ *is an edge clique cover of G;*
(iii) $H \lhd \mathcal{CH}(D)$ *if and only if* $\{\mathrm{N}_D^-(v) \cap \mathrm{V}(H) :\ v \in \mathrm{V}(D), |\mathrm{N}_D^-(v) \cap \mathrm{V}(H)| \geq 2\} = \mathscr{E}(H)$*;*
(iv) $H \lhd \mathcal{PH}(D)$ *if and only if* $\{\big(\mathrm{N}_D^-(v) \cup \{v\}\big) \cap \mathrm{V}(H) :\ v \in \mathrm{V}(D), |\big(\mathrm{N}_D^-(v) \cup \{v\}\big) \cap \mathrm{V}(H)| \geq 2\} = \mathscr{E}(H)$*.*

Lemma 2 *(i) For every graph G, it holds $\phi(G) - \kappa(G) + 1 \geq 0$.*
(ii) For every hypergraph H, it holds $\phi_{\mathrm{ST}}(H) - \kappa_{\mathrm{ST}}(H) + 1 \geq 0$.

Proof (i). Let D_1 be an acyclic digraph such that $G \lhd \mathcal{P}(D_1)$ and $|\mathrm{V}(D_1)| - |\mathrm{V}(G)| = \phi(G)$. Let $\prec$ be an acyclic ordering of D_1 and we enumerate the set of vertices in D_1 with at least one in-neighbor as $v_1, \ldots, v_\ell$, where $v_1 \prec \cdots \prec v_\ell$. Define $S_i = \big(N_{D_1}^-(v_i) \cup \{v_i\}\big) \cap \mathrm{V}(G)$ for $i \in [\ell]$. By Lemma 1 (ii), $\{S_i\}_{i=1}^{\ell}$ is an edge clique cover of G.

Let D_2 be the digraph with $\mathrm{V}(D_2) = \mathrm{V}(D_1) \sqcup \{u\}$ and $\mathrm{A}(D_2) = \{(v,u) :\ v \in S_\ell\} \cup \big(\bigcup_{i=1}^{\ell-1}\{(v, v_{i+1}) :\ v \in S_i\}\big)$. It is clear that D_2 is acyclic. By Lemma 1 (i) and the fact that $\{S_i\}_{i=1}^{\ell}$ is an edge clique cover of G, $G \lhd \mathcal{C}(D_2)$ holds. This then implies $\kappa(G) \leq |\mathrm{V}(D_2)| - |\mathrm{V}(G)| = \phi(G) + 1$, as required.

(ii). Let D_1 be an acyclic digraph such that $H \lhd \mathcal{PH}(D_1)$ and $|\mathrm{V}(D_1)| - |\mathrm{V}(H)| = \phi_{\mathrm{ST}}(H)$. Assume that $v_1, \ldots, v_\ell$ are all the vertices in D_1 such that $|\big(N_{D_1}^-(v_i) \cup \{v_i\}\big) \cap \mathrm{V}(H)| \geq 2$ for $i \in [\ell]$. Let $\prec$ be an acyclic ordering of D_1 and we may suppose $v_1 \prec \cdots \prec v_\ell$. Let $S_i = \big(N_{D_1}^-(v_i) \cup \{v_i\}\big) \cap \mathrm{V}(H)$ for $i \in [\ell]$. By Lemma 1 (iv), $\{S_i :\ i \in [\ell]\} = \mathscr{E}(H)$.

Let D_2 be the digraph with $\mathrm{V}(D_2) = \mathrm{V}(D_1) \sqcup \{u\}$ and $\mathrm{A}(D_2) = \{(v,u) :\ v \in S_\ell\} \cup \big(\bigcup_{i=1}^{\ell-1}\{(v, v_{i+1}) :\ v \in S_i\}\big)$. It is apparent that D_2 is acyclic. By Lemma 1 (iii) and the fact that $\{S_i :\ i \in [\ell]\} = \mathscr{E}(H)$, we obtain $H \lhd \mathcal{CH}(D_2)$. Therefore, $\kappa_{\mathrm{ST}}(H) \leq |\mathrm{V}(D_2)| - |\mathrm{V}(G)| = \phi_{\mathrm{ST}}(H) + 1$, finishing the proof. □

Lemma 3 *(Kim-Sano [6, Lemma 4]) For every positive integer $n \geq 4$, the complete tripartite graph K_3^n possesses an edge clique cover $\{S_\ell\}_{\ell=1}^{n^2}$ such that*

$$\begin{cases} S_1 &= \{v_1^1, v_2^1, v_3^1\}, & S_2 &= \{v_1^2, v_2^n, v_3^1\}, & S_3 &= \{v_1^1, v_2^n, v_3^n\}, \\ S_4 &= \{v_1^n, v_2^1, v_3^n\}, & S_5 &= \{v_1^n, v_2^2, v_3^1\}, & S_6 &= \{v_1^1, v_2^2, v_3^2\}, \\ S_7 &= \{v_1^{n-1}, v_2^2, v_3^n\}, & S_8 &= \{v_1^2, v_2^{n-1}, v_3^n\}, & S_9 &= \{v_1^1, v_2^{n-1}, v_3^{n-1}\}, \\ & & & \cdots & & \\ S_{3s+4} &= \{v_1^{n-s}, v_2^2, v_3^{n-s+1}\}, & S_{3s+5} &= \{v_1^2, v_2^{n-s}, v_3^{n-s+1}\}, & S_{3s+6} &= \{v_1^1, v_2^{n-s}, v_3^{n-s}\}, \\ & & & \cdots & & \\ S_{3n-5} &= \{v_1^3, v_2^2, v_3^4\}, & S_{3n-4} &= \{v_1^2, v_2^3, v_3^4\}, & S_{3n-3} &= \{v_1^1, v_2^3, v_3^3\}. \end{cases} \tag{1}$$

Proof (Proof of Theorem 2) (i). It follows from Theorem 1 (i) that $\kappa(K_m^2) = 2$ for $m \geq 2$. Consequently, by Lemma 2 (i), our task is to show that $\phi(K_m^2) \leq 1$ for $m \geq 2$.

When $m = 2$, let D be the digraph with $\mathrm{V}(D) = \mathrm{V}(K_2^2) \sqcup \{u\}$ and $\mathrm{A}(D) = \{(v_1^1, v_2^1), (v_2^1, v_1^2), (v_1^2, v_2^2), (v_2^2, u), (v_1^1, u)\}$. It is clear that D is acyclic and $K_2^2 \lhd \mathcal{P}(D)$. This gives $\phi(K_2^2) \leq 1$.

When $m \geq 3$, we consider the following edge clique cover of K_m^2:

$$\begin{cases} S_1 &= \{v_1^2, v_2^1, v_3^1, \ldots, v_{m-1}^1, v_m^1\}, \\ S_2 &= \{v_1^1, v_2^2, v_3^1, \ldots, v_{m-1}^1, v_m^1\}, \\ & \cdots \\ S_m &= \{v_1^1, v_2^1, v_3^1, \ldots, v_{m-1}^1, v_m^2\}, \\ S_{m+1} &= \{v_1^2, v_2^2, v_3^2, \ldots, v_{m-1}^2, v_m^2\}. \end{cases} \tag{2}$$

Let D be the digraph with $\mathrm{V}(D) = \mathrm{V}(K_m^2) \sqcup \{u\}$ and

$$\mathrm{A}(D) = \{(v, u) : \ v \in S_{m+1}\} \cup \Big(\bigcup_{i=1}^{m} \big\{(v, v_i^2) : \ v \in S_i \setminus \{v_i^2\}\big\} \Big).$$

Observe that the digraph D is acyclic with an acyclic ordering $\prec$ such that

$$v_1^1 \prec v_2^1 \prec \cdots \prec v_m^1 \prec v_1^2 \prec v_2^2 \prec \cdots \prec v_m^2 \prec u.$$

Note that $\big\{\big(\mathrm{N}_D^-(v) \cup \{v\}\big) \cap \mathrm{V}(K_m^2) : \ v \in \mathrm{V}(D)\big\}$ is the edge clique cover of K_m^2 as demonstrated in (2). By Lemma 1 (ii), $K_m^2 \lhd \mathcal{P}(D)$ holds, which then implies that $\phi(K_m^2) \leq 1$ when $m \geq 3$, as expected.

(ii). In view of Theorem 1 (ii), $\kappa(K_m^3) = 4$. By Lemma 2 (i), it remains to show that $\phi(K_m^3) \leq 3$.

For $m \geq 3$, there exists a unique positive integer t and a unique integer $r \in \{0,1,2\}$ such that $m = 3t + r$. For each $i \in [t]$, let

$$\begin{cases} S(v^1_{3(i-1)+1}) = \{v^1_{3(i-1)+1}, v^2_{3(i-1)+2}, v^2_{3(i-1)+3}\}, \\ S(v^1_{3(i-1)+2}) = \{v^2_{3(i-1)+1}, v^1_{3(i-1)+2}, v^2_{3(i-1)+3}\}, \\ S(v^1_{3(i-1)+3}) = \{v^2_{3(i-1)+1}, v^2_{3(i-1)+2}, v^1_{3(i-1)+3}\}, \\ S(v^2_{3(i-1)+1}) = \{v^2_{3(i-1)+1}, v^3_{3(i-1)+2}, v^3_{3(i-1)+3}\}, \\ S(v^2_{3(i-1)+2}) = \{v^3_{3(i-1)+1}, v^2_{3(i-1)+2}, v^3_{3(i-1)+3}\}, \\ S(v^2_{3(i-1)+3}) = \{v^3_{3(i-1)+1}, v^3_{3(i-1)+2}, v^2_{3(i-1)+3}\}, \\ S(v^3_{3(i-1)+1}) = \{v^3_{3(i-1)+1}, v^1_{3(i-1)+2}, v^1_{3(i-1)+3}\}, \\ S(v^3_{3(i-1)+2}) = \{v^1_{3(i-1)+1}, v^3_{3(i-1)+2}, v^1_{3(i-1)+3}\}, \\ S(v^3_{3(i-1)+3}) = \{v^1_{3(i-1)+1}, v^1_{3(i-1)+2}, v^3_{3(i-1)+3}\}. \end{cases}$$

When $r = 0$, we put

$$\begin{aligned} S(v^1_{3t+1}) &= S(v^2_{3t+1}) = S(v^3_{3t+1}) = S(v^1_{3t+2}) = S(v^2_{3t+2}) \\ = S(v^3_{3t+2}) &= S(v^1_{3t+3}) = S(v^2_{3t+3}) = S(v^3_{3t+3}) = \emptyset. \end{aligned}$$

When $r = 1$, we put

$$\begin{aligned} S(v^1_{3t+1}) = S(v^3_{3t+2}) = S(v^3_{3t+3}) = \{v^1_{3t+1}\}, \\ S(v^2_{3t+1}) = S(v^1_{3t+2}) = S(v^1_{3t+3}) = \{v^2_{3t+1}\}, \\ S(v^3_{3t+1}) = S(v^2_{3t+2}) = S(v^2_{3t+3}) = \{v^3_{3t+1}\}. \end{aligned}$$

When $r = 2$, we put

$$\begin{aligned} S(v^1_{3t+1}) = \{v^1_{3t+1}, v^2_{3t+2}\}, S(v^2_{3t+1}) = \{v^2_{3t+1}, v^3_{3t+2}\}, S(v^3_{3t+1}) = \{v^3_{3t+1}, v^1_{3t+2}\}, \\ S(v^1_{3t+2}) = \{v^2_{3t+1}, v^1_{3t+2}\}, S(v^2_{3t+2}) = \{v^3_{3t+1}, v^2_{3t+2}\}, S(v^3_{3t+2}) = \{v^1_{3t+1}, v^3_{3t+2}\}, \\ S(v^1_{3t+3}) = \{v^2_{3t+1}, v^2_{3t+2}\}, S(v^2_{3t+3}) = \{v^3_{3t+1}, v^3_{3t+2}\}, S(v^3_{3t+3}) = \{v^1_{3t+1}, v^1_{3t+2}\}. \end{aligned}$$

We demonstrate $9t + 9$ cliques of K^3_m as below:

$$\begin{cases} C_{1,i} = S(v^1_{3(i-1)+1}) \cup \left(\bigcup_{\ell=i+1}^{t+1} S(v^1_{3(\ell-1)+1})\right), \\ C_{2,i} = S(v^1_{3(i-1)+2}) \cup \left(\bigcup_{\ell=i+1}^{t+1} S(v^2_{3(\ell-1)+1})\right), \\ C_{3,i} = S(v^1_{3(i-1)+3}) \cup \left(\bigcup_{\ell=i+1}^{t+1} S(v^3_{3(\ell-1)+1})\right), \\ C_{4,i} = S(v^2_{3(i-1)+1}) \cup \left(\bigcup_{\ell=i+1}^{t+1} S(v^1_{3(\ell-1)+1})\right), \\ C_{5,i} = S(v^2_{3(i-1)+2}) \cup \left(\bigcup_{\ell=i+1}^{t+1} S(v^2_{3(\ell-1)+1})\right), \\ C_{6,i} = S(v^2_{3(i-1)+3}) \cup \left(\bigcup_{\ell=i+1}^{t+1} S(v^3_{3(\ell-1)+1})\right), \\ C_{7,i} = \mathrm{V}(K^3_m) \cap \left(\{v^3_{3(i-2)+1}\} \cup S(v^3_{3(i-1)+1}) \cup \left(\bigcup_{\ell=i+1}^{t+1} S(v^1_{3(\ell-1)+1})\right)\right), \\ C_{8,i} = \mathrm{V}(K^3_m) \cap \left(\{v^3_{3(i-2)+2}\} \cup S(v^3_{3(i-1)+2}) \cup \left(\bigcup_{\ell=i+1}^{t+1} S(v^2_{3(\ell-1)+1})\right)\right), \\ C_{9,i} = \mathrm{V}(K^3_m) \cap \left(\{v^3_{3(i-2)+3}\} \cup S(v^3_{3(i-1)+3}) \cup \left(\bigcup_{\ell=i+1}^{t+1} S(v^3_{3(\ell-1)+1})\right)\right), \end{cases} \tag{3}$$

where $i \in [t+1]$. We claim that these $9t+9$ cliques form an edge clique cover of K^3_m. Take $v^{j_1}_{i_1} v^{j_2}_{i_2} \in \mathrm{E}(K^3_m)$ with $i_1 < i_2$. We need to prove that $v^{j_1}_{i_1} v^{j_2}_{i_2}$ is contained in one of the cliques as listed in (3). Assume that $i_1 = 3t_1 + r_1, i_2 =$

$3t_2 + r_2$ where $t_1, t_2 \in \{0, 1, \ldots, t\}$ and $r_1, r_2 \in [3]$. If $t_1 = t_2$, it is clear that $v_{i_1}^{j_1} v_{i_2}^{j_2} \subseteq S(v_{3t_1+r}^{\ell})$ for some $\ell, r \in [3]$, and $v_{i_1}^{j_1} v_{i_2}^{j_2}$ is contained in the clique $C_{3(\ell-1)+r, t_1+1}$. We next turn to the case of $t_1 < t_2$. We can find that $v_{i_1}^{j_1} \in S(v_{3t_1+1}^{s_1}) \cap S(v_{3t_1+2}^{s_2}) \cap S(v_{3t_1+3}^{s_3})$ for some $s_1, s_2, s_3 \in [3]$. Note that $v_{i_2}^{j_2} \in S(v_{3t_2+1}^{\ell})$ for some $\ell \in [3]$. Therefore, $v_{i_1}^{j_1} v_{i_2}^{j_2}$ is contained in the clique $C_{3(s_\ell - 1)+\ell, t_1+1}$.

Define D to be the digraph with $\mathrm{V}(D) = \mathrm{V}(K_m^3) \sqcup \{v_0^3, v_{-1}^3, v_{-2}^3\}$ and $\mathrm{A}(D) = \bigcup_{\ell=1}^{9} A_\ell$ where

$$\begin{cases} A_1 = \bigcup_{i=1}^{t+1} \big\{(v, v_{3(i-1)+1}^1) : \ v \in C_{1,i} \setminus \{v_{3(i-1)+1}^1\}\big\}, \\ A_2 = \bigcup_{i=1}^{t+1} \big\{(v, v_{3(i-1)+2}^1) : \ v \in C_{2,i} \setminus \{v_{3(i-1)+2}^1\}\big\}, \\ A_3 = \bigcup_{i=1}^{t+1} \big\{(v, v_{3(i-1)+3}^1) : \ v \in C_{3,i} \setminus \{v_{3(i-1)+3}^1\}\big\}, \\ A_4 = \bigcup_{i=1}^{t+1} \big\{(v, v_{3(i-1)+1}^2) : \ v \in C_{4,i} \setminus \{v_{3(i-1)+1}^2\}\big\}, \\ A_5 = \bigcup_{i=1}^{t+1} \big\{(v, v_{3(i-1)+2}^2) : \ v \in C_{5,i} \setminus \{v_{3(i-1)+2}^2\}\big\}, \\ A_6 = \bigcup_{i=1}^{t+1} \big\{(v, v_{3(i-1)+3}^2) : \ v \in C_{6,i} \setminus \{v_{3(i-1)+3}^2\}\big\}, \\ A_7 = \bigcup_{i=1}^{t+1} \big\{(v, v_{3(i-2)+1}^3) : \ v \in C_{7,i} \setminus \{v_{3(i-2)+1}^3\}\big\}, \\ A_8 = \bigcup_{i=1}^{t+1} \big\{(v, v_{3(i-2)+2}^3) : \ v \in C_{8,i} \setminus \{v_{3(i-2)+2}^3\}\big\}, \\ A_9 = \bigcup_{i=1}^{t+1} \big\{(v, v_{3(i-2)+3}^3) : \ v \in C_{9,i} \setminus \{v_{3(i-2)+3}^3\}\big\}. \end{cases}$$

The digraph D is acyclic as it has the following acyclic ordering $\prec$:

$$v_m^3 \prec \cdots \prec v_{3i+3}^3 \prec v_{3i+2}^3 \prec v_{3i+1}^3 \prec v_{3i+3}^2 \prec v_{3i+2}^2 \prec v_{3i+1}^2 \prec v_{3i+3}^1 \prec v_{3i+2}^1 \\ \prec v_{3i+1}^1 \prec v_{3(i-1)+3}^3 \prec v_{3(i-1)+2}^3 \prec v_{3(i-1)+1}^3 \prec \cdots \prec v_1^1 \prec v_0^3 \prec v_{-1}^3 \prec v_{-2}^3.$$

Note that the edge clique cover of K_m^3 as shown in (3) is nothing but $\big\{\big(\mathrm{N}_D^-(v) \cup \{v\}\big) \cap \mathrm{V}(K_m^3) : \ v \in \mathrm{V}(D)\big\}$. By Lemma 1 (ii), we have $K_m^3 \lhd \mathcal{P}(D)$. Therefore, $\phi(K_m^3) \leq 3$ for $m \geq 3$, as expected.

(iii). Thanks to (i) and (ii), we can restrict our attention to the case of $n \geq 4$. Theorem 1 (iii) says that $\kappa(K_3^n) = n^2 - 3n + 4$. By Lemma 2 (i), it suffices to show $\phi(K_3^n) \leq n^2 - 3n + 3$.

Let $\mathscr{S} = \{S_1, \ldots, S_{3n-3}, T_1, \ldots, T_{n^2-3n+3}\}$ be an edge clique cover of K_3^n satisfying the condition of Lemma 3, namely $S_1, \ldots, S_{3n-3}$ are specified by (1). Construct D to be the digraph with $\mathrm{V}(D) = \mathrm{V}(K_3^n) \sqcup \{z_1, \ldots, z_{n^2-3n+3}\}$ and $\mathrm{A}(D) = (\bigcup_{\ell=1}^{3n-3} A_\ell) \cup \big(\bigcup_{\ell=1}^{n^2-3n+3} \{(v, z_\ell) : \ v \in T_\ell\}\big)$ where

$$\begin{array}{llll} A_1 = \{(v_1^1, v_3^1), (v_2^1, v_3^1)\}, & A_2 = \{(v_1^2, v_2^n), (v_3^1, v_2^n\}, \\ A_3 = \{(v_1^1, v_3^n), (v_2^n, v_3^n)\}, & A_4 = \{(v_2^1, v_1^n), (v_3^n, v_1^n)\}, \\ A_5 = \{(v_1^n, v_2^2), (v_3^1, v_2^2)\}, & A_6 = \{(v_1^1, v_3^2), (v_2^2, v_3^2)\}, \\ A_7 = \{(v_2^2, v_1^{n-1}), (v_3^n, v_1^{n-1})\}, & A_8 = \{(v_1^2, v_2^{n-1}), (v_3^n, v_2^{n-1})\}, \\ A_9 = \{(v_1^1, v_3^{n-1}), (v_2^{n-1}, v_3^{n-1})\}, & \\ & \cdots \\ A_{3s+4} = \{(v_2^2, v_1^{n-s}), (v_3^{n-s+1}, v_1^{n-s})\}, & A_{3s+5} = \{(v_1^2, v_2^{n-s}), (v_3^{n-s+1}, v_2^{n-s})\}, \\ A_{3s+6} = \{(v_1^1, v_3^{n-s}), (v_2^{n-s}, v_3^{n-s})\}, & \\ & \cdots \\ A_{3n-5} = \{(v_2^2, v_1^3), (v_3^4, v_1^3)\}, & A_{3n-4} = \{(v_1^2, v_2^3), (v_3^4, v_2^3)\}, \\ A_{3n-3} = \{(v_1^1, v_3^3), (v_2^3, v_3^3)\}. & \end{array}$$

The digraph D is acyclic due to the existence of the acyclic ordering $\prec$:

$$v_1^1 \prec v_2^1 \prec v_3^1 \prec v_1^2 \prec v_2^n \prec v_3^n \prec v_1^n \prec v_2^2 \prec v_3^2 \prec v_1^{n-1} \prec v_2^{n-1} \prec v_3^{n-1} \\ \prec v_1^{n-2} \prec v_2^{n-2} \prec v_3^{n-2} \prec \cdots \prec v_1^3 \prec v_2^3 \prec v_3^3 \prec z_1 \prec \cdots \prec z_{n^2-3n+3}.$$

It is easy to see that $\big\{\big(\mathrm{N}_D^-(v) \cup \{v\}\big) \cap \mathrm{V}(K_3^n) :\ v \in \mathrm{V}(D)\big\}$ coincides with the edge clique cover $\mathscr{S}$ of K_3^n. By Lemma 1 (ii), $K_3^n \lhd \mathcal{P}(D)$ holds. Therefore, $\phi(K_3^n) \le n^2 - 3n + 3$ for $n \ge 4$, finishing the proof. □

Proof (Proof of Theorem 3) We assume $n_1 \le \cdots \le n_m$ and write k for $\prod_{\ell=1}^m n_\ell - \sum_{\ell=1}^m n_\ell + m$. When $n_m = 1$, H is the clique hypergraph of the complete graph $K_m^1 = K_m$ and so $\phi_{\mathrm{ST}}(H) + 1 = \kappa_{\mathrm{ST}}(H) = 1 = k$. In the following, we will assume $n_m \ge 2$.

First of all, we prove that $\phi_{\mathrm{ST}}(H) \ge \kappa_{\mathrm{ST}}(H) - 1 \ge k - 1$. Suppose D is an acyclic digraph such that $\mathcal{CH}(D) = \mathrm{I}_{\kappa_{\mathrm{ST}}(H)}(H)$. The hypergraph H has $\sum_{\ell=1}^m n_\ell$ vertices and $\prod_{\ell=1}^m n_\ell$ hyperedges of the same size m. Therefore, in D there are at least $\prod_{\ell=1}^m n_\ell$ vertices which have m in-neighbors. Meanwhile, since D is acyclic, it possesses an acyclic ordering $\prec$ and so every vertex among the smallest m elements in $\prec$ has less than m in-neighbors in D. Therefore, $\kappa_{\mathrm{ST}}(H) \ge \prod_{\ell=1}^m n_\ell + m - \sum_{\ell=1}^m n_\ell = k$. By Lemma 2 (ii), $\phi_{\mathrm{ST}}(H) \ge \kappa_{\mathrm{ST}}(H) - 1 \ge k - 1$.

Secondly, we intend to show $\kappa_{\mathrm{ST}}(H) = k$. For this purpose, what remains is to get $\kappa_{\mathrm{ST}}(H) \le k$. We need to find an acyclic digraph D_1 with $\mathrm{V}(D_1) = \mathrm{V}(H) \sqcup \{z_t :\ t \in [k]\}$ such that $H \lhd \mathcal{CH}(D_1)$. Let us specify a total order $\prec$ on $\mathrm{V}(D_1)$ such that

$$v_1^1 \prec \cdots \prec v_m^1 \prec v_1^2 \prec \cdots \prec v_1^{n_1} \prec \cdots \prec v_m^2 \prec \cdots \prec v_m^{n_m} \\ \prec z_1 \prec \cdots \prec z_k.$$

For every $v_i^j \in \mathrm{V}(H)$, let $\xi(v_i^j)$ be the element covering v_i^j in $\prec$, namely $\xi(v_i^j)$ is the minimum element in $\prec$ which is bigger than v_i^j. Let $\mathrm{A}(D_1) = \big\{\big(u, \xi(v_i^j)\big) :\ u \in \big(\{v_1^1, \ldots, v_m^1\} \cup \{v_i^j\}\big) \setminus \{v_i^1\}, v_i^j \in \mathrm{V}(H) \setminus \{v_1^1, \ldots, v_{m-1}^1, v_m^{n_m}\}\big\} \cup \big(\bigcup_{t=1}^k \{(w, z_t) :\ w \in e_t\}\big)$, where $e_1, \ldots, e_k$ are the k hyperedges from the set

$$\mathscr{E}(H) \setminus \big\{\big(\{v_1^1, \ldots, v_m^1\} \cup \{v_i^j\}\big) \setminus \{v_i^1\} :\ v_i^j \in \mathrm{V}(H) \setminus \{v_1^1, \ldots, v_{m-1}^1, v_m^{n_m}\}\big\}.$$

It is easy to see $\mathrm{A}(D_1) \subseteq \{(v, w) \in \mathrm{V}(D_1) \times \mathrm{V}(D_1) :\ v \prec w\}$ and so $\prec$ is an acyclic ordering of D_1. Moreover, one can check that $\{\mathrm{N}_{D_1}^-(v) \cap \mathrm{V}(H) :\ v \in \mathrm{V}(D_1), |\mathrm{N}_{D_1}^-(v) \cap \mathrm{V}(H)| \ge 2\} = \mathscr{E}(H)$. By Lemma 1 (iii), we obtain $H \lhd \mathcal{CH}(D_1)$, as expected.

Finally, we want to prove $\phi_{\mathrm{ST}}(H) = k - 1$. It now suffices to show $\phi_{\mathrm{ST}}(H) \le k - 1$. That is, we are going to construct an acyclic digraph D_2 with $\mathrm{V}(D_2) = \mathrm{V}(H) \sqcup \{\bar{z}_t :\ t \in [k-1]\}$ such that $H \lhd \mathcal{PH}(D_2)$. Let $\bar{e}_1, \ldots, \bar{e}_{k-1}$ be the $k - 1$ hyperedges from $\mathscr{E}(H) \setminus \big\{\big(\{v_1^1, \ldots, v_m^1\} \cup \{v_i^j\}\big) \setminus \{v_i^1\} :\ v_i^j \in \mathrm{V}(H) \setminus \{v_1^1, \ldots, v_{m-1}^1\}\big\}$. Define $\mathrm{A}(D_2) = \big\{(u, v_i^j) :\ u \in \{v_1^1, \ldots, v_m^1\} \setminus \{v_i^1\}, v_i^j \in \mathrm{V}(H) \setminus \{v_1^1, \ldots, v_{m-1}^1\}\big\} \cup \big(\bigcup_{t=1}^{k-1} \{(w, \bar{z}_t) :\ w \in \bar{e}_t\}\big)$. We can verify that D_2 has an acyclic ordering $\prec$ such that

$$v_1^1 \prec \cdots \prec v_m^1 \prec v_1^2 \prec \cdots \prec v_1^{n_1} \prec \cdots \prec v_m^2 \prec \cdots \prec v_m^{n_m} \\ \prec z_1 \prec \cdots \prec z_{k-1}.$$

Note that $\{(\mathrm{N}^-_{D_2}(v) \cup \{v\}) \cap \mathrm{V}(H) : v \in \mathrm{V}(D_2), |(\mathrm{N}^-_{D_2}(v) \cup \{v\}) \cap \mathrm{V}(H)| \geq 2\} = \mathscr{E}(H)$. It follows from Lemma 1 (iv) that $H \lhd \mathcal{PH}(D_2)$, as was to be shown. □

Lemma 4 *(Park-Kim-Sano [10, Theorem 2.1]) Let m and n be positive integers such that $3 \leq m \leq \mathcal{L}(n) + 2$ and let $L = \{L_1, \ldots, L_{m-2}\}$ be a set of MOLS of order n and size $m-2$. Then $\theta_e(K_m^n) = n^2$ and an edge clique cover of K_m^n can be enumerated as*

$$\mathcal{S}_{x,y} = \{v_1^x, v_2^y, v_3^{L_1(x,y)}, v_4^{L_2(x,y)}, \ldots, v_m^{L_{m-2}(x,y)}\} \tag{4}$$

where x and y run through all elements of $[n]$.

Lemma 5 *Let m and n be positive integers such that $3 \leq m \leq \mathcal{L}(n) + 1$. There exists an edge clique cover $\mathscr{S}$ of K_m^n that satisfies the following two properties:*

(i) $|\mathscr{S}| = n^2 = \theta_e(K_m^n)$;
(ii) $\mathscr{S}$ can be partitioned into $\mathcal{F}_1, \ldots, \mathcal{F}_n$ such that $\mathcal{F}_\ell$ is a vertex clique cover of K_m^n for all $\ell \in [n]$.

Proof Let $L = \{L_1, \ldots, L_{m-1}\}$ be a set of MOLS of order n and let $\mathcal{S}_{x,y}$ be defined as in (4) for $x, y \in [n]$. Let $\mathscr{S} = \{\mathcal{S}_{x,y} : x, y \in [n]\}$. By Lemma 4, $\mathscr{S}$ is an edge clique cover of K_m^n with size $|\mathscr{S}| = n^2 = \theta_e(K_m^n)$.

For each $\ell \in [n]$, define $\mathcal{F}_\ell = \{\mathcal{S}_{x,y} : L_{m-1}(x, y) = \ell\}$. It is clear that $\mathcal{F}_1, \ldots, \mathcal{F}_n$ is a partition of $\mathscr{S}$. Pick $v_i^j \in \mathrm{V}(K_m^n)$ and take $\ell \in [n]$. Our goal is to prove that v_i^j appears in some clique from $\mathcal{F}_\ell$.

We first address the case of $i = 1$. Since L_{m-1} is a Latin square, there exists $y \in [n]$ such that $L_{m-1}(j, y) = \ell$. This means that $\mathcal{S}_{j,y}$ is a clique from $\mathcal{F}_\ell$ which contains v_i^j.

Next assume that $i = 2$. Since L_{m-1} is a Latin square, there exists $x \in [n]$ such that $L_{m-1}(x, j) = \ell$. This means that $\mathcal{S}_{x,j}$ is a clique from $\mathcal{F}_\ell$ which contains v_i^j.

Now, we deal with the remaining case that $3 \leq i \leq m$. Note that L_{i-2} and L_{m-1} are orthogonal. Henceforth, there is a unique $(x, y) \in [n] \times [n]$ such that $L_{i-2}(x, y) = j$ and $L_{m-1}(x, y) = \ell$. We thus see that $\mathcal{S}_{x,y}$ is a clique from $\mathcal{F}_\ell$ which contains v_i^j. □

The statement of Lemma 5 can be found basically in [5, p. 1177]. For a proof of the statement there, Kim, Park and Sano [5, p. 1177] suggested that the required partition $\mathcal{F}_1, \ldots, \mathcal{F}_n$ of the edge clique cover $\{\mathcal{S}_{x,y} : x, y \in [n]\}$ of K_m^n as shown in (4) can be chosen as $\mathcal{F}_t = \{\mathcal{S}_{1,1+t}, \ldots, \mathcal{S}_{n,n+t}\}$ for $t \in [n]$, where the subscripts should be read modular n. We point out here that this partition does not always work. Take $n = 3$ and $m = 3$. Let

$$L_1 = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \text{ and } L_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}.$$

Clearly, $\{L_1, L_2\}$ is a set of MOLS of order $n = 3$ and size $\mathcal{L}(3) = 2$. Observe that $3 \leq m \leq 3 = \mathcal{L}(n) + 1$. Following the construction from [5, p. 1177], we have $\mathcal{F}_1 = \{\mathcal{S}_{1,2}, \mathcal{S}_{2,3}, \mathcal{S}_{3,1}\} = \big\{\{v_1^1, v_2^2, v_3^3\}, \{v_1^2, v_2^3, v_3^3\}, \{v_1^3, v_2^1, v_3^3\}\big\}$. But $\mathcal{F}_1$ is not a vertex clique cover of K_3^3 as it covers neither v_3^1 nor v_3^2.

For every positive integer n, let Γ_n be the set of all positive integers k for which we can find a digraph D such that $\mathcal{C}(D) = \mathrm{I}_k(K_n^n)$ and that

$$\mathscr{S} = \{\mathrm{N}_D^-(v) \cap \mathrm{V}(K_n^n) : \ v \in \mathrm{V}(D), |\,\mathrm{N}_D^-(v) \cap \mathrm{V}(K_n^n)| \geq 1\}$$

is an edge clique cover of K_n^n satisfying the ensuing properties:

(i) $|\,\mathscr{S}\,| = n^2$;
(ii) $\mathscr{S}$ can be partitioned into $\mathcal{F}_1, \ldots, \mathcal{F}_n$ such that $\mathcal{F}_\ell$ is a vertex clique cover of K_n^n for every $\ell \in [n]$.

According to Lemma 5, we know that Γ_n is always nonempty.

Lemma 6 *(Kim-Park-Sano [5, Theorem 2]) Let m, n and k be positive integers such that $n \geq 2$, $k \in \Gamma_n$ and $m \geq n$. Then it holds* $\kappa(K_m^n) \leq \max\{k, \max_{t \in [n-1]} \kappa(K_t^n) - 1\}$.

Lemma 7 *Let n be a positive integer such that $\mathcal{L}(n) = n - 1$. Then $n^2 - 2n + 2 \in \Gamma_n$.*

Proof Taking $m = n$, an application of Theorem 4 and Lemma 5 ensures $n^2 - 2n + 2 \in \Gamma_n$.

Proof (Proof of Theorem 6) When $n = 2$, Theorem 1 (i) says that $\kappa(K_m^2) = 2 = n^2 - 2n + 2$ when $m \geq 2$.

When $(n, m) = (3, 2)$, Remark 1 gives $\kappa(K_m^n) = n^2 - 2n + 2$. When $n = 3$ and $m \geq 3$, Theorem 1 (ii) implies that $\kappa(K_m^3) = 4 < 5 = n^2 - 2n + 2$.

We now assume $n \geq 4$ and $\mathcal{L}(n) = n - 1$. By virtue of Theorem 4, Lemma 6 and Lemma 7, $\kappa(K_m^n) \leq n^2 - 2n + 2$ for $m \geq n$. Coupled with Theorem 4 again, this completes the proof. □

Proof (Proof of Theorem 8) For $\ell = 1, 2$, let D_ℓ be an acyclic digraph such that $H_\ell \lhd \mathcal{PH}(D_\ell)$ and $|\,\mathrm{V}(D_\ell)| - |\,\mathrm{V}(H_\ell)| = \phi_{\mathrm{ST}}(H_\ell)$. Clearly, $(H_1 \sqcup H_2) \lhd \mathcal{PH}(D_1 \sqcup D_2)$, which then implies $\phi_{\mathrm{ST}}(H_1 \sqcup H_2) \leq \phi_{\mathrm{ST}}(H_1) + \phi_{\mathrm{ST}}(H_2)$.

Suppose D' is an acyclic digraph such that $(H_1 \sqcup H_2) \lhd \mathcal{PH}(D')$ and $|\,\mathrm{V}(D')| - |\,\mathrm{V}(H_1 \sqcup H_2)| = \phi_{\mathrm{ST}}(H_1 \sqcup H_2)$. For $\ell = 1, 2$, let D'_ℓ be the subgraph of D' induced by $\mathrm{V}(H_\ell) \cup \{w : \ (u, w) \in \mathrm{A}(D'), u \in \mathrm{V}(H_\ell)\}$. It is not hard to see that $H_\ell \lhd \mathcal{PH}(D'_\ell)$ and $\mathrm{V}(D'_1) \cap \mathrm{V}(D'_2) = \emptyset$. Therefore, $\phi_{\mathrm{ST}}(H_1 \sqcup H_2) = |\,\mathrm{V}(D')| - |\,\mathrm{V}(H_1 \sqcup H_2)| = (|\,\mathrm{V}(D'_1)| - |\,\mathrm{V}(H_1)|) + (|\,\mathrm{V}(D'_2)| - |\,\mathrm{V}(H_2)|) \geq \phi_{\mathrm{ST}}(H_1) + \phi_{\mathrm{ST}}(H_2)$, and so we are done. □

Lemma 8 *(i) For positive integers n and k such that $n \geq 2$ and $0 \leq k \leq n^2 - 3n + 4$, it holds $\phi\left(\mathrm{I}_k(K_3^n)\right) - \kappa\left(\mathrm{I}_k(K_3^n)\right) + 1 = k$.*
(ii) For positive integers n, m and k such that $m \geq 2$ and $0 \leq k \leq n^m - mn + m$, it holds $\phi_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(K_m^n)\right)\right) - \kappa_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(K_m^n)\right)\right) + 1 = k$.

Proof (i). Theorem 1 (iii) implies $\kappa\left(\mathrm{I}_k(K_3^n)\right) = \kappa(K_3^n) - k = n^2 - 3n + 4 - k$ when $k \le n^2 - 3n + 4$. Meanwhile, taking into account Theorem 2 (iii) and Theorem 7, we obtain $\phi\left(\mathrm{I}_k(K_3^n)\right) = \phi(K_3^n) = n^2 - 3n + 3$. Therefore, it holds $\phi\left(\mathrm{I}_k(K_3^n)\right) - \kappa\left(\mathrm{I}_k(K_3^n)\right) + 1 = k$, as required.

(ii). We deduce from Theorem 3 that

$$\kappa_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(K_m^n)\right)\right) = \kappa_{\mathrm{ST}}\left(\mathcal{K}(K_m^n)\right) - k = n^m - mn + m - k.$$

Moreover, Theorem 3 and Theorem 8 lead to

$$\phi_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(K_m^n)\right)\right) = \phi_{\mathrm{ST}}\left(\mathcal{K}(K_m^n)\right) = n^m - mn + m - 1.$$

We thus get to $\phi_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(K_m^n)\right)\right) - \kappa_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(K_m^n)\right)\right) + 1 = k$, finishing the proof. □

Proof (Proof of Theorem 9) (i). By Lemma 2 (i), $\phi(G) - \kappa(G) + 1 \ge 0$ holds for every graph G. Take any nonnegative integer k. There exists a positive integer n satisfying $k \le n^2 - 3n + 4$. By Lemma 8 (i), $\phi\left(\mathrm{I}_k(K_3^n)\right) - \kappa\left(\mathrm{I}_k(K_3^n)\right) + 1 = k$.

(ii). Thanks to Lemma 2 (ii), $\phi_{\mathrm{ST}}(H) - \kappa_{\mathrm{ST}}(H) + 1 \ge 0$ for every hypergraph H. Let k be any nonnegative integer. We can find positive integers n and m such that $m \ge 2$ and $k < n^m - mn + m$. By Lemma 8 (ii), we have $\phi_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(K_m^n)\right)\right) - \kappa_{\mathrm{ST}}\left(\mathrm{I}_k\left(\mathcal{K}(K_m^n)\right)\right) + 1 = k$, which ends the proof. □

4 Further research

Inspired by Theorem 2, we would be interested to know whether or not $\phi(G) - \kappa(G) + 1 = 0$ holds for all uniform complete multipartite graphs. What happens if G is a strongly regular graph?

Theorem 7 and Theorem 8 are about phylogeny numbers and ST-phylogeny numbers. Is there a parallel result for competition numbers and ST-competition numbers? It is easy to deduce $\kappa(G_1 \sqcup G_2) \le \kappa(G_1) + \kappa(G_2)$ and $\kappa_{\mathrm{ST}}(H_1 \sqcup H_2) \le \kappa_{\mathrm{ST}}(H_1) + \kappa_{\mathrm{ST}}(H_2)$. But we do not know when equality will occur.

It may deserve to consider the counterpart of Theorem 9 when restricting to smaller classes of graphs and hypergraphs. What is the ranges of $\phi(G) - \kappa(G) + 1$ and $\phi_{\mathrm{ST}}(H) - \kappa_{\mathrm{ST}}(H) + 1$ when the graphs G and hypergraphs H are connected?

A graph G can be regarded as a hypergraph H_G with $\mathrm{V}(\mathrm{H}_G) = \mathrm{V}(G)$ and $\mathscr{E}(\mathrm{H}_G) = \mathrm{E}(G)$. How to understand the structure of those graphs G satisfying $\kappa_{\mathrm{ST}}(\mathrm{H}_G) - \kappa(G) = 0$ or $\phi_{\mathrm{ST}}(\mathrm{H}_G) - \phi(H_G) = 0$?

Some of these questions will be addressed in [17].

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