A five-element transformation monoid on labelled trees[☆]

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Abstract

For each tree T on n vertices, a labelling of T is a bijective map from the vertex set of T to the first n positive integers. We consider two maps, which send the labellings of T to labellings of T for all trees T. We show that the transformation monoid generated by these two maps has exactly five elements and we analyze the dynamical behaviours of the action of this monoid on the set of labellings of trees.

Keywords: consecutive vertex ordering, Dénes permutation, local order, monad, phase space, transposition

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1. Transformation monoids and discrete dynamical systems

A *transformation monoid* [1, 2] on a set S is a subset of S^S that is closed under composition. We think of the identity map on S as the composition of no map and so every transformation monoid on S contains the identity map on S. For each set V, a bijection from V to V is known as a *permutation*, and all the permutations on V form the *symmetric group* Sym_V, which is a special transformation monoid. A transformation monoid generated by one element is usually much easier to understand than the general case. For example, a Markov chain corresponds to a transformation monoid generated by one probability transition matrix while an inhomogeneous Markov chain represents a transformation monoid generated by several probability transition matrices and we can say much less on the latter than the former; see [3, §3.1] for some relevant discussions.

Transformation monoids are closely related to dynamical systems in which we are interested in the transformations among all possible states when time flows. Let us now explain a basic framework of modelling discrete dynamical systems; the readers are also referred to [4, §3] for a general discussion from some different perspective. A *deterministic automaton* is a triple $\mathcal{A} = (\mathrm{S}_{\mathcal{A}}, \mathrm{C}_{\mathcal{A}}, \delta_{\mathcal{A}})$, where $\mathrm{S} = \mathrm{S}_{\mathcal{A}}$ is the *state* set, $\mathrm{C} = \mathrm{C}_{\mathcal{A}}$ is the *input* set and $\delta_{\mathcal{A}}$ is the *transition function* which sends each $c \in \mathrm{C}$ to an element $\delta^c_{\mathcal{A}} \in \mathrm{S}^{\mathrm{S}}$. Every deterministic discrete time dynamical system can be thought of as a deterministic automaton and the local evolving rule of the system determines the transition function of the automaton. We reserve the notation $\mathrm{T}(\mathcal{A})$ for the transformation monoid generated by $\delta^c_{\mathcal{A}}$, $c \in C$. Indeed, it is convenient to identify a deterministic automaton with a transformation monoid given by a set of its generators. To have a good knowledge of the global dynamical behaviour of the system modelled by an automaton $\mathcal{A}$, we should determine the structure $\mathrm{T}(\mathcal{A})$ as a monoid and the action of $\mathrm{T}(\mathcal{A})$ on the set $\mathrm{S}_{\mathcal{A}}$ as a transformation monoid. If $\mathrm{S}_{\mathcal{A}}$ is finite and $\mathrm{C}_{\mathcal{A}}$ is a singleton set, the deterministic automaton $\mathcal{A}$ is called a *monad* by Arnold [5, 6]. Let us call $\mathcal{A}$ a k-*monad* provided $\mathrm{S}_{\mathcal{A}}$ is finite and $\mathrm{C}_{\mathcal{A}}$ is a set of size k. A k-monad $\mathcal{A}$ can be given more conveniently as

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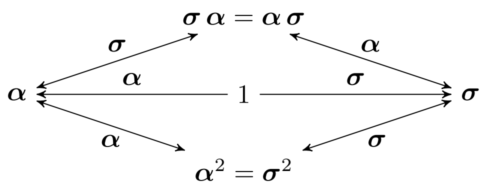


Figure 1: Cayley digraph (Phase space) of the 5-element monoid $\mathbb{M}$.

$(\mathrm{S}_{\mathcal{A}}; \delta_{\mathcal{A}}^{c_1}, \ldots, \delta_{\mathcal{A}}^{c_k})$ when $\{c_1, \ldots, c_k\} = \mathrm{C}_{\mathcal{A}}$, namely by identifying the input c_i with the corresponding transformation $\delta_{\mathcal{A}}^{c_i}$. Note that what is called a monad by Arnold will be called by us a 1-monad later on and we use the term of a monad in this note to refer to a k-monad for any $k \geq 1$. The reader should not confuse a monad here with a monad over a category [7].

The *phase space* of a deterministic automaton $\mathcal{A}$, denoted by $\mathcal{PS}(\mathcal{A})$, is the arc-labelled digraph with vertex set $\mathrm{S}_{\mathcal{A}}$ for which there is an arc labelled by $c \in \mathrm{C}_{\mathcal{A}}$ from u to v if and only if $v = \delta_{\mathcal{A}}^{c}(u)$. The space of walks in the phase space $\mathcal{PS}(\mathcal{A})$ corresponds to the dynamical behaviour of the automaton $\mathcal{A}$. Let $\mathbb{M}$ be the 5-element monoid given by the generators and relations as follows:

$$\mathbb{M} = \langle \boldsymbol{\alpha}, \boldsymbol{\sigma} :\ \boldsymbol{\alpha}\,\boldsymbol{\sigma} = \boldsymbol{\sigma}\,\boldsymbol{\alpha}, \boldsymbol{\alpha}^2 = \boldsymbol{\sigma}^2, \boldsymbol{\alpha}^3 = \boldsymbol{\alpha}, \boldsymbol{\sigma}^3 = \boldsymbol{\sigma} \rangle. \tag{1}$$

Its Cayley digraph is the arc-labelled digraph shown in Fig. 1. Note that the left-regular action of $\mathbb{M}$ on itself shows that $\mathbb{M}$ is isomorphic with the endomorphism monoid of its Cayley digraph. Also observe that the Cayley digraph depicted in Fig. 1 is just the phase space of the automaton $(\mathbb{M}; \boldsymbol{\alpha}, \boldsymbol{\sigma})$.

The phase space of a 1-monad looks to be very simple: It is a digraph with constant out-degree 1 and so each of its weakly connected components is a cycle of length 1 or bigger together with in-trees planted on its vertices [5, Theorem 1]. However, to make this simple picture more precise, say to determine the lengths of those cycles in the phase space (periods of the automaton) and the shape of those in-trees, will often be a challenging task. Thinking of the difference between Markov chains and inhomogeneous Markov chains, one should agree that to understand the phase space and transformation monoid of a k-monad for $k > 1$ will be even more complicated and so also more intriguing. Parallel to geometric group theory, for any transformation monoid we will anyway expect a rich connection between its algebraic property and the geometry of its phase space.

There are two basic ways of constructing dynamical systems: Either take an already existing "natural" system from the vast reserve arising in fields like biology, physics and economics [8–12], or build a "simple" system by hand using some basic tools like strings of letters, trees and permutations [13, Preface]. It may be astonishing to many children that the "simple systems" from the latter approach may not be really easier to analyze than the "complex systems" from the first approach and the simple systems do attract the attentions of many adults in various contexts as well; see, for example, [14–19]. Indeed, to understand the world we are living in, which consists of various dynamical systems of different scales, a common belief is that there should exist simple local dynamical mechanisms behind and the iterations of those hidden local rules create the observable complex phenomena. On the one hand, people will examine the complex evolving pictures in front of them and try to search for some possible explanations for them, namely the underlying local rules which control the changes over time; on the other hand, people may try to analyze various simple local rules and then check if the pattern generated by this man-made rules can match some natural patterns appeared around. Surely, for the latter approach, very often we will find it difficult to rigorously predict the dynamical behaviour of the system driven by a simple rule; the famous $3x+1$ problem [17] is one such example. Maybe, for almost all dynamical systems, it is far beyond human abilities to do any mathematical analysis of their dynamical behaviours. Accordingly, to see an analysis of the dynamical behaviours of some toy models of simple nontrivial dynamical systems may give you the joy of knowing the methods behind a good magic trick. The success in understanding any simple monad may be a help in getting future generations of mathematicians ready for tackling some puzzles on important or interesting dynamical processes.

For any positive integer n, we designate by $[n]$ the set $\{1,\ldots,n\}$. Let S be a set of size n. We write $\mathbb{TO}(S)$ for the set of bijections from S to $[n]$ and we call each element of $\mathbb{TO}(S)$ a *total order* on S. We take the convention that a total order $f\in\mathbb{TO}(S)$ is represented by the string of letters

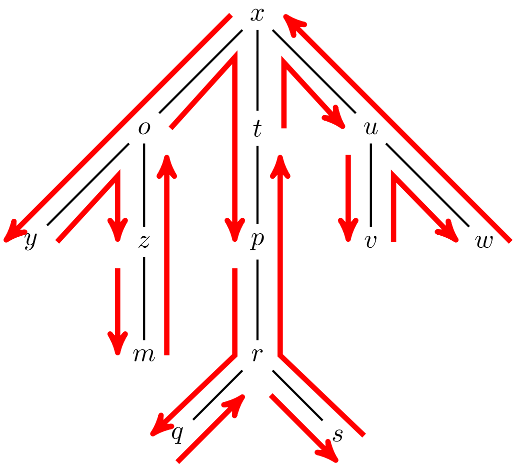
$$f^{-1}(1)f^{-1}(2)\cdots f^{-1}(n).$$

For any total order $f\in\mathbb{TO}(S)$ and for every $s,s'\in S$, we often write $s<_f s'$ whenever $f(s)<f(s')$, namely when s appears earlier than s' in the word $f^{-1}(1)f^{-1}(2)\cdots f^{-1}(n)$, and this comparison relationship surely specifies the total order f. For a graph G, we use $\mathrm{V}(G)$ and $\mathrm{E}(G)$ to denote its vertex set and edge set. For a given graph G, a graph search/traversal on G is a mechanism to generate a total order on $\mathrm{V}(G)$. Two traditional graph search strategies are breadth-first search (BFS) and depth-first search (DFS). There are many other useful classes of vertex orderings of graphs, say LexBFS, which is a BFS procedure where ties are broken to favour vertices with earlier visited neighbours. If one generates the next total order of the vertex set based on the information of the previous total orders, we have the so-called multi-sweep graph searches. A simple way of doing another sweep on a graph based on an existing total order of its vertex set is to adopt the so-called "the $^+$ rule" [20]. The execution of multi-sweep graph searches on a graph G can naturally be viewed as a dynamical system on $\mathbb{TO}(\mathrm{V}(G))$. The understanding of such kind of dynamical processes not only satisfies our curiosity but also can be useful in designing various graph algorithms [20–23]. In principle, the structure of a network should be embodied in a wide range of processes taking place on it. But, it is a nontrivial mathematics question to find an explicit way of relating the graph structure to some global dynamics on the graph. We mention that Charbit, Habib, Mouatadid and Naserasr [20] propose to use the maximum period of the automaton on the total orders of a graph generated by LexBFS and the $^+$ rule, which they call LexCycle, as a measure of the linear structure of the graph.

The aim of this paper is to examine a special 2-monad acting on labelled trees. It is to our surprise that we can understand its dynamical behaviour quite well but we fail to find any pattern for any variant of it in a big family of such 2-monads. We illustrate this simple automaton and provide a description of its phase space in Section 2. We reveal the secret behind its very regular dynamical behaviours in Section 3. Note that Theorem 2.3 (3) explains the title of this note. In addition, Theorem 2.3 (8) presents a picture which is similar to the conjecture in [20] that the LexCycle of an AT-free graph is at most two.

2. Cyclic permutations and labelled trees

For any positive integer n, we write Sym_n for $\mathrm{Sym}_{[n]}$ and we use both two-line notation and cycle notation to record a permutation. A *circular permutation of length m* in Sym_n [24, p. 120] is an element with one orbit of size



$$(x\ u)(u\ w)(u\ v)(t\ p)(r\ q)(p\ r)(r\ s)(x\ t)(o\ z)(z\ m)(o\ y)(x\ o) = (u\ v\ w\ x\ y\ z\ m\ o\ p\ q\ r\ s\ t)$$

Figure 2: An Eulerian tour on a tree.

m and all other orbits (if any) of size 1; A *cyclic permutation* in Sym_n is a circular permutation of length n, namely a permutation with only one orbit. For example, $\tau_1 = \left(\begin{smallmatrix}1&2&3&4&5\\2&3&4&5&1\end{smallmatrix}\right)$ is cyclic while $\tau_2 = \left(\begin{smallmatrix}1&2&3&4&5\\2&5&4&3&1\end{smallmatrix}\right)$ is a permutation with two orbits $\{1,2,5\}$ and $\{3,4\}$. In this paper, the composition of permutations is taken from right to left. For example, $\tau_2\tau_1 = \left(\begin{smallmatrix}1&2&3&4&5\\5&4&3&1&2\end{smallmatrix}\right)$ is a circular permutation of length 4. We mention that cyclic permutations in symmetric groups are just Coxeter elements of type A.

A *tree* is a connected graph without cycles. A *rooted tree* $\mathbb{T}$ is a pair (T,r) where T is a tree and r is a vertex of T, called the *root* of $\mathbb{T}$. Associated with the rooted tree $\mathbb{T} = (T,r)$ is the poset $\mathbf{P}(\mathbb{T})$ on $\mathrm{V}(T)$ such that $w \leq v$ if and only if v is on the unique path in T from w to r. If v covers w in the poset $\mathbf{P}(\mathbb{T})$, we call v the *father* of w and w a *child* of v in the rooted tree $\mathbb{T}$. The *descendants* of a vertex v in $\mathbb{T}$ are all those vertices of T which are less than or equal to it in $\mathbf{P}(\mathbb{T})$. For a rooted tree $\mathbb{T} = (T,r)$ and any $v \in \mathrm{V}(T)$, let $\mathbb{D}_{\mathbb{T}}\langle v\rangle$ be the set of descendants of v in $\mathbb{T}$, let $\mathbb{D}_{\mathbb{T}}(v)$ be the set of children of v in $\mathbb{T}$, and let $\mathbb{D}_{\mathbb{T}}[v]$ denote $\mathbb{D}_{\mathbb{T}}(v) \cup \{v\}$, which we call the *family* of v in $\mathbb{T}$. If $v \in \mathrm{V}(T) \setminus \{r\}$, we use $\mathbb{F}_{\mathbb{T}}(v)$ for the father of v in $\mathbb{T}$. When we write in subscripts or superscripts a rooted tree (T,r), we often omit the brackets and so $\mathbb{D}_{T,r}$ represents $\mathbb{D}_{(T,r)}$, and so on. For every graph G and $v \in \mathrm{V}(G)$, $\deg_G(v)$ refers to the number of edges incident to v in G.

Each edge $e = uv$ of a graph G gives us a transposition $(u,v) \in \mathrm{Sym}_{\mathrm{V}(G)}$ which swaps u and v. A classic result of Dénes [24, Theorem 2] [25] says that multiplying all the transpositions corresponding to edges of a tree T in any order will always end up with a cyclic permutation of $\mathrm{V}(T)$; we will call such a cyclic permutation a *Dénes permutation of the tree* T. For example, it holds

$$\begin{cases} (n\ 1\ 2\ \cdots\ n-1) = (1\ n) \circ (1\ n-1) \circ \cdots \circ (1\ 2), \\ (n\ n-1\ n-2\ \cdots\ 1) = (n-1\ n) \circ \cdots \circ (2\ 3) \circ (1\ 2). \end{cases} \tag{2}$$

In Fig. 2 we supply another example for the interest of readers. We mention that the set of Dénes permutations of a tree T corresponds to the set of planar drawings of the tree T and the associated Eulerian tours (circular traversings of all edges in both directions) of T as well as the Yushmanov ordering of the leaves of T [24, 26–30]: Indeed, there are $\prod_{v\in\mathrm{V}(T)}(\deg_T(v)-1)!$ such Yushmanov ordering of the leaves of T and each such leaves cyclic ordering is embedded in $\prod_{v\in \mathrm{V}(T)} \deg_T(v)$ Dénes permutations of T. Dénes [24, Corollary 5] [25] further shows that the number of representations of a cyclic permutation of $[n]$ as a product of $n-1$ transpositions is equal to the number of trees on n labelled vertices, and so, by Cayley's formula [30, Chapter 9], is equal to n^{n-2}. Since the set of labelled trees and the set of decompositions into transpositions have the same cardinality, Dénes [25] suggests to find an explicit bijection between them. Thirty years later, Moszkowski [31] [32, Theorem 3.16] discovers a beautiful simple bijection, solving the open problem of Dénes. Related literature on combinatorics of permutations are quite huge; see, e.g., [33–44]. To indicate the diverse directions to go, let us mention Frobenius's formula [45, Theorem A.1.9], a classic result in character theory, which implies that the number of ways of expressing the identity element in Sym_n as $c_1\circ\cdots\circ c_n$, where c_1 is a cyclic permutation and $c_2,\ldots,c_n$ are all transpositions, can be given by a formula involving the characters of Sym_n. There is a great ocean of truth lay undiscovered behind the number n^n, and the accompanying numbers $\frac{1}{n+1}\binom{2n}{n}$ and $n!$, which count the size of the full transformation monoid on $[n]$, the size of the monoid of decreasing monotone transformations on $[n]$ (as well as the size of the inverse semigroup of all strictly decreasing monotone partial permutations on $[n]$), and the size of the full symmetric group on $[n]$, respectively. In the course of a singular promenade on the beach, we are led to some simple transformations on the set of total orders on a fixed finite set, which we explain below.

Assume that T is a tree on n vertices and $f\in \mathbb{TO}(\mathrm{V}(T))$. We use $\epsilon_{T,f}$ to denote the element in Sym_n given by

$$\epsilon_{T,f}=\lambda_{n-1}\circ\cdots\circ\lambda_1,$$

where for each $i\in[n-1]$, λ_i stands for the transposition in Sym_n which swaps i and the label of the father of the vertex labelled i in the tree T with root labelled n, namely λ_i switches i and $f(\mathbb{F}_{T,f^{-1}(n)}(f^{-1}(i)))$. By virtue of Dénes's result, $\epsilon_{T,f}$ is a cyclic permutation and so we can use the two-line notation to define the following element $\kappa_{T,f}$ in Sym_n:

$$\kappa_{T,f}=\begin{pmatrix} \epsilon^0_{T,f}(n) & \epsilon_{T,f}(n) & \cdots & \epsilon^{n-2}_{T,f}(n) & \epsilon^{n-1}_{T,f}(n)\\ n & n-1 & \cdots & 2 & 1\end{pmatrix}.$$

We know that a permutation is a comparison between two total orders [37, p. 18] [46, §1] and so we are at the stage to present the transformations on the set of total orders. For any permutation $\tau\in\mathrm{Sym}_n$, let us consider the transformations T_τ, T'_τ, T''_τ and T'''_τ on $\mathbb{TO}(\mathrm{V}(T))$ such that, for any given total order f on $\mathrm{V}(T)$,

$$\begin{cases} T_\tau(f)=\tau\circ\kappa_{T,f}\circ f,\\ T'_\tau(f)=\tau\circ\kappa^{-1}_{T,f}\circ f,\\ T''_\tau(f)=\kappa_{T,f}\circ\tau\circ f,\\ T'''_\tau(f)=\kappa^{-1}_{T,f}\circ\tau\circ f.\end{cases}\tag{3}$$

If we take any k maps of the form as given in Eq. (3), maybe for different τ, do you expect a good luck that the dynamics of the resulting k-monad can have a good regularity (as in the case of the $3x+1$ problem) and can even be mathematically explained? Let $\nu_n\in\mathrm{Sym}_n$ be the permutation that fixes n and swaps i and $n-i$ for all $i\in[n-1]$, and let $\iota_n\in\mathrm{Sym}_n$ be the identity permutation. We write $\boldsymbol{\sigma}_T$ for $T_{\iota_n}=T''_{\iota_n}$ and we write $\boldsymbol{\alpha}_T$ for T_{ν_n}. That is,

$$\boldsymbol{\sigma}_T(f)=\begin{pmatrix} \epsilon^0_{T,f}(n) & \epsilon_{T,f}(n) & \cdots & \epsilon^{n-2}_{T,f}(n) & \epsilon^{n-1}_{T,f}(n)\\ n & n-1 & \cdots & 2 & 1\end{pmatrix}\circ f,$$

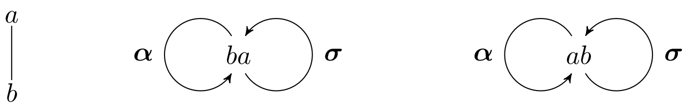


Figure 3: The 1-path T and the phase space of the 2-monad $(\mathbb{TO}(\mathrm{V}(T)); \boldsymbol{\alpha}_T, \boldsymbol{\sigma}_T)$.

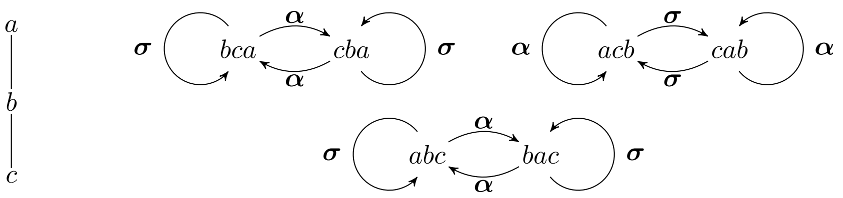


Figure 4: The 2-path T and the phase space of the 2-monad $(\mathbb{TO}(\mathrm{V}(T)); \boldsymbol{\alpha}_T, \boldsymbol{\sigma}_T)$.

$$\boldsymbol{\alpha}_T(f) = \begin{pmatrix} \epsilon^0_{T,f}(n) & \epsilon_{T,f}(n) & \cdots & \epsilon^{n-2}_{T,f}(n) & \epsilon^{n-1}_{T,f}(n) \\ n & 1 & \cdots & n-2 & n-1 \end{pmatrix} \circ f,$$

and

$$\boldsymbol{\alpha}_T = \nu_n \circ \boldsymbol{\sigma}_T \,. \tag{4}$$

A tree is called a *star* if it has one vertex which is adjacent to all other vertices. Stars and paths are often the two extremal families among trees for many problems. By Eq. (2), you can say that $\boldsymbol{\sigma}_T$ and $\boldsymbol{\alpha}_T$ correspond to stars and paths, respectively. Will this be a sign of some regularity for their dynamical behaviours?

Given a finite tree T and $f \in \mathbb{TO}(\mathrm{V}(T))$, we can regard (T, f) as the tree obtained from T by renaming each vertex $v \in \mathrm{V}(T)$ as $f(v)$. In this way, we think of $\boldsymbol{\sigma}_T$ and $\boldsymbol{\alpha}_T$ as transformations among labelled trees with vertex set $[|\mathrm{V}(T)|]$. The small ambiguity here is that two different total orders of $\mathrm{V}(T)$ may give you two labelled trees which are isomorphic to each other as labelled graphs. If we view $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$ as transformations whose actions on a tree T on $[n]$ are $\boldsymbol{\sigma}_T$ and $\boldsymbol{\alpha}_T$, respectively, we can basically say that they are transformations on all labelled trees modulo the above-mentioned ambiguity. Let us probe an example to see the actions of $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$.

Example 2.1. *In Figs. 3 to 5, we draw several trees and part of the phase spaces of the corresponding transformations* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\sigma}$*. To help the reader understand the definition, here are some calculation details. Let* T *be the length-three path as shown on the left of Fig. 5 and let* $f = cdba$ *be a total order on* $\mathrm{V}(T)$*. Then* $\epsilon_{T,f} = (4\ 3) \circ (3\ 2) \circ (4\ 1) = (4\ 1\ 3\ 2)$,

$$\boldsymbol{\sigma}_T(f) = \begin{pmatrix} 4 & 1 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} \circ f = \begin{pmatrix} 4 & 1 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} c & d & b & a \\ 1 & 2 & 3 & 4 \end{pmatrix} = dbca$$

and

$$\boldsymbol{\alpha}_T(f) = \begin{pmatrix} 4 & 1 & 3 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix} \circ f = \begin{pmatrix} 4 & 1 & 3 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} c & d & b & a \\ 1 & 2 & 3 & 4 \end{pmatrix} = cbda.$$

These two local phase transitions are indicated in the figure on the right of Fig. 5.

The goal of this paper is to describe the surprisingly simple dynamical behaviour of the 2-monad $(\mathbb{TO}(\mathrm{V}(T));$ $\boldsymbol{\alpha}_T,\ \boldsymbol{\sigma}_T)$ for any tree T. When $|\mathrm{V}(T)| \leq 2$, it holds $\boldsymbol{\alpha}_T = \boldsymbol{\sigma}_T = 1$ and the corresponding phase spaces are quite trivial, each of the weakly connected components being two loops attached to one vertex; see Fig. 3. When $|\mathrm{V}(T)| = 3$, the phase space of the 2-monad $(\mathbb{TO}(\mathrm{V}(T));\ \boldsymbol{\alpha}_T,\ \boldsymbol{\sigma}_T)$ is depicted in Fig. 4. The reader can check that

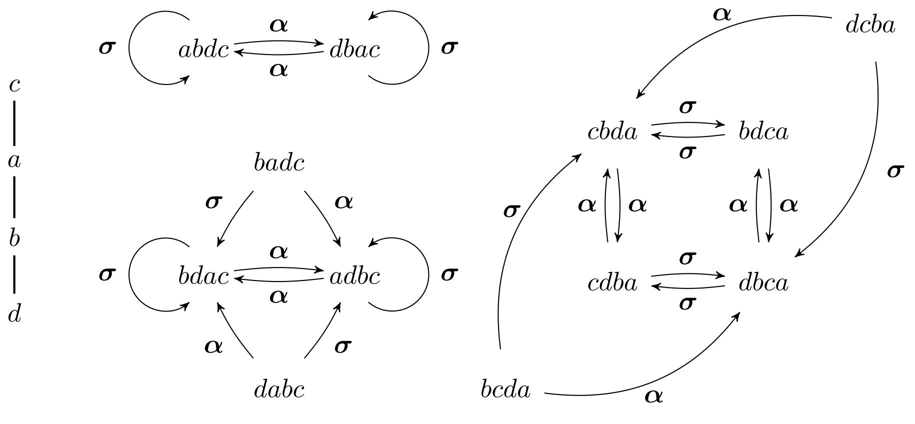


Figure 5: The 3-path T and several weakly connected components of the phase space of the 2-monad $(\mathbb{TO}(\mathrm{V}(T)); \boldsymbol{\alpha}_T, \boldsymbol{\sigma}_T)$.

the transformation monoid generated by $\boldsymbol{\alpha}_T$ and $\boldsymbol{\sigma}_T$ is composed of exactly four elements, $1 = \boldsymbol{\alpha}_T^2 = \boldsymbol{\sigma}_T^2, \boldsymbol{\alpha}_T, \boldsymbol{\sigma}_T$ and $\boldsymbol{\alpha}_T \circ \boldsymbol{\sigma}_T = \boldsymbol{\sigma}_T \circ \boldsymbol{\alpha}_T$. When $|\mathrm{V}(T)| > 3$, our understanding of $(\mathbb{TO}(\mathrm{V}(T));\ \boldsymbol{\alpha}_T,\ \boldsymbol{\sigma}_T)$ will be formulated in Theorem 2.3, the main result of this note.

Before presenting Theorem 2.3, we recall some concepts and introduce some notations. For any graph G on n vertices, an element $f \in \mathbb{TO}(\mathrm{V}(G))$ [21, Theorem 2.5] is a *depth-first search* (DFS) vertex ordering of G if we cannot find $1 \leq i < j < k \leq n$ such that $f^{-1}(i)f^{-1}(k) \in \mathrm{E}(G)$ but $\{f^{-1}(\ell)f^{-1}(j) :\ i \leq \ell \leq j-1\} \cap \mathrm{E}(G) = \emptyset$. Let $\mathcal{A}$ be an automaton. A *periodic point* of $\mathcal{A}$ is any vertex on a cycle of the digraph $\mathcal{PS}(\mathcal{A})$ and a *fixed point* of $\mathcal{A}$ is any element of $\mathrm{S}_{\mathcal{A}}$ to which a loop of $\mathcal{PS}(\mathcal{A})$ is attached. A positive integer is a *period* of $\mathcal{A}$ if it is the length of a simple cycle in $\mathcal{PS}(\mathcal{A})$. We use $\mathrm{Per}(\mathcal{A})$, $\mathrm{Fix}(\mathcal{A})$ and $\mathrm{per}(\mathcal{A})$ for the set of periodic points, the set of fixed points and the set of periods of $\mathcal{A}$, respectively.

Let n be a positive integer and let T be a tree with $|\mathrm{V}(T)| = n$. For each $x \in \mathrm{V}(T)$, a *local order* on the rooted tree $\mathbb{T} = (T, x)$ is a map $\mathbb{f}$ from each $u \in \mathrm{V}(T)$ to an element $\mathbb{f}_u$ in $\mathbb{TO}(\mathbb{D}_{\mathbb{T}}[u])$. Given $f \in \mathbb{TO}(\mathrm{V}(T))$ and $x \in \mathrm{V}(T)$, for the rooted tree $\mathbb{T} = (T, x)$ we designate by $f^{\mathbb{T}} = f^{T,x}$ the local order on $\mathbb{T}$ which maps each $u \in \mathrm{V}(T)$ to the total order $f_u^{\mathbb{T}} = f_u^{T,x}$ that is the restriction of f to $\mathbb{D}_{\mathbb{T}}[u]$. We point out that the definition of local order here resembles the concept of a rotation scheme (also called combinatorial embedding) in combinatorial map theory [47, Definition 5.1.2] [48, 6-6, 16-1]. A subset X of $[n]$ is *consecutive* provided X consists of all integers inbetween $\min X$ and $\max X$, namely $X = [\min\{X\}, \max\{X\}]$. We call $f \in \mathbb{TO}(\mathrm{V}(T))$ a *consecutive vertex ordering* of (T, x) if $\{f(u) :\ u \in \mathbb{D}_{T,x}\langle v\rangle\}$ is a consecutive subset of $[n]$ for all $v \in \mathrm{V}(T)$. We remark that $f \in \mathbb{TO}(\mathrm{V}(T))$ is a DFS ordering of T if and only if it is a consecutive vertex ordering of $\mathbb{T} = (T, f^{-1}(1))$ and u is the minimum element in the total order $f_u^{\mathbb{T}}$ for all $u \in \mathrm{V}(T)$. When f is a consecutive vertex ordering of $(T, f^{-1}(n))$, we simply say f is *consecutive for* T. The set of consecutive vertex orderings of the tree T is denoted by $\mathrm{CS}(T)$. We say that $f, g \in \mathbb{TO}(\mathrm{V}(T))$ are *locally equivalent* with respect to T whenever $f^{-1}(n) = g^{-1}(n)$ and $f^{T,f^{-1}(n)} = g^{T,g^{-1}(n)}$. We write $[f]_T$ for the set of all $g \in \mathbb{TO}(\mathrm{V}(T))$ which are locally equivalent with f with respect to T and call it a *local ordering class of* T.

Example 2.2. *Let T be the tree shown in Fig. 2. Corresponding to the same Eulerian trail as displayed in Fig. 2, there are three consecutive vertex orderings of T: $uvwxyzmopqrst$, $vwxyzmopqrstu$ and $yzmopqrstuvwx$. For $f = yzmopqrstuvwx \in \mathbb{TO}(\mathrm{V}(T))$, $f_u^{T,x}$ is the total order $u < v < w$, namely uvw is the subword of the word f with*

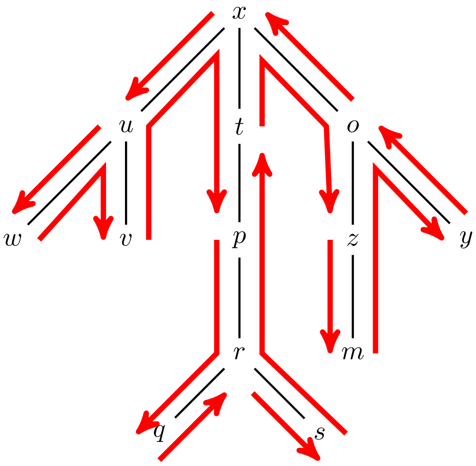


Figure 6: $\boldsymbol{\sigma}_T(ypzmoqrtsuvwx)$ as embedded in an Eulerian tour of T.

support $\mathbb{D}_{T,x}[u]$. *The reader can also check that, with respect to the tree* T, $g = ypzmoqrtsuvwx$ *is locally equivalent with the consecutive vertex ordering* f. *In addition,* $\boldsymbol{\alpha}_T(g) = oymztsrqpvwux$, $\boldsymbol{\sigma}_T(g) = uwvpqrstzmyox$, *and they can be read clockwise or counterclockwise alongside the boundary of the plane tree shown in Fig. 6. Note that we only indicate how to read* $\boldsymbol{\sigma}_T(g)$ *in Fig. 6.*

Michel Deza (27 April 1939 – 23 November 2016) was a spirited mathematician and a poet with a colorful personality. He was a passionate man who often spoke very frankly and openly. However, after the talk of the first-named author in the Workshop on Metric Graph Theory (11 – 13 November 2009, Kanazawa), Deza pointed out to Wu his wrong pronunciation of the word "equality" but only after everyone else in the audience was absent. The first-named author will remember forever how Michel taught him to speak aloud the word "equality" with a wide open mouth. We were encouraged by Michel to venture out into various wild caves for possible mushrooms, beautiful or ordinary. We dedicate the following theorem to his memory, which was brought back from one such "wild cave."

Theorem 2.3. *Let* T *be a tree with* $n \geq 4$ *vertices and* m *leaves. Let* $\mathcal{A} = (\mathbb{TO}(\mathrm{V}(T)); \boldsymbol{\alpha}_T, \boldsymbol{\sigma}_T)$, $\mathcal{A}_1 = (\mathbb{TO}(\mathrm{V}(T)); \boldsymbol{\alpha}_T)$ *and* $\mathcal{A}_2 = (\mathbb{TO}(\mathrm{V}(T)); \boldsymbol{\sigma}_T)$. *Let* $\boldsymbol{\xi}_T : \mathbb{TO}(\mathrm{V}(T)) \to \mathbb{TO}(\mathrm{V}(T))$ *be the map which sends* $f \in \mathbb{TO}(\mathrm{V}(T))$ *to* $\boldsymbol{\xi}_T(f) = \nu_n \circ f$, *where* ν_n *is the map defined below Eq.* (3).

(1) $\boldsymbol{\alpha}_T = \boldsymbol{\xi}_T \circ \boldsymbol{\sigma}_T$ *and* $\boldsymbol{\sigma}_T = \boldsymbol{\xi}_T \circ \boldsymbol{\alpha}_T$.

(2) *The map* $\boldsymbol{\xi}_T$ *is an involution map on both* $\mathrm{Per}(\mathcal{A})$ *and* $\mathbb{TO}(\mathrm{V}(T)) \setminus \mathrm{Per}(\mathcal{A})$, *and is an automorphism of the arc-labelled digraph* $\mathcal{PS}(\mathcal{A})$, *namely* $\boldsymbol{\alpha}_T \circ \boldsymbol{\xi}_T = \boldsymbol{\xi}_T \circ \boldsymbol{\alpha}_T$ *and* $\boldsymbol{\sigma}_T \circ \boldsymbol{\xi}_T = \boldsymbol{\xi}_T \circ \boldsymbol{\sigma}_T$.

(3) *The transformation monoid* $\mathrm{T}(\mathcal{A})$ *generated by* $\boldsymbol{\alpha}_T$ *and* $\boldsymbol{\sigma}_T$ *comprises exactly five elements* 1, $\boldsymbol{\alpha}_T$, $\boldsymbol{\alpha}_T^2$, $\boldsymbol{\sigma}_T$, $\boldsymbol{\alpha}_T \circ \boldsymbol{\sigma}_T$. *As a monoid, it is just the monoid* $\mathbb{M}$ *given in Eq.* (1) *with a Cayley digraph as demonstrated in Fig. 1.*

(4) $\mathrm{Im}(\boldsymbol{\alpha}_T) = \mathrm{Im}(\boldsymbol{\sigma}_T) = \mathrm{Per}(\mathcal{A}_1) = \mathrm{Per}(\mathcal{A}_2) = \mathrm{Per}(\mathcal{A}) = \mathrm{CS}(T)$. *They all coincide with the set of fixed points of* $\boldsymbol{\alpha}_T^2$ *and also the set of fixed points of* $\boldsymbol{\sigma}_T^2$.

(5) $\mathrm{per}(\mathcal{A}_1) \cup \mathrm{per}(\mathcal{A}_2) \subseteq \{1, 2\}$ *and* $\mathrm{per}(\mathcal{A}) \subseteq \{1, 2, 4\}$.

(6) $|\operatorname{Per}(\mathcal{A})| = |\operatorname{Per}(\mathcal{A}_1)| = |\operatorname{Per}(\mathcal{A}_2)| = n \prod_{v\in \mathrm{V}(T)} \deg_T(v)!$.

(7) *Let k be any positive integer. For any f and f' in $\mathbb{TO}(\mathrm{V}(T))$, $[f]_T = [f']_T$ if and only if $\boldsymbol{\alpha}_T^k(f) = \boldsymbol{\alpha}_T^k(f')$ and if and only if $\boldsymbol{\sigma}_T^k(f) = \boldsymbol{\sigma}_T^k(f')$.*

(8) *All DFS vertex orderings of T are fixed points of both $\boldsymbol{\alpha}_T^2$ and $\boldsymbol{\sigma}_T^2$.*

(9) $|\operatorname{Fix}(\mathcal{A})| = |\operatorname{Fix}(\mathcal{A}_1)| + |\operatorname{Fix}(\mathcal{A}_2)|$ *where*

$$|\operatorname{Fix}(\mathcal{A}_1)| = \begin{cases} (n-1)!, & \text{if } T \text{ is a star;} \\ 0, & \text{else;} \end{cases} \quad \text{and } |\operatorname{Fix}(\mathcal{A}_2)| = \begin{cases} m2^{n-m}, & \text{if } \max_{v\in\mathrm{V}(T)} \deg_T(v) \le 3; \\ 0, & \text{else.} \end{cases}$$

Forgetting the arc-labelling, the digraph in Fig. 1 has the Klein four-group, the smallest non-cyclic group, as its automorphism group. For each strongly connected component of $\mathcal{PS}(\mathbb{TO}(\mathrm{V}(T)); \boldsymbol{\alpha}_T, \boldsymbol{\sigma}_T)$ as an arc-labelled digraph, its automorphism group must be a subgroup of the Klein four-group. Actually, a strongly connected component outside of $\operatorname{Per}(\mathcal{A})$ must have size 1, while each strongly connected component inside $\operatorname{Per}(\mathcal{A})$, thanks to Theorem 2.3 (5), can only have sizes 1, 2 or 4. For any component inside $\operatorname{Per}(\mathcal{A})$, Theorem 2.3 (4) says that each $\boldsymbol{\alpha}_T$-arc ($\boldsymbol{\sigma}_T$-arc) is either a loop or is paired with another $\boldsymbol{\alpha}_T$-arc ($\boldsymbol{\sigma}_T$-arc) of different orientation (and hence they form an edge); this means that the automorphism of that component must be generated by the reflection over the edges labelled by $\boldsymbol{\alpha}_T$ or $\boldsymbol{\sigma}_T$. In the middle of Fig. 5, we only find reflection over $\boldsymbol{\alpha}_T$-edges while on the right of Fig. 5 we can see reflections over both $\boldsymbol{\alpha}_T$-edges and $\boldsymbol{\sigma}_T$-edges and so we arrive at the Klein four-group. However, for the weakly connected component as shown on the right of Fig. 5, neither the reflection over $\boldsymbol{\alpha}_T$-edges nor the reflection over the $\boldsymbol{\sigma}_T$-edges can be extended to an automorphism of the whole weakly connected component: Only the identity and the composition of the $\boldsymbol{\sigma}_T$-reflection with the $\boldsymbol{\alpha}_T$-reflection (which equals the map $\boldsymbol{\xi}_T$ mentioned in Theorem 2.3) are the possible automorphism of that weakly connected component. This symmetry breaking is due to the fact that the numbers of incoming arcs at different periodic points of that component are not the same, or, as seen from Theorem 2.3 (7), the sizes of local ordering classes containing those periodic points there are not uniform. Instead, in Fig. 7 we see a weakly connected component which possesses the largest possible symmetry inherited from that of its periodic points, namely the whole Klein four-group. Besides that, considering a set of vertices which are not periodic but are mapped by $\boldsymbol{\alpha}_T$ or $\boldsymbol{\sigma}_T$ to the same point, an arbitrary permutation of this set must be an automorphism of that weakly connected component, as guaranteed by Theorem 2.3 (7).

It would be interesting to see if there is any formula for the size of each weakly connected component of $\mathcal{PS}(\mathbb{TO}(\mathrm{V}(T)); \boldsymbol{\alpha}_T, \boldsymbol{\sigma}_T)$, $\mathcal{PS}(\mathbb{TO}(\mathrm{V}(T)); \boldsymbol{\alpha}_T)$ and $\mathcal{PS}(\mathbb{TO}(\mathrm{V}(T)); \boldsymbol{\sigma}_T)$. For this purpose, maybe we should try to determine the size of $[f]_T$ for all $f \in \mathbb{TO}(\mathrm{V}(T)$. Hopefully, this line of study may give us more insight on the automorphism group of $\mathcal{PS}(\mathbb{TO}(\mathrm{V}(T)); \boldsymbol{\alpha}_T, \boldsymbol{\sigma}_T)$, which can be viewed as a digraph with or without arc-labelling. Especially, how is this automorphism group determined by the symmetry of the tree T and the symmetry of the Cayley digraph in Fig. 1?

3. Proofs

We embark on our voyage towards Theorem 2.3. It is much shorter than we originally expected.

Lemma 3.1. *For every tree T, it holds $\boldsymbol{\sigma}_T \circ \boldsymbol{\xi}_T = \boldsymbol{\xi}_T \circ \boldsymbol{\sigma}_T$, where $\boldsymbol{\xi}_T$ is the map defined in Theorem 2.3.*

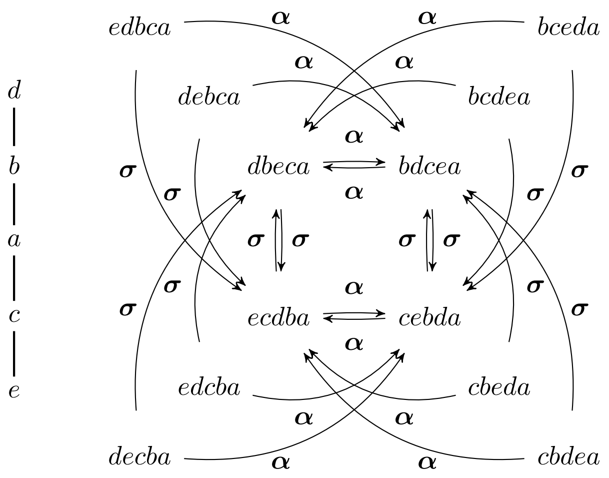


Figure 7: The 4-path T and one weakly connected component of the phase space of the 2-monad $(\mathbb{TO}(\mathrm{V}(T)); \boldsymbol{\alpha}_T, \boldsymbol{\sigma}_T)$.

Proof. Let $n = |\,\mathrm{V}(T)|$, $f \in \mathbb{TO}(\mathrm{V}(T))$ and $r = f^{-1}(n)$. Set $\lambda = \lambda_{n-1} \circ \cdots \circ \lambda_1$ and $\mu = \mu_{n-1} \circ \cdots \circ \mu_1$, where $\lambda_i = (\mathbb{F}_{T,r}(f^{-1}(i))\ f^{-1}(i)) \in \mathrm{Sym}_{\mathrm{V}(T)}$ and $\mu_i = (\mathbb{F}_{T,r}(f^{-1}(n-i))\ f^{-1}(n-i)) \in \mathrm{Sym}_{\mathrm{V}(T)}$ for all $i \in [n-1]$. Note that $\lambda_i = \mu_{n-i}^{-1}$ for $i \in [n-1]$ and so $\lambda = \mu^{-1}$. Moreover, both λ and μ are Dénes permutations of T and hence cyclic permutations of $\mathrm{V}(T)$. This allows us to proceed with

$$\begin{aligned}
\boldsymbol{\xi}_T \circ \boldsymbol{\sigma}_T(f) &= \boldsymbol{\xi}_T \circ \begin{pmatrix} \lambda^0(r) & \lambda(r) & \cdots & \lambda^{n-2}(r) & \lambda^{n-1}(r) \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \lambda^0(r) & \lambda(r) & \cdots & \lambda^{n-2}(r) & \lambda^{n-1}(r) \\ n & 1 & \cdots & n-2 & n-1 \end{pmatrix} \\
&= \begin{pmatrix} \mu^0(r) & \mu(r) & \cdots & \mu^{n-2}(r) & \mu^{n-1}(r) \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix} \\
&= \boldsymbol{\sigma}_T \circ \boldsymbol{\xi}_T(f),
\end{aligned}$$

completing the proof. □

In the proof of [49, Lemma 11.3], we need to find a consecutive vertex ordering of a tree. Lemmas 3.2 and 3.3 below both provide methods of obtaining such orderings. It would be interesting to connect the study here to the active study of the so-called consecutive-ones property [50].

Lemma 3.2. *Let T be a tree on n vertices.*

(1) *For every $x \in \mathrm{V}(T)$ and every local order $\mathbb{g}$ on (T, x), there is a unique consecutive vertex ordering f of (T, x) such that $f^{T,x} = \mathbb{g}$.*

(2) *The number of consecutive vertex orderings of T and the number of local ordering classes of T are both equal to $n \prod_{v \in \mathrm{V}(T)} \deg_T(v)!$.*

Proof. (1). Let $\mathbb{T} = (T, x)$. Let $v_1 \cdots v_n$ be a linear extension of the poset $\mathbf{P}(\mathbb{T})$, and hence $v_n = x$. For each $i \in [n]$, the local order $\mathbb{g}$ sends v_i to the total order $\mathbb{g}_{v_i}$ on $\mathbb{D}_{\mathbb{T}}[v_i]$ and we will read $\mathbb{g}_{v_i}$ as a word which lists elements of

$\mathbb{D}_{\mathbb{T}}[v_i]$ according to this total order. We start with g_{v_n}, then substitute the letter v_{n-1} by the word $\mathrm{g}_{v_{n-1}}$ to get a new word, and then substitute the letter v_{n-2} by the word $\mathrm{g}_{v_{n-2}}$, and so on. After finishing these steps we yield an element $f \in \mathbb{TO}(\mathrm{V}(T))$, which is the unique required element.

In Example 2.2, we indicate that $g = ypzmoqrtsuvwx$ is locally equivalent with the consecutive vertex ordering $f = yzmopqrstuvwx$ with respect to the tree T. We now take $sqmrwvpzyutox$ as a linear extension of $\mathbf{P}(\mathbb{T})$. Based on the previous linear extension and the given element $\mathrm{g} = g^{T,x}$, here is the sequence of growing words to get f according to the above proof: $otux$, $yzotux$, $yzoptux$, $yzoptuvwx$, $yzmoptuvwx$, $yzmoprtuvwx$, $yzmopqrstuvwx = f$.

(2). By (1), every local order on (T, x) is of the form $f^{T,x}$ for some $f \in \mathbb{TO}(\mathrm{V}(T))$. Moreover, each local ordering class of T consists of a class of $f \in \mathbb{TO}(\mathrm{V}(T))$ with a fixed root $x = f^{-1}(n)$ and a fixed local order $\mathbb{f} = f^{T,x}$ for which $\mathbb{f}_x$ has x as the maximum element. Note that there are n choices for the root x, and that $|\,\mathbb{D}_{T,x}(x)| = \deg_T(x)$ while, for each vertex y from $\mathrm{V}(T) \setminus \{x\}$, $|\,\mathbb{D}_{T,x}[y]| = \deg_T(y)$. This implies that the number of local ordering classes of T equals to $n \prod_{v \in \mathrm{V}(T)} \deg_T(v)!$. By (1), every local ordering class of T contains a unique consecutive vertex ordering of T and hence the two sets have the same cardinality. This completes the proof. $\square$

Let T be a tree on n vertices and $g \in \mathbb{TO}(\mathrm{V}(T))$. Note that each edge of T corresponds to two arcs of different directions and so we can think of T as an Eulerian digraph. For $i \in [n]$, let $P_i(T, g)$ be the directed path from $g^{-1}(i)$ to $g^{-1}(i+1)$ in T, where we regard $n+1$ as 1. The cyclic sequence of arcs of T obtained by concatenating $P_1(T, g), \ldots, P_n(T, g)$ in that cyclic order gives rise to a closed walk in the digraph T, which we record as $\mathcal{T}_{T,g}$. We call g an *Eulerian ordering of* T provided $\mathcal{T}_{T,g}$ is an Eulerian tour of T. See Figs. 2 and 6. It is noteworthy that every Eulerian ordering of T must be a consecutive vertex ordering. For the proof of Lemma 3.3 below, the readers should find it quite related to [26, Theorem 6] and [29, Theorem 3.2, Lemma 4.1].

Lemma 3.3. *Let T be a tree. For every $f \in \mathbb{TO}(\mathrm{V}(T))$, both $\boldsymbol{\alpha}_T(f)$ and $\boldsymbol{\sigma}_T(f)$ are Eulerian orderings of T, and hence also consecutive vertex orderings of T.*

Proof. Let $n = |\,\mathrm{V}(T)|$ and $h = \boldsymbol{\sigma}_T(f)$, thereby $\boldsymbol{\alpha}_T(f) = \nu_n \circ h$. If $\mathcal{T}_{T,h}$ is the cyclic ordering of arcs $\overrightarrow{a_1a_2}$, $\overrightarrow{a_2a_3}$, $\ldots$, $\overrightarrow{a_{2n-2}a_1}$, then $\mathcal{T}_{T,\nu_n \circ h}$ can be read cyclically as $\overrightarrow{a_1a_{2n-2}}, \overrightarrow{a_{2n-2}a_{2n-3}}, \ldots, \overrightarrow{a_2a_1}$. Henceforth, it suffices to show that h is an Eulerian ordering of T. Let r denote $f^{-1}(n)$ and let $\lambda = \lambda_{n-1} \circ \cdots \circ \lambda_1$, where $\lambda_i = (\mathbb{F}_{T,r}(f^{-1}(i))\; f^{-1}(i)) \in \mathrm{Sym}_{\mathrm{V}(T)}$ for all $i \in [n-1]$. We then arrive at

$$h = \begin{pmatrix} \lambda^0(r) & \lambda(r) & \cdots & \lambda^{n-2}(r) & \lambda^{n-1}(r) \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}. \tag{5}$$

Take any two adjacent vertices a and b of T. Our task is to show that there is a unique $i \in [n]$ such that the directed path from $\lambda^i(r)$ to $\lambda^{i-1}(r)$ in T passes through the arc $\overrightarrow{ab}$ leading from a to b. Let $j \in [n-1]$ be the unique index such that $\{a, b\} = \{\mathbb{F}_{\mathbb{T}}(f^{-1}(j)), f^{-1}(j)\}$. Then on the way from y to $\lambda^{-1}(y) = \lambda_1 \circ \cdots \circ \lambda_{j-1} \circ \lambda_j \circ \lambda_{j+1} \circ \cdots \circ \lambda_{n-1}(y)$, one will walk across $\overrightarrow{ab}$ if and only if $\lambda_{j+1} \circ \cdots \circ \lambda_{n-1}(y) = a$. This says that the unique $i \in [n]$ with the claimed property is the one satisfying $\lambda^i(r) = y = \lambda_{n-1} \circ \cdots \circ \lambda_{j+1}(a)$, finishing the proof. $\square$

Let f be a total order on a set S. We often write $\nu \circ f$ for $\nu_{|S|} \circ f$ when there is no necessity to specify the size of S, and we write $\overleftarrow{f}$ for the reverse of the total order f, namely $a < b$ in f if and only if $b < a$ in $\overleftarrow{f}$. For any subset R of S, we adhere to the convention that $f|_R$ stands for the restriction of f on R, which is an element of $\mathbb{TO}(R)$.

Lemma 3.4. *Let T be a tree with n vertices, $f \in \mathbb{TO}(\mathrm{V}(T))$, $r = f^{-1}(n)$, $\mathbb{T} = (T, r)$, $g = \boldsymbol{\alpha}_T(f)$ and $h = \boldsymbol{\sigma}_T(f)$.*

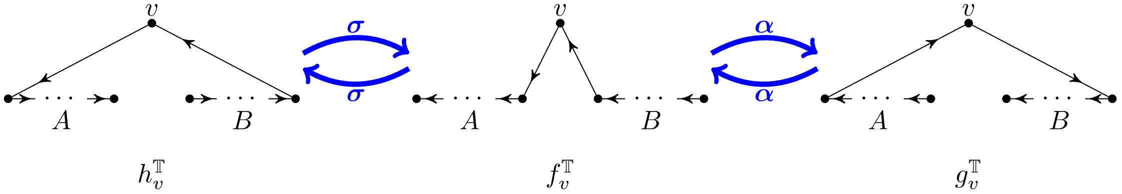


Figure 8: Update of the local order at a non-root non-leaf vertex v. The symbol $x \to y$ indicates that x is bigger than y according to the total order assigned to the family of v in $\mathbb{T}$.

(1) *Take $v \in \mathrm{V}(T) \setminus \{r\}$ with $\mathbb{D}_{\mathbb{T}}(v) \neq \emptyset$, namely $\deg_T(v) > 1$. Let A be the set of elements in $\mathbb{D}_{\mathbb{T}}(v)$ which are less than v in f and let B be the set of elements in $\mathbb{D}_{\mathbb{T}}(v)$ which are bigger than v in f. Then, as demonstrated in Fig. 8, we have the following facts:*

- $g|_A = f|_A$, $g|_B = f|_B$ *and* $B <_g v <_g A$;
- $h|_A = \overleftarrow{f}|_A$, $h|_B = \overleftarrow{f}|_B$ *and* $A <_h v <_h B$.

(2) *It holds $g_r^{\mathbb{T}} = f_r^{\mathbb{T}}$ and $h_r^{\mathbb{T}} = \nu \circ f_r^{\mathbb{T}}$.*

Proof. For all $u \in \mathrm{V}(T) \setminus \{r\}$, let us reserve the notation $\bar{u}$ for $\mathbb{F}_{\mathbb{T}}(u)$, the father of u in $\mathbb{T}$. For any $1 \leq i \leq j \leq n-1$, we call the edge e connecting $f^{-1}(i)$ and $\overline{f^{-1}(i)}$ an edge of type i, denoted by $i = \mathbb{t}_e$, we write $\pi_{[i]}$ for the transposition $(\overline{f^{-1}(i)}\ f^{-1}(i))$ and use the shorthand $\pi_{[i,j]}$ for

$$\pi_{[j]} \circ \cdots \circ \pi_{[i+1]} \circ \pi_{[i]} \in \mathrm{Sym}_{\mathrm{V}(T)}, \tag{6}$$

with the convention that $\pi_{[1,0]}$ is the identity map in $\mathrm{Sym}_{\mathrm{V}(T)}$. The map $\pi_{[1,n-1]}$, which is a Dénes permutation of T, will be dubbed π. Similar to Eq. (5), we have

$$g = \boldsymbol{\alpha}_T(f) = \begin{pmatrix} r & \pi(r) & \cdots & \pi^{n-2}(r) & \pi^{n-1}(r) \\ n & 1 & \cdots & n-2 & n-1 \end{pmatrix}. \tag{7}$$

For all $i \in [n]$, let P_i be the directed path $P_i(T, g)$ from $g^{-1}(i) = \pi^i(r)$ to $g^{-1}(i+1) = \pi^{i+1}(r)$ as defined before Lemma 3.3. Let us call any directed path among $P_1, P_2, \ldots, P_n$ a segment. For each path P_i, we assume that it goes through the vertices $g^{-1}(i) = \pi^i(r) = p_{i,1}, p_{i,2}, \ldots, g^{-1}(i+1) = \pi^{i+1}(r) = p_{i,\Omega_i+1}$ in that order and the edges $p_{i,1}p_{i,2}, \ldots, p_{i,\Omega_i}p_{i,\Omega_i+1}$ are referred to as $e_{i,1}, \ldots, e_{i,\Omega_i}$, respectively. For any $v \in \mathrm{V}(T)$ and integer k, let $\mathbb{t}_{v,k} = \min\{\mathbb{t}_{vw} : vw \in \mathrm{E}(T), \mathbb{t}_{vw} > k\}$. We write $\mathbb{t}_v$ for $\mathbb{t}_{v,0}$ and we regard $\mathbb{t}_{v,k}$ as ∞ when $k \geq \max\{\mathbb{t}_{vw} : vw \in \mathrm{E}(T)\}$. For all $i \in [n]$, the definition of π ensures the following:

$$\mathbb{t}_{e_{i,1}} = \mathbb{t}_{p_{i,1}}, \mathbb{t}_{e_{i,2}} = \mathbb{t}_{p_{i,2}, \mathbb{t}_{e_{i,1}}}, \ldots, \mathbb{t}_{e_{i,\Omega_i}} = \mathbb{t}_{p_{i,\Omega_i}, \mathbb{t}_{e_{i,\Omega_i - 1}}}, \mathbb{t}_{p_{i,\Omega_i+1}, \mathbb{t}_{e_{i,\Omega_i}}} = \infty. \tag{8}$$

From Lemma 3.3 we know that the closed walk $P_1 P_2 \cdots P_n$ starting and ending at r produces an Eulerian tour $\mathcal{T}_{T,g}$ of T. We are now ready for the proof of (1) and (2).

(1). Since $h = \nu_n \circ g$, we only need to establish the first half of the claim.

Let $k = f(v)$, $w = \pi^{-1}_{[1,k-1]}(\bar{v})$, $g(w) = \ell$, $u = \pi^{-1}_{[1,k-1]}(v)$, $g(u) = m$. We find that P_ℓ goes from w to w' and passes by the arc $\overrightarrow{\bar{v}v}$ while P_m goes from u to u' and passes by the arc $\overrightarrow{v\bar{v}}$. If we walk around $P_1P_2\cdots P_n$ from r, we will have to traverse $\overrightarrow{\bar{v}v}$ before visiting $\overrightarrow{v\bar{v}}$ and so we know that $\ell < m$. Let P'_ℓ be the directed path obtained from P_ℓ by removing its initial part of walking from w to v, and let P'_m be the directed path obtained from P_m by removing its terminal part of walking from v to u'. Let P be the walk $P'_\ell P_{\ell+1}\cdots P_{m-1}P'_m$. Since $\mathcal{T}_{T,g}$ is an Eulerian tour of T, the walk P gives rise to an Eulerian tour of the subtree of T induced by $\mathbb{D}_{\mathbb{T}}\langle v\rangle$; see Fig. 9.

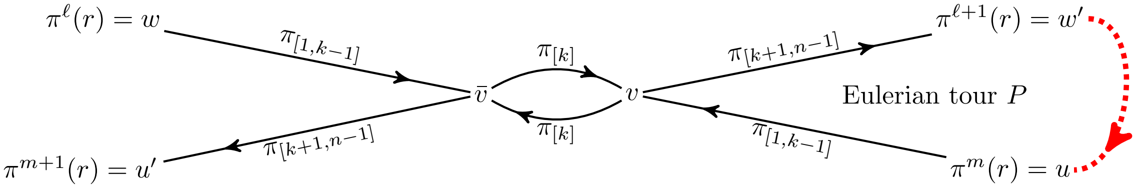


Figure 9: The directed paths P_ℓ and P_m traverse through the edge $v\bar{v}$ in opposite directions.

By Eq. (7), the total order $g_v^{\mathbb{T}}$ is given by the word $\pi^{\ell+1}(r)\pi^{\ell+2}(r)\cdots\pi^m(r)$, namely the sequence of the terminal vertices of $P_\ell, P_{\ell+1}, \ldots, P_{m-1}$. We assume that $a_1 <_f \cdots <_f a_s$ are all the elements of A and $b_1 <_f \cdots <_f b_t$ are all the elements of B. Having in mind Eqs. (6) to (8), one can check the following facts about P.

- If $B \neq \emptyset$, then $\bar{v}, v, b_1$ are three consecutive vertices in that order on P_ℓ, while for each $i \in [t-1]$, b_i, v, b_{i+1} are three consecutive vertices in that order in one segment, and $\overrightarrow{b_t v}$ is the last arc in one segment.
- If $B = \emptyset$, then $w' = v$ and $\overrightarrow{va_1}$ is the first arc of $P_{\ell+1}$;
- If $A \neq \emptyset$, then, for each $i \in [s-1]$, the vertices a_i, v, a_{i+1} appear consecutively in a segment, and $\overrightarrow{va_1}$ is the first arc of one segment.
- For any $x \in A \cup B$, after the first visit of the walk P to x it will first run through an Eulerian tour of the subtree formed by the descendants of x in $\mathbb{T}$ before visiting $\bar{x} = v$.

Combining what were described above, we find that $g|_A = f|_A$, $g|_B = f|_B$ and $B <_g v <_g A$, as was to be shown.

(2). As in (1), we only need to verify $g_r^{\mathbb{T}} = f_r^{\mathbb{T}}$. We assume that $a_1 <_f \cdots <_f a_s <_f r$ are all elements in $\mathbb{D}_{\mathbb{T}}[r]$. By Eq. (8), $\overrightarrow{ra_1}$ is the first arc of P_n; for each $i \in [s-1]$, $\overrightarrow{a_i r}$ is followed by $\overrightarrow{ra_{i+1}}$ in one segment among $P_1, \ldots, P_n$. Now an application of Eq. (7) yields $g_r^{\mathbb{T}} = f_r^{\mathbb{T}}$, finishing the proof. ☐

Lemma 3.5. *Let T be a tree and pick $f \in \mathbb{TO}(\mathrm{V}(T))$. Then,* $\boldsymbol{\alpha}_T^3 = \boldsymbol{\alpha}_T$, $\boldsymbol{\sigma}_T^3 = \boldsymbol{\sigma}_T$, $\boldsymbol{\alpha}_T\boldsymbol{\sigma}_T = \boldsymbol{\sigma}_T\boldsymbol{\alpha}_T$ *and* $\boldsymbol{\alpha}_T^2 = \boldsymbol{\sigma}_T^2$.

Proof. From Lemma 3.4 we obtain $[\boldsymbol{\alpha}_T\boldsymbol{\sigma}_T(f)]_T = [\boldsymbol{\sigma}_T\boldsymbol{\alpha}_T(f)]_T = [\nu \circ f]_T$ and $[\boldsymbol{\alpha}_T^2(f)]_T = [\boldsymbol{\sigma}_T^2(f)]_T = [f]_T$. The latter equality further implies $[\boldsymbol{\alpha}_T^3(f)]_T = [\boldsymbol{\alpha}_T(f)]_T$ and $[\boldsymbol{\sigma}_T^3(f)]_T = [\boldsymbol{\sigma}_T(f)]_T$. These observations, along with Lemma 3.2 (1) and Lemma 3.3, prove the result. ☐

Lemma 3.6. *Let T be a tree. For every $f \in \mathrm{CS}(T)$, it holds $f = \boldsymbol{\alpha}_T^2(f) = \boldsymbol{\sigma}_T^2(f)$. Consequently,* $\mathrm{CS}(T) \subseteq \mathrm{Im}(\boldsymbol{\alpha}_T) \cap \mathrm{Im}(\boldsymbol{\sigma}_T)$.

Proof. Lemma 3.3 says $\boldsymbol{\alpha}_T^2(f) \in \mathrm{CS}(T)$ while Lemma 3.4 gives $[\boldsymbol{\alpha}_T^2(f)]_T = [f]_T$. We now utilize Lemma 3.2 (1) to obtain $f = \boldsymbol{\alpha}_T^2(f)$. Analogously, we can derive $f = \boldsymbol{\sigma}_T^2(f)$. ☐

A rooted tree (T, r) is a *star tree* if its root r is adjacent to all other vertices in T. A rooted tree (T, r) is a *binary tree* provided every vertex has at most two children and is a *leafy binary tree* provided it is a binary tree with $\deg_T(r) \leq 1$. For any binary tree (T, r), we call $f \in \mathbb{TO}(\mathrm{V}(T))$ a *balanced vertex ordering* of (T, r) if, for every $v \in \mathrm{V}(T) \setminus \{r\}$, there are no two elements in $\mathbb{D}_{\mathbb{T}}(v)$ which are less than v in f and there are no two elements in $\mathbb{D}_{\mathbb{T}}(v)$ which are bigger than v in f.

Lemma 3.7. *Let T be a tree with n vertices, $f \in \mathbb{TO}(\mathrm{V}(T))$, $r = f^{-1}(n)$ and $\mathbb{T} = (T, r)$. Then the following hold.*

(1) $\boldsymbol{\alpha}_T(f) = f$ *if and only if* $\mathbb{T}$ *is a star tree.*

(2) $\boldsymbol{\sigma}_T(f) = f$ *if and only if* $\mathbb{T}$ *is a leafy binary tree,* $f \in \mathrm{CS}(T)$*, and* f *is a balanced vertex ordering of* $\mathbb{T}$*.*

Proof. By Lemma 3.3, $\boldsymbol{\alpha}_T(f), \boldsymbol{\sigma}_T(f) \in \mathrm{CS}(T)$. Therefore, neither $\boldsymbol{\alpha}_T(f) = f$ nor $\boldsymbol{\sigma}_T(f) = f$ is possible when $f \notin \mathrm{CS}(T)$. On the other hand, when $f \in \mathrm{CS}(T)$, Lemma 3.2 (1) implies that $[\boldsymbol{\alpha}_T(f)]_T = [f]_T$ if and only if $\boldsymbol{\alpha}_T(f) = f$, and that $[\boldsymbol{\sigma}_T(f)]_T = [f]_T$ if and only if $\boldsymbol{\sigma}_T(f) = f$.

(1). If $\mathbb{T}$ is a star tree, $f \in \mathrm{CS}(T)$ is immediate, and from Lemma 3.4 (2) we get $[\boldsymbol{\alpha}_T(f)]_T = [f]_T$. If $\mathbb{T}$ is not a star tree, it has at least four vertices and has a vertex v which is neither the root nor any leaf. In light of Lemma 3.4 (1), $\boldsymbol{\alpha}_T(f)^{\mathbb{T}}_v \neq f^{\mathbb{T}}_v$.

(2). From Lemma 3.4 (2) we see that $\boldsymbol{\alpha}_T(f)^{\mathbb{T}}_r = f^{\mathbb{T}}_r$ if and only if $\deg_T(r) \leq 1$; Lemma 3.4 (1) further implies that $\boldsymbol{\alpha}_T(f)^{\mathbb{T}}_v = f^{\mathbb{T}}_v$ for all $v \in \mathrm{V}(T) \setminus \{r\}$ if and only if $\mathbb{T}$ is a binary tree and f is a balanced vertex ordering of $\mathbb{T}$. $\square$

Proof of Theorem 2.3. (1). By Eq. (4).

(2). The map $\boldsymbol{\xi}_T$ is surely an involution on $\mathbb{TO}(\mathrm{V}(T))$. The relation of $\boldsymbol{\sigma}_T \circ \boldsymbol{\xi}_T = \boldsymbol{\xi}_T \circ \boldsymbol{\sigma}_T$ is guaranteed by Lemma 3.1. Owing to (1), we can further get

$$\boldsymbol{\alpha}_T \circ \boldsymbol{\xi}_T = (\boldsymbol{\xi}_T \circ \boldsymbol{\sigma}_T) \circ \boldsymbol{\xi}_T = \boldsymbol{\xi}_T \circ (\boldsymbol{\sigma}_T \circ \boldsymbol{\xi}_T) = \boldsymbol{\xi}_T \circ (\boldsymbol{\xi}_T \circ \boldsymbol{\sigma}_T) = \boldsymbol{\xi}_T \circ \boldsymbol{\alpha}_T .$$

Therefore, we now see that $\boldsymbol{\xi}_T$ is an automorphism of $\mathcal{PS}(\mathcal{A})$. As an automorphism, it must send periodic points to periodic points and so $\boldsymbol{\xi}_T$ is an involution on both $\mathrm{Per}(\mathcal{A})$ and $\mathbb{TO}(\mathrm{V}(T)) \setminus \mathrm{Per}(\mathcal{A})$, as wanted.

(3). By Lemma 3.5, the monoid $\mathrm{T}(\mathcal{A})$ contains at most 5 elements: $1, \boldsymbol{\alpha}_T, \boldsymbol{\sigma}_T, \boldsymbol{\alpha}_T\, \boldsymbol{\sigma}_T$ and $\boldsymbol{\alpha}_T^2$. It remains to show that they are really five different transformations on $\mathbb{TO}(T)$.

As $n \geq 4$, there exists at least one vertex ordering of T which is not in $\mathrm{CS}(T)$: Indeed, any total ordering of T which maps a leaf of T to 1 and the adjacent vertex of that leaf to n will be such a candidate. According to Lemma 3.3, this means that none of the four maps $\boldsymbol{\alpha}_T, \boldsymbol{\sigma}_T, \boldsymbol{\alpha}_T\, \boldsymbol{\sigma}_T$ and $\boldsymbol{\alpha}_T^2$ can be the identity map 1. To finish the proof, we now intend to show that

$$|\{\boldsymbol{\alpha}_T, \boldsymbol{\sigma}_T, \boldsymbol{\alpha}_T\, \boldsymbol{\sigma}_T, \boldsymbol{\alpha}_T^2\}| = 4. \tag{9}$$

We first assume that T is not a path. We choose a leaf r and an inner vertex v of T. Consider the rooted tree $\mathbb{T} = (T, r)$ and take two different vertices $a_1, a_2 \in \mathbb{D}_{\mathbb{T}}(v)$. We choose an $f \in \mathbb{TO}(\mathrm{V}(T))$ such that $f(r) = n$ and $f(a_2) < f(a_1) < f(v)$. In view of Lemma 3.4, we have $\boldsymbol{\alpha}_T(v) < \boldsymbol{\alpha}_T(a_2) < \boldsymbol{\alpha}_T(a_1)$, $\boldsymbol{\sigma}_T(a_1) < \boldsymbol{\sigma}_T(a_2) < \boldsymbol{\sigma}_T(v)$, $\boldsymbol{\alpha}_T\, \boldsymbol{\sigma}_T(f)(v) < \boldsymbol{\alpha}_T\, \boldsymbol{\sigma}_T(f)(a_1) < \boldsymbol{\alpha}_T\, \boldsymbol{\sigma}_T(f)(a_2)$ and $\boldsymbol{\alpha}_T^2(a_2) < \boldsymbol{\alpha}_T^2(a_1) < \boldsymbol{\alpha}_T^2(v)$, which leads to Eq. (9).

We next consider the case that T is a path. We take four vertices a, b, c, d of T such that $ca, ab, bd \in \mathrm{E}(T)$ and $\deg_T(c) = 1$. Pick $f \in \mathbb{TO}(\mathrm{V}(T))$ which fulfils $f(a) = n$ and $f(c) < f(b) < f(d)$. According to Lemma 3.4, we have $\boldsymbol{\alpha}_T(c) < \boldsymbol{\alpha}_T(b) > \boldsymbol{\alpha}_T(d)$, $\boldsymbol{\sigma}_T(c) > \boldsymbol{\sigma}_T(b) < \boldsymbol{\sigma}_T(d)$, $\boldsymbol{\alpha}_T\,\boldsymbol{\sigma}_T(f)(c) > \boldsymbol{\alpha}_T\,\boldsymbol{\sigma}_T(f)(b) > \boldsymbol{\alpha}_T\,\boldsymbol{\sigma}_T(f)(d)$ and $\boldsymbol{\alpha}_T^2(c) < \boldsymbol{\alpha}_T^2(b) < \boldsymbol{\alpha}_T^2(d)$; see Fig. 5. This gives Eq. (9) again!

(4). The claim will follow once we can draw the ensuing conclusion:

$$\begin{cases} \mathrm{CS}(T) \subseteq \mathrm{Fix}(\boldsymbol{\alpha}_T^2) \subseteq \mathrm{Im}(\boldsymbol{\alpha}_T) \subseteq \mathrm{Per}(\mathcal{A}_1) \subseteq \mathrm{Per}(\mathcal{A}) \subseteq \mathrm{Im}(\boldsymbol{\alpha}_T) \cup \mathrm{Im}(\boldsymbol{\sigma}_T) \subseteq \mathrm{CS}(T); \\ \mathrm{CS}(T) \subseteq \mathrm{Fix}(\boldsymbol{\sigma}_T^2) \subseteq \mathrm{Im}(\boldsymbol{\sigma}_T) \subseteq \mathrm{Per}(\mathcal{A}_2) \subseteq \mathrm{Per}(\mathcal{A}) \subseteq \mathrm{Im}(\boldsymbol{\alpha}_T) \cup \mathrm{Im}(\boldsymbol{\sigma}_T) \subseteq \mathrm{CS}(T). \end{cases} \tag{10}$$

By symmetry, we only prove the first half of Eq. (10). Lemma 3.6 ensures the truth of $\mathrm{CS}(T) \subseteq \mathrm{Fix}(\boldsymbol{\alpha}_T^2) \subseteq \mathrm{Im}(\boldsymbol{\alpha}_T)$. As seen in Lemma 3.5, we have $\boldsymbol{\alpha}_T^3 = \boldsymbol{\alpha}_T$ and so $\mathrm{Im}(\boldsymbol{\alpha}_T) \subseteq \mathrm{Per}(\mathcal{A}_1)$ is obtained. It is trivial that $\mathrm{Per}(\mathcal{A}_1) \subseteq \mathrm{Per}(\mathcal{A}) \subseteq \mathrm{Im}(\boldsymbol{\alpha}_T) \cup \mathrm{Im}(\boldsymbol{\sigma}_T)$. Finally, by Lemma 3.3, $\mathrm{Im}(\boldsymbol{\alpha}_T) \cup \mathrm{Im}(\boldsymbol{\sigma}_T) \subseteq \mathrm{CS}(T)$, which ends the proof.

(5). A monad is just a homomorphism of a monoid to the full transformation monoid of a set. So, each length-k simple cycle of the phase space of a monad can be lifted to a length-k simple cycle or a length-k walk containing $k+1$ different vertices in the Cayley digraph of the corresponding monoid. Taking (3) into account, an examination of Fig. 1 brings to us $\mathrm{per}(\mathcal{A}_1) \cup \mathrm{per}(\mathcal{A}_2) \subseteq \{1, 2\}$ and $\mathrm{per}(\mathcal{A}) \subseteq \{1, 2, 3, 4\}$.

To conclude the proof, we need to find a contradiction under the assumption that $3 \in \mathrm{per}(\mathcal{A})$. Take a simple cycle of length three in $\mathcal{PS}(\mathcal{A})$. By the pigeon-hole principle, two arcs of the cycle are assigned the same color $C \in \{\boldsymbol{\alpha}_T, \boldsymbol{\sigma}_T\}$ and so we can assume that $f, C(f)$ and $C^2(f)$ are the three distinct vertices on the simple cycle. But (4) claims that $C^2(f) = f$, which violates our assumption.

(6). This is a consequence of (4) and Lemma 3.2 (2).

(7). Assume that $[f]_T = [f']_T$. Lemma 3.4 then gives $[\boldsymbol{\alpha}_T^k(f)]_T = [\boldsymbol{\alpha}_T^k(f')]_T$. But Lemma 3.3 asserts that both $[\boldsymbol{\alpha}_T^k(f)]_T$ and $[\boldsymbol{\alpha}_T^k(f')]_T$ are consecutive vertex orderings of T. Thus, Lemma 3.2 (1) shows that $\boldsymbol{\alpha}_T^k(f) = \boldsymbol{\alpha}_T^k(f')$. The same reasoning leads to $\boldsymbol{\sigma}_T^k(f) = \boldsymbol{\sigma}_T^k(f')$.

For the reverse direction, let us just assume $\boldsymbol{\alpha}_T^k(f) = \boldsymbol{\alpha}_T^k(f')$ and aim to deduce $[f]_T = [f']_T$. Making use of Lemma 3.4 repeatedly, we have $[f]_T = \cdots = [\boldsymbol{\alpha}_T^{2k-2}(f)]_T = [\boldsymbol{\alpha}_T^{2k}(f)]_T = [\boldsymbol{\alpha}_T^{2k}(f')]_T = [\boldsymbol{\alpha}_T^{2k-2}(f')]_T = \cdots = [f']_T$, as desired.

(8). Note that all DFS orderings of T are Eulerian orderings and hence consecutive vertex orderings. Accordingly, the result follows from Lemma 3.6.

(9). Let $\mathbb{T} = (T, f^{-1}(n))$.

We first consider $\mathrm{Fix}(\mathcal{A}_1)$. By Lemma 3.7 (1), we know that $\boldsymbol{\alpha}_T(f) = f$ if and only if $\mathbb{T}$ is a star tree. Obviously, the number of such kind of $f \in \mathbb{TO}(\mathrm{V}(T))$ is $(n-1)!$ when T is a star and zero otherwise.

Then we consider $\mathrm{Fix}(\mathcal{A}_2)$. By Lemma 3.7 (2), $\boldsymbol{\sigma}_T(f) = f$ if and only if $\mathbb{T}$ is a leafy binary tree, $f \in \mathrm{CS}(T)$, and f is a balanced vertex ordering of $\mathbb{T}$. Such an f exists if and only if $\max_{v \in \mathrm{V}(T)} \deg_T(v) \leq 3$. We choose any of the m leaves of T to be $f^{-1}(n)$ and further suppose that $\mathbb{T}$ is a leafy binary tree. Thanks to Lemma 3.2 (1), what is left is to count the number of local orders on this tree $\mathbb{T}$ such that its corresponding consecutive vertex ordering is balanced. Notice that at each vertex the local order there can only have one or two possibilities and the possible local order is unique if and only if that vertex is a leaf of T. This means that the number of such local orders is 2^{n-m}. Considering the m choices of the root of $\mathbb{T}$, we find $|\,\mathrm{Fix}(\mathcal{A}_2)| = m2^{n-m}$, as wanted.

Note that no rooted tree on at least four vertices can be both a star tree and a leafy binary tree. This gives $|\,\mathrm{Fix}(\mathcal{A})| = |\,\mathrm{Fix}(\mathcal{A}_1)| + |\,\mathrm{Fix}(\mathcal{A}_2)|$ and so we are done. ☐

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