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Does the lit-only restriction make any difference for the σ-game and σ^+-game?

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ABSTRACT

Each vertex in a simple graph is in one of two states: "on" or "off". The set of all on vertices is called a configuration. In the σ-game, "pushing" a vertex v changes the state of all vertices in the open neighborhood of v, while in the σ^+-game pushing v changes the state of all vertices in its closed neighborhood. The reachability question for these two games is to decide whether a given configuration can be reached from a given starting configuration by a sequence of pushes. We consider the lit-only restriction on these two games where a vertex can be pushed only if it is in the on state. We show that the lit-only restriction can make a big difference for reachability in the σ-game, but has essentially no effect in the σ^+-game.

1. Introduction

1.1. Reachability problem

Let $G = (V(G), E(G))$ be a *graph*, where $V(G)$ is its set of vertices and $E(G) \subseteq \binom{V(G)}{2}$ its set of edges. We write uv for any edge $\{u, v\} \in E(G)$. In general, we write $u_1u_2 \cdots u_n$ for a *path* of length $n - 1$, namely a graph with vertex set $\{u_1, \ldots, u_n\}$ and edge set $\{u_1u_2, u_2u_3, \ldots, u_{n-1}u_n\}$. The *open neighborhood* of $v \in V(G)$ in G is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v in G is $N_G[v] = N_G(v) \cup \{v\}$.

For any set S, $\mathbb{F}_2^S$ is the set of functions from S to the binary field $\mathbb{F}_2$. For ease of notation, we simply identify $\mathbb{F}_2^S$ with 2^S through identifying a function with its support. Thus, we are allowed to use the

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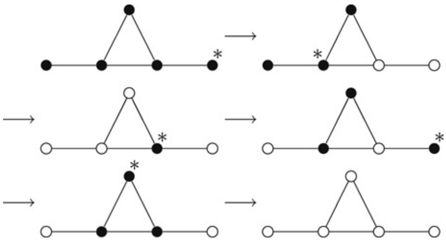

Fig. 1. From all-on to all-off in the lit-only σ^+-game.

expression $A + B$ for $A, B \subseteq S$, standing both for the sum of two functions and for the symmetric difference of two sets. With this understanding, we call $A \subseteq V(G)$ a *configuration* on the graph G and say that, when A is the given configuration, a vertex v is *on* (*lit*) if $v \in A$ and that v is *off* otherwise. More generally, we say that $B \subseteq V(G)$, as a set of vertices, is *on* (*lit*) in the configuration $A \subseteq V(G)$ if $B \subseteq A$ and is *off* if $B \cap A = \emptyset$. You can think of a configuration on G as setting one of the two *states*, on or off, to each of its vertices. In what follows we often call $\emptyset$ the *all-off* configuration and $V(G)$ the *all-on* configuration when it is clear from text that we are talking about configurations on G.

Let G be a graph. In the *σ-game on G*, given any configuration $A \subseteq V(G)$ one can *toggle* (*push*) any vertex v and that changes the state of each vertex in $N_G(v)$. The basic questions for the σ-game include to determine the existence of a sequence of toggles which changes a given configuration to another one, to find a best configuration one can reach starting from a given one, and to find an optimal sequence of toggles to move from one configuration to another. We define the *σ^+-game on G* similarly, the difference being that a toggling of $v \in V(G)$ changes the states of each vertex in $N_G[v]$. For the lit-only σ-game, the effect of toggling a vertex is the same as in the σ-game but we are allowed only to toggle a lit vertex at each step. Analogously, we define the *lit-only σ^+-game* to be the σ^+-game played under the lit-only restriction.

The σ-game was first introduced by Sutner [35]. We consider here the σ-game and σ^+-game only in a narrow sense, both of which were uniformly treated as the σ-game for general digraphs in a general sense [11,38]. We refer the reader to [2–7,9–26,28,29,32,34–42] and their references for many variations and generalizations of the σ-game and σ^+-game.

As with the reachability problem for digraphs, the *reachability problem* for a game is to determine whether or not one can reach some game position (configuration) from some given game position (configuration) in that game [1,12,36]. The task of this paper is to investigate how much do reachability situations change when we only allow lit-only toggling in the σ-game and σ^+-game. We will show that the lit-only restriction can make a big difference in the σ-game, but has virtually no effect in the σ^+-game.

Let us recall a famous result on the reachability of all-off from all-on in the σ^+-game.

Theorem 1 (*Sutner, [35]*). *For any graph G, we can always transform the all-on configuration to the all-off configuration in the σ^+-game on G.*

Many different proofs to Theorem 1 are known; See, e.g., [9,11,29,32,35,39,41]. Based on some observations on small graphs, Goldwasser and Klostermeyer [18, Question 7] wondered whether Theorem 1 still holds when the σ^+-game is replaced by the lit-only σ^+-game. Indeed, before they posed their question, Jaap Scherphuis [32] showed that the answer is 'yes' by presenting what he called a "remarkably hard" proof. However, even earlier, an alleged 'counterexample' to the result of Scherphuis was presented by Eriksson, Eriksson and Sjöstrand [11, Section 5, p. 362]. In the same paper they also proved that the answer is 'yes' provided the graph is bipartite [11, Theorem 5.1]. Let us take a look at the 'counterexample' given in [11] and show that it is not a counterexample at all. To depict a configuration on a graph, we use a bullet for an on vertex and a circle for an off vertex.

Example 2. We display in Fig. 1 a sequence of toggles turning the all-on configuration to the all-off configuration for the lit-only σ^+-game on a special graph, which was asserted to be impossible in [11]. We put a star beside an on vertex to indicate that it is to be pushed at that moment.

It is easy to see that in the lit-only σ^+-game one cannot go from all-off to any other configuration and cannot go from any other configuration to all-on. We show that this is the only difference between the lit-only σ^+-game and the σ^+-game regarding the reachability problem. More precisely, we have the following theorem whose proof is deferred until Section 2.

Theorem 3. *Let $B \neq \emptyset$ and $C \neq V(G)$ be configurations on a connected graph G. Then it is possible to go from B to C in the σ^+-game if and only if it is possible to do so in the lit-only σ^+-game.*

Corollary 4 (*Scherphuis, [32]*). *We can always go from all-on to all-off in the lit-only σ^+-game on any graph.*

Proof. This is a consequence of Theorems 1 and 3. □

Answering the following basic problem would advance our understanding of the lit-only restriction on the σ-game.

Problem 5. Let G be a graph, $B \subseteq V(G)$, and $v \in V(G) \setminus B$. Let C be the configuration obtained from the configuration B by pushing v in the σ-game. When is it possible to reach C from B in the lit-only σ-game?

We say a graph G is *nonsingular* if and only if its adjacency matrix is nonsingular over $\mathbb{F}_2$ (and *singular* otherwise). The next result, which will be proved with a long parity argument in Section 3.1, shows that the lit-only restriction causes a substantial difference for reachability in the σ-game.

Theorem 6. *Let B be a set of vertices in a nonsingular graph G and v be a vertex of G not in B. Let C be the configuration obtained from configuration B by pushing v in the σ-game. Then C cannot be reached from B in the lit-only σ-game.*

In Section 1.2, we rephrase the reachability problem in the framework of automata theory and establish the language of combinatorics on words. In Section 1.3, we further illustrate some observations on the influence of the lit-only restriction.

1.2. Words and automata

Let V be any set. A *word* over the alphabet V is a finite sequence of elements of V. We denote the empty sequence by ϵ, which is also called the *empty word*. For any word W over V and any $u, v \in V$, let $|W|_u$ stand for the number of occurrences of u in W, $|W|_{v,u}$ the number of occurrences of v before the first occurrence of u in W. We set $|W|_{v,u} = 0$ if $|W|_u = 0$. Designate by *alph*(W) the set of letters $v \in V$ satisfying $|W|_v > 0$. We use $|W|$ for the *length* of W, which is surely just $\sum_{u \in V} |W|_u$. For any $1 \le i \le j \le |W|$, define $W_{[i,j]}$ to be the *subword* of W which is obtained from W by deleting the first $i-1$ and the last $|W|-j$ elements. We adopt the convention that $W_{[i,j]} = \epsilon$ for any $i > j$. The set of words over V form a free monoid V^* under the *concatenation product* which associates with two words W_1 and W_2 their product W_1W_2, which is the word of length $|W_1| + |W_2|$ such that

$$\begin{cases} W = W_1, & \text{if } W_2 = \epsilon; \\ W = W_2, & \text{if } W_1 = \epsilon; \\ W_{[1,|W_1|]} = W_1,\ W_{[|W_1|+1,|W_1|+|W_2|]} = W_2, & \text{if } W_1, W_2 \neq \epsilon. \end{cases}$$

Due to the introduction of this product, we often write a word $(a_1, a_2, \ldots, a_n)$ of length $n \ge 1$ by mere juxtaposition $a_1a_2\cdots a_n$, regarding it as the product of n words of length one. It will always be clear from the context whether $u_1u_2\cdots u_n$ means a word over some alphabet or a path in some graph.

We mention that the four games on G, σ-game, σ^+-game, lit-only σ-game, and lit-only σ^+-game, naturally correspond to four kinds of automata on the alphabet $V(G)$ [30], which we denote by $\mathcal{A}_1(G)$, $\mathcal{A}_2(G)$, $\mathcal{A}_3(G)$ and $\mathcal{A}_4(G)$, respectively. These four automata all have $\mathbb{F}_2^{V(G)}$ as *state spaces* and for each v from the alphabet $V(G)$ we have the set of *transitions*

$$\begin{cases} \{(A, A + N_G(v)) : A \subseteq V(G)\}, & \text{for } \mathcal{A}_1(G); \\ \{(A, A + N_G[v]) : A \subseteq V(G)\}, & \text{for } \mathcal{A}_2(G); \\ \{(A, A + N_G(v)) : v \in A \subseteq V(G)\}, & \text{for } \mathcal{A}_3(G); \\ \{(A, A + N_G[v]) : v \in A \subseteq V(G)\}, & \text{for } \mathcal{A}_4(G). \end{cases}$$

Note that we can represent these automata $\mathcal{A}_i(G)$ as arc labeled digraphs $\mathcal{G}_i$ for $i = 1, 2, 3, 4$, where the state spaces correspond to the vertex sets and those transitions given by $v \in V(G)$ correspond to arcs labeled by v. In game theory, these digraphs are called the *game graph* of the four combinatorial games [14]; in discrete dynamical system theory, they are called the *phase space* of the four games [8].

Let G be a graph. We define inductively two monoid homomorphisms σ and σ^+ from the free monoid $V(G)^*$ to the monoid of all affine transformations on the space $\mathbb{F}_2^{V(G)}$ of configurations: Both $\sigma(\epsilon)$ and $\sigma^+(\epsilon)$ act as the identity action on $\mathbb{F}_2^n$; For any $W \in V(G)^*$ of length $n > 0$ and any $A \subseteq V(G)$, we define $A^{\sigma(W)} = A^{\sigma(W_{[1,n-1]})} + N_G(v)$ and $A^{\sigma^+(W)} = A^{\sigma^+(W_{[1,n-1]})} + N_G[v]$, where $v = W_{[n,n]}$. Notice that applying a sequence of toggles according to a word W in the σ-game and σ^+-game is just to apply the action which is the image of the word under the homomorphism σ and σ^+, respectively. For any $W \in V(G)^*$ and $u \in V(G)$, let $\mathcal{F}_u^G(W) = \sum_{v \in N_G(u)} |W|_{v,u}$. For any $A \subseteq V(G)$ and $W \in V(G)^*$, we say that

- W is *good* for G and A if $\mathcal{F}_u^G(W)$ is even for any $u \in A \cap alph(W)$ and $\mathcal{F}_u^G(W)$ is odd for any $u \in alph(W) \setminus A$;
- W is *lit-only σ-allowable* for G provided that we find in W for any $u \in V(G)$ an even number of occurrences of elements of $N_G(u)$ between any two occurrences of u in W; (This property was first formulated by Chuah and Hu [6, Lemma A.1(a)].)
- W is *lit-only σ-allowable* for G and A if it is lit-only σ-allowable for G and if W is good for G and A;
- W is *lit-only σ^+-allowable* for G provided that we find in W for any $u \in V(G)$ an odd number of occurrences of elements of $N_G(u)$ between any two occurrences of u in W;
- W is *lit-only σ^+-allowable* for G and A if it is lit-only σ^+-allowable for G and if W is good for G and A.

We now come to two easy observations which provide the tangible criteria for checking the reachability of one configuration from another. They are especially useful for our analysis in Section 3.1 and enable us in turn to check the local vertex distribution of a word (in one dimensional world) rather than chase the global configuration evolution (in a seemingly uncontrolled world).

Observation 7. *For the σ-game on a graph G and any $A \subset V(G)$, the set of configurations which we can reach from A is just $\{A^{\sigma(W)} : W \in V(G)^*\}$; for the σ^+-game on a graph G and any $A \subseteq V(G)$, the set of configurations which we can reach from A is just $\{A^{\sigma^+(W)} : W \in V(G)^*\}$.*

Observation 8. *For the lit-only σ-game on a graph G and any $A \subseteq V(G)$, the set of configurations which we can reach from A is just $\{A^{\sigma(W)} : W$ is lit-only σ-allowable for G and $A\}$; for the lit-only σ^+-game on a graph G and any $A \subseteq V(G)$, the set of configurations which we can reach from A is $\{A^{\sigma^+(W)} : W$ is lit-only σ^+-allowable for G and $A\}$.*

Let G be a graph. For any $A, B \subseteq V(G)$, the *language* recognized by the four automata introduced above with respect to A and B are $L^*(\mathcal{A}_1(G), A, B) = \{W : A^{\sigma(W)} = B\}$, $L^*(\mathcal{A}_2(G), A, B) = \{W : A^{\sigma^+(W)} = B\}$, $L^*(\mathcal{A}_3(G), A, B) = \{W : A^{\sigma(W)} = B, W$ is lit-only σ-allowable for G and $A\}$, $L^*(\mathcal{A}_4(G), A, B) = \{W : A^{\sigma^+(W)} = B, W$ is lit-only σ^+-allowable for G and $A\}$. For $i = 1, 2, 3, 4$ and any $A \in V(G)$ we use the notation $\mathcal{O}_i^G(A)$ to mean the set of configurations B for which $L^*(\mathcal{A}_i(G), A, B) \neq \emptyset$, namely $\mathcal{O}_i^G(A)$ is the set of reachable configurations starting from A in the corresponding game.

The relation $\sim_i$ defined by $A \sim_i B$ if and only if $B \in \mathcal{O}_i^G(A)$ is clearly an equivalence relation for $i = 1, 2, 3$, because we can reverse any sequence of toggles. In other words, the σ-game, σ^+-game and lit-only σ-game are all *reversible* [24,34], namely if A is reachable from B then B is also reachable from A in these games. So we will call $\mathcal{O}_i^G(A)$ an *orbit* of the corresponding game for $i = 1, 2, 3$. In the lit-only σ^+-game obviously no sequence of toggles which goes from A to B can be executed in the reverse order to go from B to A. It is hence unclear at first sight whether or not $A \sim_4 B$ and $B \sim_4 A$ can happen simultaneously. Consequently, it is interesting to note that Theorem 3 implies that $\sim_4$ is an equivalence relation on the set of configurations on a connected graph G not including $\emptyset$ and $V(G)$.

1.3. Background

To introduce more background on the influence of the lit-only restriction on the reachability problem, we prepare some notation. Let G be a graph. For any $A \subseteq V(G)$ and for $i = 1, 2, 3, 4$, define $w(\mathcal{O}_i^G(A)) = \min\{|B| : B \in \mathcal{O}_i^G(A)\}$ and define $W_i(G) = \max\{w(\mathcal{O}_i^G(A)) : A \subseteq V(G)\}$. Following [18], we also use W, WC, WL and WCL for W_1, W_2, W_3 and W_4, respectively, where C comes from closed neighborhood and L comes from lit-only.

By Theorem 1, any configuration and its complement are in the same orbit of the σ^+-game and hence it holds for any graph G that

$$WC(G) \leq \frac{|V(G)|}{2}. \tag{1}$$

Furthermore, Goldwasser and Klostermeyer [18] introduced the construction of the *closed doubling* of any graph G (replacing each vertex of G by a pair of adjacent vertices) and showed that equality holds in (1) if and only if G is the closed doubling of some graph. It follows immediately from Theorem 3 that these results hold under the lit-only restriction as well:

Corollary 9. *For any graph G, $WCL(G) = WC(G)$. Moreover, $w(\mathcal{O}_2^G(A)) = w(\mathcal{O}_4^G(A))$ for any $A \subseteq V(G)$.*

For any graph G without isolated vertices, it is well-known that [11,18,37]

$$W(G) \leq \frac{|V(G)|}{2}. \tag{2}$$

Goldwasser and Klostermeyer [18] showed that equality holds in (2) if and only if G is the *doubling* of some graph (the doubling of a graph H is the graph obtained by replacing each vertex of H by a pair of non-adjacent vertices). The orbit partition in the lit-only σ-game is clearly a refinement of that for the σ-game and hence $W(G) \leq WL(G)$ holds for any graph G. Goldwasser, Wang and Wu [20] proved that

$$WL(G) \leq \frac{2|V(G)|}{3} \tag{3}$$

with equality holding if and only if G is a complete tripartite graph with each of the three partite sets having equal size. Wang and Wu [37,38] showed that if T is a tree with $\ell \geq 2$ leaves then

$$W(T) \leq \left\lfloor \frac{\ell}{2} \right\rfloor \quad \text{and} \quad WL(T) \leq \left\lceil \frac{\ell}{2} \right\rceil.$$

They gave examples to show that both inequalities are sharp. It seems to be still open to solve the problem of finding the maximum of $WL(T) - W(T)$ over all trees T and the maximum of $WL(G) - W(G)$ over all graphs with n vertices without isolated vertices.

In the remainder of this paper, we will prove Theorem 3 in Section 2, prove Theorem 6 in Section 3.1 (it is a special case of Proposition 20), and give a precise answer to Problem 5 when G is the line graph of a tree in Section 3.2 (Theorem 21).

2. Lit-only σ^+-game and σ^+-game

We start this section with the following three lemmas, each of which shows that if a graph G has a certain local structure then there is a sequence of toggles in the lit-only σ^+-game which has the same effect as toggling a certain off vertex v in the σ^+-game.

Lemma 10. *If G is a graph with an induced path $vx_1x_2 \cdots x_t$ where t is any positive integer not equal to 2, and if all vertices in the path are off except x_t, then there is a sequence of lit-only vertex toggles that has the same result as just toggling v in the σ^+-game.*

Proof. Put

$$W_1 = x_{2m-1}x_{2m-2}\cdots x_1, \qquad W_2 = x_1x_3x_5\cdots x_{2m-1}, \qquad W_3 = x_2x_4x_6\cdots x_{2m-2}.$$

If $t = 2m$ for $m \geq 2$, $x_{2m}W_1vW_2W_3x_{2m-1}x_{2m}x_{2m-1}$ finishes it; if $t = 2m-1$, $W_1vW_2W_3$ does it. □

Lemma 11. *If G has an induced path $vx_1x_2\cdots x_t$ where $t \geq 3$ and if all vertices in the path are on except v, x_1 and x_t, then there is a sequence of lit-only vertex toggles that has the same result as just toggling v in the σ^+-game.*

Proof. Take $m \geq 2$. Let $W_1 = x_{2m-2}x_{2m-4}\cdots x_2$, $W_2 = x_{2m+1}x_{2m-1}x_{2m-3}\cdots x_3$, $W_3 = x_1x_2x_3\cdots x_{2m}$, $W_4 = x_3x_5x_7\cdots x_{2m-1}$. If $t = 2m+1$, we apply the sequence of toggles according to $x_{2m}W_1x_1vW_2W_3x_{2m+1}$; if $t = 2m$, we apply the sequence of toggles according to $x_{2m-1}x_{2m}x_{2m-1}W_1x_1vW_4W_3$. □

Lemma 12. *If G has a vertex v such that all vertices in $N_G[v]$ are off and there exist two vertices at distance two from v, one off and one on, then there is a sequence of lit-only toggles that has the same result as just toggling v in the σ^+-game.*

Proof. Let u be an on vertex at distance two from v and let z be an off vertex at distance two from v.

Case 1: If $uz \in E(G)$, then the sequence of vertex toggles $uzuyvyz$ does what is required, where $y \in N_G(v) \cap N_G(z)$.

Case 2: If $uz \notin E(G)$ and $N_G(u), N_G(v), N_G(z)$ contain a common element, say y, then the vertex toggles $uyuvzyz$ finishes it.

Case 3: $uz \notin E(G)$ and $N_G(u) \cap N_G(v) \cap N_G(z) = \emptyset$.

Subcase 3.1: G contains three induced paths $uwyz, uwv$ and zyv. The sequence of toggles $uwvwywuwzyz$ does it.

Subcase 3.2: G contains an induced path $uwvyz$. By Lemma 10, a lit-only sequence of toggles gives the same result as just toggling y. Then toggle $vzyz$ and we are done. □

The next lemma is concerned with the process of moving from one configuration to another in the σ^+-game.

Lemma 13. *Suppose there is a sequence of toggles in the σ^+-game on a connected graph G which takes you from configuration B to configuration C. Then there is a sequence of toggles in the σ^+-game from B to C where none of the intermediate configurations are either all-off or all-on.*

Proof. The statement in the lemma is obviously true if G is a complete graph. We now assume that the connected graph G is not a complete graph, which means G has an induced path with three vertices and hence G has three vertices with distinct closed neighborhoods. Suppose, contrary to our claim, that for every word W satisfying $B^{\sigma^+(W)} = C$ there exists a positive integer $t \in [1, |W|-1]$ such that $B^{\sigma^+(W_{[1,t]})} \in \{\emptyset, V(G)\}$ and we denote the minimal such t by $t(W)$. Let $W = x_1x_2\cdots x_{|W|}$ be a word from $L^*(\mathscr{A}_2(G), B, C)$ such that $|W| - t(W)$ achieves the minimum possible value. In the following we write t for $t(W)$.

Case 1: $N_G[x_t] + N_G[x_{t+1}] \neq V(G)$. Let $W' = W_{[1,t-1]}x_{t+1}x_tW_{[t+2,|W|]}$. Note that $W' \in L^*(\mathscr{A}_2(G), B, C)$. But it is clear that $B^{\sigma^+(W'_{[1,k]})} \notin \{\emptyset, V(G)\}$ for any $k \in [1, t]$ and hence $|W'| - t(W') < |W| - t$, which is impossible by our assumption on W.

Case 2: $N_G[x_t] + N_G[x_{t+1}] = V(G)$. Let $y \in V(G)$ be such that $N_G[y]$ is neither equal to $N_G[x_t]$ nor to $N_G[x_{t+1}]$. We also know that $N_G[y] \neq V(G)$ as otherwise the word $W'' = W_{[1,t-1]}yW_{[t+2,|W|]} \in L^*(\mathscr{A}_2(G), B, C)$ fulfils $|W''| - t(W'') \leq (|W|-1) - t < |W| - t$, a contradiction. Let $W''' = W_{[1,t-1]}yx_tx_{t+1}yW_{[t+2,|W|]}$. Clearly, $W''' \in L^*(\mathscr{A}_2(G), B, C)$. Since $|W'''| = |W| + 2$, to derive a contradiction it suffices to show $t(W''') \geq t + 3$. We only need to check that

$$B^{\sigma^+(W'''_{[1,k]})} \notin \{\emptyset, V(G)\} \tag{4}$$

for $k = t, t+1, t+2$. Remember that $B^{\sigma^+(W_{[1,t]})} = \emptyset$ and $N_G[x_t] + N_G[x_{t+1}] = V(G)$. Therefore, Eq. (4) is valid for $k = t$ and $k = t+2$ because $N_G[y] \notin \{N_G[x_t], N_G[x_{t+1}]\}$ and is true for $k = t+1$ because $N_G[y] \notin \{\emptyset, V(G)\}$. □

There is not much more to do to prove one of our main results.

Proof of Theorem 3. Suppose there is sequence W of vertex toggles from $B \neq \emptyset$ to $C \neq V(G)$ in the σ^+-game. By Lemma 13, there is also such a sequence W' which avoids all-off and all-on at all intermediate stages. Suppose W' calls for an off vertex v to be toggled. We show that there is a sequence of toggles in the lit-only σ^+-game which produces the same result as just toggling v. Let $N^i(v)$ be the set of vertices of G which are at distance i to v.

Case 1: There is a lit vertex in $N^1(v)$. Then we are done by Lemma 10 with $t = 1$.

Case 2: $N^1(v)$ is off, $N^2(v)$ is neither off nor on. Lemma 12 applies to settle this case.

Case 3: $N^1(v)$ is off, $N^2(v)$ is on, there exists an off vertex at distance $t \geq 3$ from v. Then we are done by Lemma 11.

Case 4: $N^1(v)$ is off, $\cup_{t\geq 2} N^t(v)$ is on. This is impossible because pushing v will result in all-on and we assumed that W' avoids all-on at all intermediate stages and the ending stage.

Case 5: $N^1(v)$ is off, $N^2(v)$ is off, there exists an on vertex at distance $t \geq 3$ from v. We treat this by Lemma 10.

Case 6: All vertices are off. This cannot happen due to our assumption on W'. □

3. Lit-only σ-game and σ-game

3.1. Words: A parity invariant

The following parity result is our main result in this section, from which we will derive several impossibility results showing the substantial effect of the lit-only restriction on the σ-game.

Lemma 14. *Let G be a graph, $W \in V(G)^*$, S and Q be two disjoint subsets of $V(G)$ such that*

(i) *S is a set of independent vertices in G, namely $uv \notin E(G)$ for any $u, v \in S$;*
(ii) *For any $v \in Q$, we have $N_G(v) \subseteq S$;*
(iii) *$\{v \in V(G) : |W|_v$ is odd $\} = S \cup Q$.*

If W is lit-only σ-allowable for G, then $\sum_{u\in S\cup Q} \mathcal{F}_u^G(W)$ has the same parity as $|\{v \in Q : |N_G(v)|$ is odd $\}|$.

Proof. Let $m = |W|$ and $[m] = \{1, 2, \ldots, m\}$. We use the notation W_i to mean $W_{[i,i]}$ for any $i \in [m]$. For any $P \in \binom{[m]}{2}$, we assume that the two elements in P are $P_{\min}$ and $P_{\max}$ where $P_{\min} < P_{\max}$.

Let $\mathscr{I} = \{i : i \in [m], W_i \in V(G) \setminus S\}$ and $\mathscr{J} = \{j : j \in [m], W_j \in S\}$. We choose a total ordering $\lhd$ for $V(G)$, only requiring it to fulfil that

$$u \lhd v \lhd w, \quad \forall u \in V(G) \setminus (S \cup Q), v \in Q, w \in S. \tag{5}$$

A pair $P \in \binom{[m]}{2}$ is said to be *good* if $W_{P_{\min}} W_{P_{\max}} \in E(G)$ and $W_{P_{\min}} \lhd W_{P_{\max}}$. A good pair is said to be *blue* if it intersects both $\mathscr{I}$ and $\mathscr{J}$. Denote by $\mathscr{G}$ and $\mathscr{B}$ the set of good pairs and the set of blue pairs, respectively. All calculations below are carried out over $\mathbb{F}_2$.

For any $u \in V(G)$, put $f(u)$ to be

$$|\{P \in \mathscr{G} : \exists i \in P, W_i = u\}| = |\{P \in \mathscr{G} : W_{P_{\min}} = u\}| + |\{P \in \mathscr{G} : W_{P_{\max}} = u\}|.$$

For our purpose, let us determine $f(u)$ in three cases depending on where u comes from, S, Q or $V(G) \setminus (S \cup Q)$.

We claim that $f(u) = 0$ for $u \in V(G) \setminus (S \cup Q)$. It suffices to check this for $u \in alph(W)$. By Condition (iii), the elements of $\{i : W_i = u\}$ can be enumerated as $i_1 < i_2 < \cdots < i_{2\alpha}$ for some positive integer α. Let

$$U_0 = \{j : j < i_1, \{j, i_1\} \in \mathscr{G}\},$$
$$U_{2\alpha} = \{j : j > i_{2\alpha}, \{i_{2\alpha}, j\} \in \mathscr{G}\},$$

and for $t = 1, 2, \ldots, 2\alpha - 1$, let

$$U_t^1 = \{j : i_t < j < i_{t+1}, W_j \lhd u, W_j u \in E(G)\},$$
$$U_t^2 = \{j : i_t < j < i_{t+1}, u \lhd W_j, uW_j \in E(G)\}.$$

The following calculation verifies our claim:

$$\begin{aligned} f(u) &= \sum_{j=1}^{m} \left|\left\{P \in \mathscr{G} : (P_{\min} = j, W_{P_{\max}} = u) \text{ or } (P_{\max} = j, W_{P_{\min}} = u)\right\}\right| \\ &= \sum_{j \in U_0 \cup U_{2\alpha}} 2\alpha + \sum_{t=1}^{2\alpha-1} \left((2\alpha - t)|U_t^1| + t|U_t^2|\right) = \sum_{t=1}^{2\alpha-1} t \left|U_t^1 \cup U_t^2\right| \\ &= \sum_{t=1}^{2\alpha-1} t \left|\left\{j : i_t < j < i_{t+1}, W_j u \in E(G)\right\}\right| = 0, \end{aligned} \tag{6}$$

where the last equality is due to our assumption that W is lit-only σ-allowable for G.

We next show that for any $u \in Q$ it holds

$$f(u) = \mathcal{F}_u^G(W) + |N_G(u)|. \tag{7}$$

Indeed, by Condition (iii) we can enumerate the elements of $\{i : W_i = u\}$ as $i_1 < i_2 < \cdots < i_{2\alpha-1}$ for some positive integer α. Let $V_t = \{j : i_t < j < i_{t+1}, W_j \in N_G(u)\}$ for $t = 1, 2, \ldots, 2\alpha - 2$. In addition, let $V_0 = \{j : j < i_1, W_j \in N_G(u)\}$ and $V_{2\alpha-1} = \{j : j > i_{2\alpha-1}, W_j \in N_G(u)\}$. As W is lit-only σ-allowable for G, we see that

$$|V_t| = 0, \quad t = 1, 2, \ldots, 2\alpha - 2. \tag{8}$$

Combining this with Conditions (ii) and (iii), we obtain

$$\begin{aligned} \mathcal{F}_u^G(W) &= |V_0| = |V_0| + \sum_{t=1}^{2\alpha-2} |V_t| = |V_{2\alpha-1}| + \sum_{t=0}^{2\alpha-1} |V_t| \\ &= |V_{2\alpha-1}| + \sum_{v \in N_G(u)} |W|_v = |V_{2\alpha-1}| + \sum_{v \in N_G(u)} 1 \\ &= |V_{2\alpha-1}| + |N_G(u)|. \end{aligned} \tag{9}$$

By Conditions (ii) and (iii) together with (5), we get that

$$\begin{aligned} \{P \in \mathscr{G} : \exists i \in P, W_i = u\} &= \{P \in \mathscr{G} : W_{P_{\min}} = u, W_{P_{\max}} \in S\} \\ &= \{(i, j) : 1 \le i < j \le m, W_i = u, W_j \in N_G(u)\}. \end{aligned}$$

This combined with Eq. (8) leads us to

$$\begin{aligned} f(u) &= \sum_{s=1}^{2\alpha-1} |\{j : j > i_s, W_j \in N_G(u)\}| \\ &= \sum_{s=1}^{2\alpha-1} \sum_{t=s}^{2\alpha-1} |V_t| = (2\alpha - 1)|V_{2\alpha-1}| \\ &= |V_{2\alpha-1}|. \end{aligned} \tag{10}$$

Comparing Eqs. (9) and (10) yields Eq. (7), as wanted.

Now consider $u \in S$ and let us proceed to show that

$$f(u) = \mathcal{F}_u^G(W). \tag{11}$$

Condition (iii) says that those positions i for which $W_i = u$ can be listed as $i_1 < i_2 < \cdots < i_{2\alpha-1}$ where α is a positive integer. By Condition (i), Eq. (5) and the definition of good pairs, we know that

$$\{P \in \mathscr{G} : W_{P_{\min}} \in S\} = \emptyset.$$

It follows that

$$f(u) = \sum_{j=1}^{m} |\{P \in \mathscr{G} : P_{\min} = j, W_{P_{\max}} = u\}|. \tag{12}$$

For $t = 1, \ldots, 2\alpha - 2$, we set

$$\begin{aligned} U_t &= \{j : i_t < j < i_{t+1}, \exists P \in \mathscr{G}, P_{\min} = j, W_{P_{\max}} = u\} \\ &= \{j : i_t < j < i_{t+1}, W_j \lhd u, W_j u \in E(G)\}, \end{aligned}$$

which, according to Condition (i) and Eq. (5), can be equivalently defined as

$$\{j : i_t < j < i_{t+1}, W_j u \in E(G)\}.$$

Taking account of the fact that W is lit-only σ-allowable, we then arrive at

$$|U_t| = 0, \quad t = 1, \ldots, 2\alpha - 2. \tag{13}$$

Eqs. (12) and (13) conspire to give

$$\begin{aligned} f(u) &= \sum_{j \in \mathfrak{F}_u} (2\alpha - 1) + \sum_{t=1}^{2\alpha-2} (2\alpha - 1 - t)|U_t| \\ &= |\mathfrak{F}_u|, \end{aligned}$$

where we use the notation $\mathfrak{F}_u$ for $\{j : j < i_1, W_j u \in E(G)\}$. Noting that $\mathcal{F}_u^G(W)$ just counts the size of $\mathfrak{F}_u$, this establishes Eq. (11).

After finishing the computation of $f(u)$, let us look at $|\mathscr{B}|$ as a number in $\mathbb{F}_2$. First of all, we have

$$\begin{aligned} |\mathscr{B}| &= \sum_{P \in \mathscr{B}} 1 \\ &= \sum_{P \in \mathscr{B}} \sum_{i \in P \cap \mathscr{I}} 1 \\ &= \sum_{P \in \mathscr{G}} \sum_{i \in P \cap \mathscr{I}} 1 \\ &= \sum_{u \in V(G) \setminus S} \sum_{P \in \mathscr{G} :\, \exists i \in P, W_i = u} 1 \\ &= \sum_{u \in V(G) \setminus S} f(u) \\ &= \sum_{u \in Q} (\mathcal{F}_u^G(W) + |N_G(u)|), \end{aligned} \tag{14}$$

where the last equality comes from Eqs. (6) and (7). We count $|\mathscr{B}|$ in another way:

$$\begin{aligned} |\mathscr{B}| &= \sum_{P \in \mathscr{B}} 1 \\ &= \sum_{P \in \mathscr{B}} \sum_{j \in P \cap \mathscr{J}} 1 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{P\in\mathscr{G}} \sum_{j\in P\cap \mathscr{J}} 1 \\
&= \sum_{u\in S} \sum_{P\in\mathscr{G}:\, \exists j\in P, W_j=u} 1 \\
&= \sum_{u\in S} f(u) \\
&= \sum_{u\in S} \mathcal{F}_u^G(W),
\end{aligned} \tag{15}
$$

where the last equality is a consequence of Eq. (11). Putting together Eqs. (14) and (15), we obtain the assertion of the theorem. □

Example 15. Let G be the path of length 1, say $G = u_1u_2$. Let W be the word u_1u_2. Then W is lit-only σ-allowable for G and $\mathcal{F}_{u_1}^G(W) + \mathcal{F}_{u_2}^G(W) = 1$. This is consistent with our assertion in Lemma 14 if we take $S = \{u_1\}$ and $Q = \{u_2\}$. If we take $S = \{u_1, u_2\}$ and $Q = \emptyset$, we see that the condition that S is an independent set cannot be dropped from Lemma 14.

A *bipartite graph* is a graph whose vertex set is the union of two disjoint independent sets, called *partite sets* of the graph.

Proposition 16. *Let G be a bipartite graph with partite sets V_1 and V_2, where $V_1 = \{v_1, v_2, \ldots, v_{2t-1}\}$, $t \geq 1$. Assume that $\{N_G(v) : v \in V_1\}$ is a set of linearly independent vectors. Further suppose that there are $U \subseteq V_1$ and $B \subseteq V_2$ such that*

$(*)$ $|U \cap N_G(v)|$ *is even if* $v \in B$ *and is odd if* $v \in V_2 \setminus B$.

Then there is no sequence of lit-only moves starting from B and ending at $B^{\sigma(v_1v_2\cdots v_{2t-1})}$ in the σ-game on G.

Proof. Let W be any word which is lit-only σ-allowable for G and B. Our task is to show that $B^W \neq B^{\sigma(v_1v_2\cdots v_{2t-1})}$. We proceed under the opposite assumption that $B^W = B^{\sigma(v_1v_2\cdots v_{2t-1})}$ and try to derive a contradiction. Set $S = V_1 \cap \mathscr{O}$ and $Q = V_2 \cap \mathscr{O}$, where $\mathscr{O} = \{u : |W|_u \text{ is odd}\}$. Considering the bipartiteness of G and that $N_G(v)$, $v \in V_1$, are linearly independent, we find that $S = V_1$ and that

$$|N_G(v) \cap Q| \equiv 0 \pmod 2 \tag{16}$$

for each $v \in V_1$. It follows from Eq. (16) that the number of vertices in Q with odd degrees is even and then Lemma 14 ensures that

$$\sum_{u\in S\cup Q} \mathcal{F}_u^G(W) \equiv 0 \pmod 2. \tag{17}$$

Another immediate consequence of Eq. (16) is that $\sum_{u\in Q} |N_G(u) \cap U|$ is even. By Condition (*), this means that

$$|Q \cap (V_2 \setminus B)| \equiv 0 \pmod 2. \tag{18}$$

Because W is lit-only σ-allowable for G and B, we deduce from $B \subseteq V_2$ that $\mathcal{F}_u^G(W)$ is odd for each $u \in S = V_1$ and hence $\sum_{u\in S} \mathcal{F}_u^G(W)$ is odd. Consequently, we deduce from Eq. (17) that $\sum_{u\in Q} \mathcal{F}_u^G(W)$ is odd. This says that there are an odd number of vertices from $V_2 \setminus B$ which are toggled an odd number of times according to W. This is in contradiction with Eq. (18), as desired. □

Example 17 (*[6, pp. 138–143][7, p. 826]*). Let G be the graph with $V(G) = \{v_1, v_2, \ldots, v_8\}$ and $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_4v_6, v_6v_7, v_7v_8\}$. Let $B = \{v_5\}$ and $C = \{v_1, v_8\} = B^{\sigma(v_2v_4v_7)}$. See Fig. 2. Clearly G is a bipartite graph with partite sets $V_1 = \{v_2, v_4, v_7\}$ and $V_2 = \{v_1, v_3, v_5, v_6, v_8\}$. Let $U = \{v_2, v_7\}$. It is not hard to check that Proposition 16 shows we cannot reach C from B in the lit-only σ-game. Observe that this means that we also cannot reach B from C in the lit-only σ-game and this also follows from Proposition 16 with $U = \{v_4\}$.

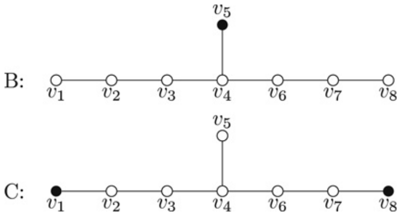

Fig. 2. B can evolve to C in the σ-game but not in the lit-only σ-game.

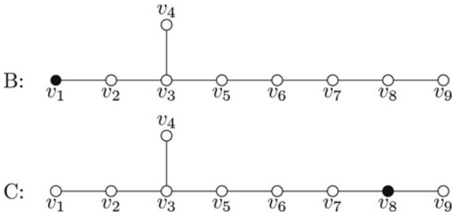

Fig. 3. B can evolve to C in the σ-game but not in the lit-only σ-game.

Proposition 18. *Let G be a graph, let $B \subseteq V(G)$, and let $W \in V(G)^*$ be lit-only σ-allowable for G and B. If $S = \{v \in V(G) : |W|_v \text{ is odd }\}$ is an independent set, then $|S \setminus B|$ is even.*

Proof. Immediate from Lemma 14 by taking $Q = \emptyset$. □

Example 19 (*[6, pp. 143–147][7, p. 826]*). Let G be the bipartite graph with $V(G) = \{v_1, v_2, \ldots, v_9\}$ and $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_3v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9\}$. Let $B = \{v_1\}$ and $C = \{v_8\}$. See Fig. 3. Note that $B^{\sigma(v_2v_5v_7)} = C$ and so B and C are in the same orbit of the σ-game on G. It is not hard to check that the only set of linearly dependent neighborhoods in G is $\{N(v_4), N(v_5), N(v_7), N(v_9)\}$. Thus, if $B^{\sigma(W)} = C$ then $S = \{v : |W|_v \text{ is odd }\}$ is equal to either $\{v_2, v_4, v_9\}$ or $\{v_2, v_5, v_7\}$. Since both are independent sets and in both cases $|S \setminus B|$ is odd, by Proposition 18, B and C are in different orbits in the lit-only σ-game.

Proposition 20. *Let G be a graph whose vertex set is partitioned into sets R and T. Denote $\{N_G(v) \in \mathbb{F}_2^{V(G)} : v \in R\}$ and $\{N_G(v) \in \mathbb{F}_2^{V(G)} : v \in T\}$ by $\mathcal{R}$ and $\mathcal{T}$, respectively. Suppose that $\mathcal{T}$ is linearly independent and $span(\mathcal{T}) \cap span(\mathcal{R}) = \{0\}$. Further suppose that there is $U \subseteq V(G)$ such that $|U \cap N_G(v)|$ is odd for each $v \in R$. Then starting from any configuration $B \subseteq T$, we cannot execute a sequence of lit-only moves whose net effect is to press an odd number of vertices in R which form an independent set in G.*

Proof. Assume to the contrary that there is a $W \in V(G)^*$ that is lit-only σ-allowable for G and B such that $B^{\sigma(W)} - B = \sum_{i=1}^{2k+1} N_G(r_i)$ where $r_1, \ldots, r_{2k+1} \in R$. Observe that $B^{\sigma(W)} - B = \sum_{v \in S} N_G(v) + \sum_{v \in S'} N_G(v)$, where $S = \{v \in R : |W|_v \text{ is odd}\}$ and $S' = \{v \in T : |W|_v \text{ is odd}\}$. Because $span(\mathcal{T}) \cap span(\mathcal{R}) = \{0\}$, we conclude that

$$\sum_{i=1}^{2k+1} N_G(r_i) = \sum_{v \in S} N_G(v) \tag{19}$$

and that

$$0 = \sum_{v \in S'} N_G(v). \tag{20}$$

As $\mathcal{T}$ is linearly independent, we deduce from Eq. (20) that $S' = \emptyset$. Taking the inner product with U on both sides of Eq. (19) reveals that S is a set of odd cardinality. By now, an application of Proposition 18 gives the desired contradiction and hence ends the proof. □

Finally, let us prove Theorem 6 as promised.

Proof of Theorem 6. Apply Proposition 20 with $R = \{v\}$ and $T = V(G) \setminus R$. $\square$

3.2. *Line graph: A lifting result*

All the above influences of the lit-only restriction on the σ-game are observed with the help of a parity result, Lemma 14. By appealing to some other technique, we can also give a precise answer to Problem 5 when G is the line graph of a tree. To do this we will need to review the notions of boundary and coboundary maps on a graph.

The *coboundary map* on a graph G is the $\mathbb{F}_2$-linear map δ_G from $\mathbb{F}_2^{V(G)}$ to $\mathbb{F}_2^{E(G)}$ satisfying $\delta_G(v) = \sum_{v\in e\in E(G)} e$ for each $v \in V(G)$. The *adjoint* of δ_G, or the *boundary map* on G, is the $\mathbb{F}_2$-linear map $\delta_G^\top$ from $\mathbb{F}_2^{E(G)}$ to $\mathbb{F}_2^{V(G)}$ satisfying $\delta_G^\top(ab) = a + b$ for each $ab \in E(G)$. The *line graph* of a graph G, denoted by $L(G)$, is the graph with $V(L(G)) = E(G)$ and $ef \in E(L(G))$ if and only if $|e \cap f| = 1$. For any $e = ab \in V(L(G))$, a basic observation is that

$$N_{L(G)}(e) = \delta_G(a + b) = \delta_G\delta_G^\top(e). \tag{21}$$

This says that if we start with $\delta_G(A)$ and push $e \in V(L(G))$ in the σ-game on $L(G)$, we will arrive at $\delta_G(A + \delta_G^\top(e))$. Note that $|A + \delta_G^\top(e)|$ and $|A|$ have the same parity. Another thing to be noticed is that for the configuration $\delta_G(A)$ of $L(G)$, the pressing of $e \in V(L(G))$ will be a lit-only toggle if and only if $|\delta_G^\top(e) \cap A| = 1$ and in such a case $|A + \delta_G^\top(e)| = |A|$. By virtue of Eq. (21), we can lift the σ-game and lit-only σ-game on $L(G)$ to corresponding games on G; see [40] for more details.

Let T be a tree. Let $\mathscr{V}_i = \{A \subseteq V(T) : |A| = i\}$ for $i = 0, \ldots, |V(T)|$. As T is connected, we know that

$$span\{\delta_T^\top(e) : e \in E(T)\} = \bigcup_{i \text{ even}} \mathscr{V}_i \tag{22}$$

is a subspace of $\mathbb{F}_2^{V(T)}$ of index two. Since T is a tree, it is easy to see that:

(**) δ_T is a two-to-one surjective map with kernel $\{\emptyset, V(T)\}$.

If $V(T) = n$, it follows immediately from Eq. (21) and the preceding discussions the following [40, Theorem 10]:

(A) When n is odd, the σ-game on $L(T)$ has only the orbit $\{\delta_T(A) : A \in \cup_{i \text{ even}} \mathscr{V}_i\} = \{\delta_T(A) : A \in \cup_{i \text{ odd}} \mathscr{V}_i\}$ and the orbits of the lit-only σ-game on $L(T)$ are $\mathscr{O}_0, \ldots, \mathscr{O}_{\frac{n-1}{2}}$, where $\mathscr{O}_i = \{\delta_T(A) : A \in \mathscr{V}_i\} = \{\delta_T(A) : A \in \mathscr{V}_{n-i}\}$, $i = 0, \ldots, \frac{n-1}{2}$;

(B) When n is even, the σ-game on $L(T)$ has exactly two orbits $\{\delta_T(A) : A \in \cup_{i \text{ even}} \mathscr{V}_i\}$ and $\{\delta_T(A) : A \in \cup_{i \text{ odd}} \mathscr{V}_i\}$ while the orbits of the lit-only σ-game on $L(T)$ are $\mathscr{O}_0, \ldots, \mathscr{O}_{\frac{n}{2}}$, where $\mathscr{O}_i = \{\delta_T(A) : A \in \mathscr{V}_i\} = \{\delta_T(A) : A \in \mathscr{V}_{n-i}\}$, $i = 0, \ldots, \frac{n}{2}$.

The next result characterizes those single moves in the σ-game on a line graph of a tree which have no way to be replaced by a series of lit-only moves.

Theorem 21. *Let T be a tree, $B \subsetneq E(T)$ and $e \in E(T) \setminus B$. It is possible to reach $B^{\sigma(e)}$ from B in the lit-only σ-game on $L(T)$ if and only if $B = \delta_T(F)$ for some $F \subseteq V(T)$ such that $\delta_T^\top(e) \cap F = \emptyset$ and $2|F| + 2 = |V(T)|$.*

Proof. Because T is a tree and $e \in E(T) \setminus B$, we deduce from (**) that there exists a unique $F \subseteq V(T)$ such that $B = \delta_T(F)$ and that $\delta_T^\top(e) \cap F = \emptyset$ (the complement of the given F is the only other set which is sent by δ_T to B). By Eq. (21), we have $\delta_T(F)^{\sigma(e)} = \delta_T(F + \delta_T^\top(e))$. Note that $|F + \delta_T^\top(e)| = |F| + 2$. According to Claims (A) and (B) above, we conclude that $B^{\sigma(e)}$ and B can lie in the same orbit of the lit-only σ-game on $L(T)$ if and only if $|F| + (|F| + 2) = |V(T)|$. $\square$

Comparing Theorems 6 and 21 leads us to the by-product that the line graph of a tree on an even number of vertices is singular. This is indeed a special case of a well-known result and several other proofs without using the result on lit-only σ-game are available. We deviate a bit to report one such proof below.

Fact 22 *([27,31,33]). The line graph of a connected graph G with $|V(G)| \geq 2$ is nonsingular if and only if G is a tree on an odd number of vertices.*

Proof. Suppose G contains a cycle and let Y be the set of edges in a cycle of G. Clearly, $\delta_G^\top(Y) = \emptyset$. We thus conclude from Eq. (21) that $L(G)$ is singular.

Assume that G is a tree T. The map $\delta_T^\top$ is clearly injective. It now follows by (**), Eqs. (21) and (22), and linearity that the kernel of the adjacency matrix of $L(T)$ contains a nonzero vector if and only if $|V(T)|$ is even. □

4. Concluding remarks

Lemma 14, which is proved by a double-counting argument, is our main tool for developing sufficient conditions for two configurations to be in different lit-only orbits in the σ-game. While we cannot hope for as succinct a characterization as Theorem 3 gives us for the σ^+-game, perhaps it would be possible to develop some necessary conditions.

Due to Theorem 1, Sutner posed the so-called minimum all-ones problem, which is to investigate how to use the minimum possible number of moves to transform the all-on configuration to the all-off configuration in the σ^+-game. This problem has been studied extensively; see [2] and its references. Due to Corollary 4, it would be interesting to study the minimum all-ones problem for the lit-only σ^+-game.

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