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Lit-only sigma game on a line graph

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Abstract

Let Γ be a connected graph. For any $x \in \mathbb{F}_2^{V(\Gamma)}$, a move of the lit-only sigma game on Γ consists of choosing some $v \in V(\Gamma)$ with $x(v) = 1$ and changing the values of x at all those neighbors of v. An orbit is a maximal subset of $\mathbb{F}_2^{V(\Gamma)}$ any two of which can reach each other by a series of moves. The minimum light number of Γ is $\max_K \min_{x \in K} \# \mathrm{supp}(x)$, where K runs through all orbits.

Denote by $L(\Gamma)$ the line graph of Γ. The subspace of $\mathbb{F}_2^{E(\Gamma)}$ that is generated by the rows of the adjacency matrix of $L(\Gamma)$ is dubbed $\sigma^1(\Gamma)$. We view $\sigma^1(\Gamma)$ as a linear code in $\mathbb{F}_2^{E(\Gamma)}$ and let $\rho(\sigma^1(\Gamma))$ be the covering radius of $\sigma^1(\Gamma)$. For any $S \subseteq V(\Gamma)$, the edge-boundary of S is the set $EB(S)$ of edges with exactly one endpoint lying inside S. The edge isoperimetric number of Γ, denoted $b(\Gamma)$, is defined to be $\max_k \min_{\# S = k} \# EB(S)$.

The main result of this note is that the minimum light number of $L(\Gamma)$ is equal to $\max\{\rho(\sigma^1(\Gamma)), b(\Gamma)\}$. We also determine the sizes of all orbits of the lit-only sigma game on $L(\Gamma)$. Especially, when Γ is a tree, we prove that the minimum light number of $L(\Gamma)$ is $b(\Gamma)$. Moreover, if the tree Γ has $n \geq 3$ vertices, the group formed by the moves of the lit-only sigma game on $L(\Gamma)$ is shown to be the symmetric group on n elements.

1. Introduction

For any finite set S, $\mathbb{F}_2^S$ is the set of functions from S to the binary field $\mathbb{F}_2$. We always associate with $\mathbb{F}_2^S$ the standard bilinear form such that $x \cdot y = \sum_{s \in S} x(s) y(s)$ for $x, y \in \mathbb{F}_2^S$. The *support* of $x \in \mathbb{F}_2^S$, designated by $\mathrm{supp}(x)$, is the subset of S on which x takes value 1. For any $U \subseteq S$, put U^* to be the element of $\mathbb{F}_2^S$ whose support is U. Note that $\emptyset^*, S^* \in \mathbb{F}_2^S$ are just the

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constant functions 0 and 1 on S, respectively. We will talk about constant binary functions with different domains and so the reader should tell the domains from the context when we just use the notation 0 or 1. For a singleton set $\{s\} \subseteq S$, we often write s^* directly instead of the cumbersome notation $\{s\}^*$.

We only consider simple graphs, namely we view a *graph* Γ as a pair $(V(\Gamma), E(\Gamma))$, where the *edge* set $E(\Gamma)$ is a subset of the two-elements subset $\binom{V(\Gamma)}{2}$ of the *vertex* set $V(\Gamma)$. For ease of notation, we often write an edge $\{a, b\}$ as ab and call each of a and b an *endpoint* of ab. For $x \in \mathbb{F}_2^{V(\Gamma)}$, we say that $v \in V(\Gamma)$ is a *lit* vertex of x if $v \in \mathrm{supp}(x)$ and say that v is *off* otherwise. For any $v \in V(\Gamma)$, its *neighbors* in Γ are those w for which $vw \in E(\Gamma)$; we write this set of vertices as $\mathcal{N}_\Gamma(v)$ and often omit the subscript when it is clear from the context. The *line graph* of Γ is $L(\Gamma) = (E(\Gamma), E(L(\Gamma)))$ where $E(L(\Gamma)) = \{ab \in \binom{E(\Gamma)}{2} : \exists v \in V(\Gamma), v \in a \cap b\}$. In words, $L(\Gamma)$ is the graph with the edge set of Γ as vertex set, where two vertices are adjacent if the corresponding edges of Γ have exactly one common endpoint.

The sigma game [18] is a solitary combinatorial game played on a graph, which has close connections with the σ-automata [11] and has many variants which are receiving a lot of attention. A variant of it is the so-called lit-only sigma game [9,12,16,19], which has an interesting connection to Vogan diagrams [6,7]. We introduce these two combinatorial games below in an algebraic way.

Let Γ be a graph. For each $v \in V(\Gamma)$, the *lit-only operation* A_v is a linear transformation which transforms each $x \in \mathbb{F}_2^{V(\Gamma)}$ to

$$xA_v = x + x(v)\mathcal{N}(v)^* \in \mathbb{F}_2^{V(\Gamma)}, \tag{1}$$

namely A_v keeps x unchanged when v is off in x and changes the values of x in $\mathcal{N}(v)$ when v is lit in x. Let $\mathcal{A}(\Gamma) = \{A_v : v \in V(\Gamma)\}$. Clearly, $\mathcal{A}(\Gamma)$ generates a subgroup of $GL(\mathbb{F}_2^{V(\Gamma)})$, the general linear group of $\mathbb{F}_2^{V(\Gamma)}$. We call this group the *lit-only group* of Γ and denote it by $\mathbb{LOG}(\Gamma)$. We consider the group action graph $AG(\Gamma) = \mathrm{ActGrph}(\mathbb{LOG}(\Gamma), \mathbb{F}_2^{V(\Gamma)}, \mathcal{A}(\Gamma))$, whose vertex set is $\mathbb{F}_2^{V(\Gamma)}$ and there is an edge between two vertices x and y if and only if $x \neq y$ and there is $v \in V(\Gamma)$ such that $x = yA_v$. In combinatorial game theory, this is also called the *game graph* [12] of the lit-only sigma game. Note that the vertex set of each component of $AG(\Gamma)$ is just an orbit of the action of $\mathbb{LOG}(\Gamma)$ on $\mathbb{F}_2^{V(\Gamma)}$. The input of the *lit-only sigma game* is any $x \in \mathbb{F}_2^{V(\Gamma)}$ and the player is required to find a good path in $AG(\Gamma)$, say as short as possible, which starts from x and ends at some good y, say with the number of lit vertices as small as possible or as big as possible. The *sigma code* of Γ, which we dub $\sigma(\Gamma)$, is the binary subspace of $\mathbb{F}_2^{V(\Gamma)}$ that is spanned by $\{\mathcal{N}(v)^* : v \in V(\Gamma)\}$, and so is just the row space of an adjacency matrix for Γ. $\sigma(\Gamma)$ acts on $\mathbb{F}_2^{V(\Gamma)}$ by vector addition and hence, just like the action of $\mathbb{LOG}(\Gamma)$ defines the lit-only sigma game, gives rise to another combinatorial game, the so-called *sigma game* on Γ. Note that any orbit of $\mathbb{LOG}(\Gamma)$ lies entirely in some orbit of $\sigma(\Gamma)$. Though all problems addressed in this note have parallels for the sigma game, our effort will mainly focus on the lit-only sigma game.

For any graph Γ, one important graph parameter that arises is the *minimum light number* of Γ, denoted $\mathrm{LN}(\Gamma)$. This is defined to be

$$\max_{K} \min_{x \in K} \# \mathrm{supp}(x),$$

where K runs through all connected components of $AG(\Gamma)$. Note that whatever game position $x \in \mathbb{F}_2^{V(\Gamma)}$ we start with in the lit-only sigma game, we can always reach a configuration whose light number is no greater than $\mathrm{LN}(\Gamma)$. Verifying a conjecture of Chang [4], Wang and Wu obtained an estimate of $\mathrm{LN}(\Gamma)$ when Γ is a tree.

Theorem 1 (*[19]*). *The minimum light number of a tree is no greater than $\lceil \frac{\ell}{2} \rceil$, where ℓ is the number of leaves of the tree.*

Following the practice of coding theory, we refer to

$$\rho(D) = \max_{x \in \mathbb{F}_2^n} \min_{y \in D} \# \operatorname{supp}(x - y) \tag{2}$$

as the *covering radius* of a subspace D of the binary linear space $\mathbb{F}_2^n$. It is worth mentioning that the natural counterpart of $\mathrm{LN}(\Gamma)$ for the sigma game is $\rho(\sigma(\Gamma))$ and a discussion of it can be found in [20].

For the study of various problems on the lit-only sigma game, it is important to understand the orbit distribution of the action of the lit-only group. For any positive integer m, the *path* $P_{m-1} = [v_1 v_2 \ldots v_m]$ is the graph with vertex set $\{v_1, v_2, \ldots, v_m\}$ and edge set $\{v_i v_{i+1} : 1 \le i \le m-1\}$. John Goldwasser made the following conjecture after working out some examples by hand.

Conjecture 2 (*[13]*). *The action of $\mathbb{LOG}([v_1 v_2 \ldots v_m])$ on $\mathbb{F}_2^{\{v_1,\ldots,v_m\}}$ has $\lfloor \frac{m+3}{2} \rfloor$ orbits with $v_i^*, 0 \le i \le \lfloor \frac{m+1}{2} \rfloor$ being a set of orbit representatives, where v_0^* stands for the constant function 0. The size of the orbit which contains v_i^* is $\binom{m+1}{i}$ if $i < (m+1)/2$ and $\binom{m+1}{i}/2$ if $i = (m+1)/2$.*

This note is motivated by Conjecture 2 and intends to investigate the behavior of the lit-only sigma game on a line graph. Our basic observation will be that the study of the lit-only sigma game on a line graph $L(\Gamma)$ can be reduced to the study of a different game on Γ, in a way that the essence of the problem is preserved and that some seemingly difficult questions for the original problem become trivial from the point of view of the latter problem.

The balance of this paper is as follows. In the next section, we recall some concepts and facts around the coboundary map, which set the stage for our subsequent work. In Section 3, we present the key observation of this work, namely the lit-only sigma game on $L(\Gamma)$ can be lifted to be some other game on Γ. Section 4 discusses the structure of the lit-only group and determines the group when the graph is the line graph of a tree. We proceed in Section 5 to work out the orbit distribution for the lit-only sigma game on a line graph, which naturally leads to the discovery of a close connection between the minimum light number and the edge isoperimetric number. We conclude this paper with a discussion of the lit-only sigma game on a path, including a proof of Conjecture 2.

2. Coboundary map

Many graph properties are most easily understood when formulated in terms of the coboundary map on the graph. We summarize in this section some basic concepts and facts about the coboundary map which we will use later freely, referring the reader to [3] for further details.

Let Γ be a graph. The *coboundary map* on Γ is the linear map $\delta = \delta^\Gamma$ from $\mathbb{F}_2^{V(\Gamma)}$ to $\mathbb{F}_2^{E(\Gamma)}$ defined by $\delta x = \sum_{v \in \operatorname{supp}(x)} \sum_{v \in e \in E(\Gamma)} e^*$ for any $x \in \mathbb{F}_2^{V(\Gamma)}$. The adjoint of δ, called the *boundary map* and denoted by $\partial = \partial^\Gamma$, is the linear map $\delta^\top$ from $\mathbb{F}_2^{E(\Gamma)}$ to $\mathbb{F}_2^{V(\Gamma)}$ which sends $y \in \mathbb{F}_2^{E(\Gamma)}$ to $\sum_{e\in \mathrm{supp}(y)} \sum_{v\in e} v^*$. For any $v \in V(\Gamma)$, its neighbor set $\mathcal{N}(v)$ can be equivalently defined as $\{w \in V(\Gamma) : \delta(v^*)\cdot\delta(w^*) = 1\}$. Meanwhile, the structure of the line graph $L(\Gamma)$ of Γ can be conveniently characterized by the so-called *edge Laplacian* on Γ, namely $\delta\partial$, as follows:

$$\mathcal{N}_{L(\Gamma)}(e)^* = \delta(\partial e^*), \quad \forall e \in E(\Gamma) = V(L(\Gamma)). \tag{3}$$

The *cycle space* $\mathcal{C}(\Gamma)$, also called *cycle code* [17], of Γ is the kernel of ∂ and its orthogonal complement in $\mathbb{F}_2^{E(\Gamma)}$ is the *bond space* $\mathcal{B}(\Gamma)$ of Γ, which coincides with $\delta(\mathbb{F}_2^{V(\Gamma)})$. The dimension of $\mathcal{B}(\Gamma)$ is $\#V(\Gamma)-c$ where c is the number of components of Γ. Taking any spanning forest T of Γ, it is known that $\{U^*+\mathcal{B}(\Gamma) : U \subseteq E(\Gamma)\setminus E(T)\}$ is a set of coset representatives of $\mathcal{B}(\Gamma)$ in $\mathbb{F}_2^{E(\Gamma)}$, namely the cohomology group $\mathbb{F}_2^{E(\Gamma)}/\mathcal{B}(\Gamma)$ can be identified with $\mathbb{F}_2^{E(\Gamma)\setminus E(T)}$. We adopt the notation $\sigma^1(\Gamma)$ for $\sigma(L(\Gamma))$. Clearly, Eq. (3) tells us that

$$\sigma^1(\Gamma) = \delta\partial(\mathbb{F}_2^{E(\Gamma)}) \le \delta(\mathbb{F}_2^{V(\Gamma)}) = \mathcal{B}(\Gamma).$$

Example 3. It is not hard to check that for $P_{m-1} = [v_1 v_2 \dots v_m]$ we have

$$\sigma^1(P_{m-1}) = \begin{cases} \mathcal{B}(P_{m-1}) = \delta(\mathbb{F}_2^{V(P_{m-1})}) = \mathbb{F}_2^{E(P_{m-1})}, & \text{if } m \text{ is odd;} \\ \left\{ x \in \mathbb{F}_2^{E(P_{m-1})} : \sum_{1\le i\le \frac{m}{2}} x(v_{2i-1}v_{2i}) = 0 \right\}, & \text{if } m \text{ is even.} \end{cases}$$

The *edge boundary* of $W \subseteq V(\Gamma)$ is the set $EB_\Gamma(W)$ of edges of Γ with one endpoint in W and the other outside of W, i.e.,

$$EB_\Gamma(W) = \{e \in E(\Gamma) : \partial e^* \cdot W^* = 1\} = \mathrm{supp}(\delta(W^*)). \tag{4}$$

The kth *edge-isoperimetric number* of Γ is defined to be

$$b_k(\Gamma) = \min \left\{ \#EB_\Gamma(W) : W \in \binom{V(\Gamma)}{k} \right\}$$

while the *edge-isoperimetric number* of Γ [1,2,10,14] is

$$b(\Gamma) = \max_{0\le k\le \#V(\Gamma)} b_k(\Gamma) = \max_{0\le k\le \#V(\Gamma)/2} b_k(\Gamma). \tag{5}$$

We reserve the symbol $\mathcal{C}_0(\Gamma)$ for the subspace

$$\left\{ x \in \mathbb{F}_2^{V(\Gamma)} : 1 \cdot x = \sum_{v\in V(\Gamma)} x(v) = 0 \right\} \tag{6}$$

and $\mathcal{C}_1(\Gamma)$ for

$$\mathbb{F}_2^{V(\Gamma)} \setminus \mathcal{C}_0(\Gamma). \tag{7}$$

Take $U \subseteq E(\Gamma)$. Let $\delta_U : \mathbb{F}_2^{V(\Gamma)} \to \mathcal{B}(\Gamma) + U^*$ be the affine map satisfying $\delta_U x = \delta x + U^*$ for any $x \in \mathbb{F}_2^{V(\Gamma)}$. This map δ_U makes $\mathbb{F}_2^{V(\Gamma)}$ a 2^c-sheet covering space of $U^* + \mathcal{B}(\Gamma)$ where c is the number of components of Γ. We point out that Eq. (4) gives

$$\#\,\mathrm{supp}(\delta_U(W^*)) = \#\,\mathrm{supp}(EB_\Gamma(W)^* + U^*), \quad \forall W \subseteq V(\Gamma).$$

We define the *odd U-isoperimetric number* of Γ to be

$$\begin{aligned} b_U^o(\Gamma) &= \min\{\#\,\mathrm{supp}(\delta_U W^*) : \#W \text{ is odd},\ W \subseteq V(\Gamma)\} \\ &= \min_{x \in \mathcal{C}_1(\Gamma)} \#\,\mathrm{supp}(\delta_U x) \end{aligned} \tag{8}$$

and define the *even U-isoperimetric number* of Γ to be

$$\begin{aligned} b_U^e(\Gamma) &= \min\{\#\,\mathrm{supp}(\delta_U W^*) : \#W \text{ is even},\ W \subseteq V(\Gamma)\} \\ &= \min_{x \in \mathcal{C}_0(\Gamma)} \#\,\mathrm{supp}(\delta_U x). \end{aligned} \tag{9}$$

The *U-isoperimetric number* of Γ is

$$b_U(\Gamma) = \min\{b_U^o(\Gamma), b_U^e(\Gamma)\} = \min_{x \in \mathbb{F}_2^{V(\Gamma)}} \#\,\mathrm{supp}(\delta_U x). \tag{10}$$

As a consequence of Eqs. (2) and (10), for the subspace $\mathcal{B}(\Gamma)$ of $\mathbb{F}_2^{V(L(\Gamma))} = \mathbb{F}_2^{E(\Gamma)}$ and for any spanning forest T of Γ we find that

$$\rho(\mathcal{B}(\Gamma)) = \max\{b_U(\Gamma) : U \subseteq E(\Gamma) \setminus E(T)\}. \tag{11}$$

Another thing to notice is that

$$\begin{cases} 1 + \mathcal{C}_0(\Gamma) = \mathcal{C}_1(\Gamma), & 1 + \mathcal{C}_1(\Gamma) = \mathcal{C}_0(\Gamma), \quad \text{if } \#V(\Gamma) \text{ is odd;} \\ 1 + \mathcal{C}_0(\Gamma) = \mathcal{C}_0(\Gamma), & 1 + \mathcal{C}_1(\Gamma) = \mathcal{C}_1(\Gamma), \quad \text{if } \#V(\Gamma) \text{ is even.} \end{cases} \tag{12}$$

Therefore, we derive

$$b_U(\Gamma) = b_U^o(\Gamma) = b_U^e(\Gamma) \tag{13}$$

in the case that $\#V(\Gamma)$ is odd.

It is simple to see that the following three statements are pairwise equivalent:

(i) Γ is connected;
(ii) $\partial^{\Gamma}(\mathbb{F}_2^{E(\Gamma)}) = \mathcal{C}_0(\Gamma)$;
(iii) The kernel of δ^{Γ} equals $\{0, 1\}$.

Let us assume now that Γ is a connected graph. By the equivalence among (i)–(iii) above, we know that the kernel of δ is $\{0, 1\}$ and that

$$\sigma^1(\Gamma) = \delta\partial(\mathbb{F}_2^{E(\Gamma)}) = \delta\mathcal{C}_0(\Gamma). \tag{14}$$

Taking into account $\mathcal{B}(\Gamma) = \delta(\mathbb{F}_2^{V(\Gamma)}) = \delta\mathcal{C}_0(\Gamma) \cup \delta\mathcal{C}_1(\Gamma)$, it follows immediately that

$$\sigma^1(\Gamma) = \mathcal{B}(\Gamma) \Leftrightarrow \delta(\mathcal{C}_0(\Gamma)) = \delta(\mathcal{C}_1(\Gamma)) \Leftrightarrow 1 \notin \mathcal{C}_0(\Gamma) \Leftrightarrow \#V(\Gamma) \text{ is odd} \tag{15}$$

and that

$$[\mathcal{B}(\Gamma) : \sigma^1(\Gamma)] = 2 \Leftrightarrow \delta(\mathcal{C}_0(\Gamma)) \neq \delta(\mathcal{C}_1(\Gamma)) \Leftrightarrow 1 \in \mathcal{C}_0(\Gamma) \Leftrightarrow \#V(\Gamma) \text{ is even.} \tag{16}$$

This is in accordance with our Example 3. Moreover, if T is a spanning tree of Γ, it is not hard to see from Eqs. (2), (8), (9) and (13)–(16), that for the subspace $\sigma^1(\Gamma)$ of $\mathbb{F}_2^{V(L(\Gamma))} = \mathbb{F}_2^{E(\Gamma)}$ we have $\rho(\sigma^1(\Gamma)) =$

$$\begin{cases} \max\limits_{U\subseteq E(\Gamma)\setminus E(T)} \{b_U^e(\Gamma) = b_U^o(\Gamma) = b_U(\Gamma)\}, & \text{if } \#V(\Gamma) \text{ is odd,} \\ \max\limits_{U\subseteq E(\Gamma)\setminus E(T)} \{b_U^e(\Gamma), b_U^o(\Gamma)\}, & \text{if } \#V(\Gamma) \text{ is even,} \end{cases} \tag{17}$$

which reminds us of the earlier Eq. (11). This close relationship between $\sigma^1(\Gamma)$ and $\mathcal{B}(\Gamma)$ suggests that the study of the (lit-only) sigma game on a line graph has some connections with the study of the cycle code of a graph [17].

3. Lifting up the lit-only sigma game

Let Γ be a graph. For any $U \subseteq E(\Gamma)$ and $e \in E(\Gamma)$, define $\alpha_e^U : \mathbb{F}_2^{V(\Gamma)} \to \mathbb{F}_2^{V(\Gamma)}$ by requiring

$$x\alpha_e^U = \begin{cases} x + (\delta x + 1)(e)\partial e^*, & \text{if } e \in U; \\ x + (\delta x)(e)\partial e^*, & \text{if } e \notin U. \end{cases} \tag{18}$$

In words, if $e = ab$ is not in U then α_e^U swaps the values of x at a and b, while if e is in U then the values at both a and b are changed if they are equal and left the same if they are unequal. It follows from Eq. (18) instantly that it holds for any $x \in \mathbb{F}_2^{V(\Gamma)}$ that

$$x\alpha_e^U = y \Leftrightarrow (x+1)\alpha_e^U = y + 1. \tag{19}$$

Note that α_e^U is a linear map provided $e \notin U$ and is merely an affine map otherwise. We use $G_U(\Gamma)$ to denote the group generated by $\alpha_U(\Gamma) = \{\alpha_e^U : e \in E(\Gamma)\}$.

Remark 4. For any $U \subseteq E(\Gamma)$, it is clear that both $\mathcal{C}_0(\Gamma)$ and $\mathcal{C}_1(\Gamma)$, which are introduced in Eqs. (6) and (7), respectively, are $G_U(\Gamma)$-invariant.

For each $v \in V(\Gamma)$, the *off-only operation* B_v is a linear transformation which brings each $x \in \mathbb{F}_2^{V(\Gamma)}$ to

$$xB_v = x + (1 + x(v))\mathcal{N}(v)^* \in \mathbb{F}_2^{V(\Gamma)}, \tag{20}$$

namely B_v keeps x unchanged when v is lit in x while it toggles the values of x in $\mathcal{N}(v)$ when v is off in x.

From Eqs. (1), (3), (18) and (20), we obtain the following two commutative diagrams for any $x \in \mathbb{F}_2^{V(\Gamma)}$, $e \in E(\Gamma)$ and $U \subseteq E(\Gamma)$:

$$e \in U: \quad \begin{array}{ccccc} x & \xrightarrow{\delta} & \delta x & \xrightarrow{+U^*} & \delta x + U^* \\ \alpha_e^U\big\downarrow & & B_e\big\downarrow & & A_e\big\downarrow \\ x\alpha_e^U & \xrightarrow{\delta} & \delta x + (\delta x + 1)(e)\mathcal{N}_{L(\Gamma)}(e)^* & \xrightarrow{+U^*} & (\delta x + U^*)A_e \end{array} \tag{21}$$

$$e \notin U: \quad \begin{array}{ccccc} x & \xrightarrow{\delta} & \delta x & \xrightarrow{+U^*} & \delta x + U^* \\ \alpha_e^U\big\downarrow & & A_e\big\downarrow & & A_e\big\downarrow \\ x\alpha_e^U & \xrightarrow{\delta} & \delta x + (\delta x)(e)\mathcal{N}_{L(\Gamma)}(e)^* & \xrightarrow{+U^*} & (\delta x + U^*)A_e \end{array} \tag{22}$$

It is now a straightforward matter to combine Eqs. (21) and (22) to get

$$\begin{array}{ccc} x & \xrightarrow{\delta_U} & \delta x + U^* \\ \alpha_e^U \downarrow & & \downarrow A_e \\ x\alpha_e^U & \xrightarrow{\delta_U} & (\delta x + U^*)A_e \end{array}, \qquad (23)$$

which means that the action of $\mathbb{LOG}(L(\Gamma))$ on $U^* + \mathcal{B}(\Gamma)$ can be lifted to the action of $G_U(\Gamma)$ on $\mathbb{F}_2^{V(\Gamma)}$.

Theorem 5. *Let Γ be a connected graph and $U \subseteq E(\Gamma)$. For each orbit $O \subseteq U^* + \mathcal{B}(\Gamma)$ of $\mathbb{LOG}(L(\Gamma))$, $\delta_U^{-1}(O)$ is either an orbit $\mathcal{O}$ of $G_U(\Gamma)$ satisfying*

$$1 + \mathcal{O} = \mathcal{O}, \qquad \delta_U \mathcal{O} = O, \quad \text{and} \quad \#\mathcal{O} = 2\#O,$$

or the disjoint union of two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ of $G_U(\Gamma)$ such that

$$\mathcal{O}_1 = 1 + \mathcal{O}_2, \qquad \delta_U \mathcal{O}_1 = \delta_U \mathcal{O}_2 = O, \quad \text{and} \quad \#\mathcal{O}_1 = \#\mathcal{O}_2 = \#O.$$

Proof. Eq. (19) demonstrates that for any orbit $\mathcal{O}$ of $G_U(\Gamma)$, $1 + \mathcal{O} = \{1 + x : x \in \mathcal{O}\}$, which might equal to $\mathcal{O}$, is also an orbit of $G_U(\Gamma)$. Consequently, in the case that Γ is connected and hence the kernel of δ is $\{0, 1\}$, Eq. (19) along with Eq. (23) gives the result, as required. □

Following a suggestion from Xinmao Wang, we illustrate here a more algebraic way of viewing the lifting of the action of $\mathbb{LOG}(L(\Gamma))$ on $\mathcal{B}(\Gamma)$ to the action of $G_\emptyset(\Gamma)$ on $\mathbb{F}_2^{V(\Gamma)}$, which is a special case of Eq. (23) and Theorem 5. Let I_V and I_E be the identity matrices whose lines are indexed by $V(\Gamma)$ and $E(\Gamma)$, respectively. For any $e \in E(\Gamma)$, set $\mathcal{U}_e$ to be the unit matrix with lines indexed by $E(\Gamma)$ and with a one in the (e, e) position and zeros elsewhere. In view of Eq. (3), $\delta\delta^\top$ is the adjacency matrix of $L(\Gamma)$ and so the lit-only operation A_e on $L(\Gamma)$ is just $I_E + \delta(\delta^\top \mathcal{U}_e)$. On the other hand, the linear operator $\alpha_e^\emptyset$ on $\mathbb{F}_2^{V(\Gamma)}$ is nothing but $I_V + (\delta^\top \mathcal{U}_e)\delta$. It is trivial that

$$\delta(I_V + \delta^\top \mathcal{U}_e \delta) = (I_E + \delta\delta^\top \mathcal{U}_e)\delta$$

and hence the above observations show clearly that δ gives rise to a two-to-one lifting from the described action of $\mathbb{LOG}(L(\Gamma))$ to that of $G_\emptyset(\Gamma)$ when Γ is connected.

4. Lit-only group

Let $S_{V(\Gamma)}$ be the symmetric group on $V(\Gamma)$ where each $\alpha \in S_{V(\Gamma)}$ permutes $v \in V(\Gamma)$ to $v^\alpha \in V(\Gamma)$. Each edge $e = uv$ of Γ corresponds to an element $\alpha_e \in S_{V(\Gamma)}$ which swaps u and v while it fixes all remaining vertices. In cycle notation, α_e is the transposition (uv). Let $\mathbb{A}(\Gamma) = \{\alpha_e : e \in E(\Gamma)\}$. We recall an easy basic result.

Lemma 6 ([8, *Section 6.1*]). *Let Γ be a graph. Then $\mathbb{A}(\Gamma)$ generates $S_{V(\Gamma)}$ if and only if Γ is connected.*

$S_{V(\Gamma)}$ acts on $\mathbb{F}_2^{V(\Gamma)}$ linearly where $\alpha \in S_{V(\Gamma)}$ sends $x \in \mathbb{F}_2^{V(\Gamma)}$ to $x\alpha \in \mathbb{F}_2^{V(\Gamma)}$ given by $x\alpha(v) = x(v^\alpha)$ for each $v \in V(\Gamma)$. We will accordingly view each $\alpha_e, e \in E(\Gamma)$, as a member of $GL(\mathbb{F}_2^{V(\Gamma)})$ hereafter. For any $e \notin U \subseteq E(\Gamma)$, an easy but crucial observation is that

$$\alpha_e = \alpha_e^U, \qquad (24)$$

the latter being specified by Eq. (18). For any $e \in E(\Gamma)$, due to Eq. (24), taking $U = \emptyset$ in Eq. (22) yields

$$\begin{array}{ccc} y & \xrightarrow{\alpha_e \in \mathbb{A}(\Gamma)} & y\alpha_e \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow{\scriptstyle\delta} \\ \delta y & \xrightarrow{A_e \in \mathcal{A}(L(\Gamma))} & (\delta y)A_e = \delta(y\alpha_e) \end{array}. \tag{25}$$

Lemma 7. *Let Γ be a connected graph with at least three vertices. Then, upon the identification of α_e with the lit-only operation A_e for each $e \in V(L(\Gamma))$, the set of relations satisfied by $\mathcal{A}(L(\Gamma))$ when acting on $\mathcal{B}(\Gamma)$ is the same as the set of relations satisfied by $\mathbb{A}(\Gamma)$ in $S_{V(\Gamma)}$.*

Proof. For any sequence $e_1, \ldots, e_t$ over $E(\Gamma)$, we need to show that $\alpha_{e_1} \cdots \alpha_{e_t} = 1$ in $S_{V(\Gamma)}$ if and only if $A_{e_1} \cdots A_{e_t}$ fixes all elements of $\mathcal{B}(\Gamma)$. More precisely, we have to show that

$$yA_{e_1} \cdots A_{e_t} = y, \quad \forall y \in \mathcal{B}(\Gamma) \tag{26}$$

is equivalent to

$$x\alpha_{e_1} \cdots \alpha_{e_t} = x, \quad \forall x \in \mathbb{F}_2^{V(\Gamma)}. \tag{27}$$

The first thing to notice is that $\{v^* : v \in V(\Gamma)\}$ is a basis of $\mathbb{F}_2^{V(\Gamma)}$ while $\{\delta v^* : v \in V(\Gamma)\}$ span the bond space $\mathcal{B}(\Gamma)$. Consequently, as both group actions in consideration are linear representations, Eq. (27) holds for all $x \in \mathbb{F}_2^{V(\Gamma)}$ if and only if it holds for $x = v^*$ for all $v \in V(\Gamma)$, while Eq. (26) holds for all $y \in \mathcal{B}(\Gamma)$ if and only if it holds for $y = \delta v^*$ for all $v \in V(\Gamma)$. Therefore, fixing any $v \in V(\Gamma)$, our goal becomes demonstrating the equivalence between

$$\delta(v^*)A_{e_1} \cdots A_{e_t} = \delta v^* \tag{28}$$

and

$$v^*\alpha_{e_1} \cdots \alpha_{e_t} = v^*. \tag{29}$$

Successive applications of Eq. (25) leads to $\delta(v^*)A_{e_1} \cdots A_{e_t} = \delta(v^*\alpha_{e_1} \cdots \alpha_{e_t})$ and hence Eq. (28) is nothing but

$$\delta(v^*\alpha_{e_1} \cdots \alpha_{e_t}) = \delta v^*. \tag{30}$$

This reduces our task to proving the equivalence of Eqs. (29) and (30).

Clearly, Eq. (30) follows from Eq. (29). Going the other way, since Γ is connected, we see that the kernel of δ is $\{0, 1\}$ and hence deduce from Eq. (30) that $\{v^*\alpha_{e_1} \cdots \alpha_{e_t}, v^*\} \subseteq \{x, x+1\}$ for some $x \in \mathbb{F}_2^{V(\Gamma)}$. Because $\#V(\Gamma) \geq 3$, at most one of x and $x+1$ can have exactly one lit vertex. However, both $v^*\alpha_{e_1} \cdots \alpha_{e_t}$ and v^* have surely exactly one lit vertex. This then establishes Eq. (29), finishing the proof. □

Theorem 8. *Let Γ be a connected graph with $\#V(\Gamma) \geq 3$. Then $S_{V(\Gamma)}$ is a quotient group of $\mathbb{LOG}(L(\Gamma))$. In particular, the map that sends A_e to α_e for each $e \in E(\Gamma)$ induces an isomorphism from $\mathbb{LOG}(L(\Gamma))$ to $S_{V(\Gamma)}$ when Γ is a tree other than P_0 and P_1.*

Proof. Let N be the subgroup of $\mathbb{LOG}(L(\Gamma))$ consisting of all its elements which act as the identity transformation when restricted to $\mathcal{B}(\Gamma)$. It follows from Lemmas 6 and 7 that $S_{V(\Gamma)} = \mathbb{LOG}(L(\Gamma))/N$. The second claim comes from the first, because for a tree Γ we have $\mathcal{B}(\Gamma) = \mathbb{F}_2^{E(\Gamma)}$ and hence $N = \{1\}$. □

Example 9. When $\Gamma = P_1$, $\mathbb{LOG}(L(\Gamma)) = S_1$ does not have $S_2 = S_{V(\Gamma)}$ as a quotient group.

5. Orbit distribution and minimum light number

Without loss of generality, let us assume below that Γ is a connected graph. Since $\sigma^1(\Gamma) \leq \mathcal{B}(\Gamma)$ (cf. Eqs. (15) and (16)), we know that $\mathbb{F}_2^{E(\Gamma)}$ can be decomposed into a disjoint union of $\mathbb{LOG}(L(\Gamma))$-invariant sets $x + \mathcal{B}(\Gamma)$, $x \in \mathbb{F}_2^{E(\Gamma)}/\mathcal{B}(\Gamma)$. We already have within our reach the picture of the action of $\mathbb{LOG}(L(\Gamma))$ on $x + \mathcal{B}(\Gamma)$, $x \in \mathbb{F}_2^{E(\Gamma)}/\mathcal{B}(\Gamma)$ and here comes a detailed analysis of it.

We first deal with the case of $x + \mathcal{B}(\Gamma) = \mathcal{B}(\Gamma)$.

Theorem 10. *Let Γ be a connected graph and $n = \#V(\Gamma)$. The action of $\mathbb{LOG}(L(\Gamma))$ on $\mathcal{B}(\Gamma)$ has $\lfloor \frac{n+2}{2} \rfloor$ orbits with sizes $s_0, \ldots, s_{\lfloor \frac{n}{2} \rfloor}$, respectively, where $s_k = \binom{n}{k}$ if $k < n/2$ and $s_k = \binom{n}{k}/2$ if $k = n/2$.*

Proof. We first observe that $G_\emptyset(\Gamma) = \mathbb{A}(\Gamma)$. Therefore, in view of Lemma 6, the orbits of $G_\emptyset(\Gamma)$ on $\mathbb{F}_2^{V(\Gamma)}$ can be enumerated as $\mathcal{O}^0, \ldots, \mathcal{O}^n$, where

$$\mathcal{O}^k = \{x \in \mathbb{F}_2^{V(\Gamma)} : \#\operatorname{supp}(x) = k\}, \quad 0 \leq k \leq n. \tag{31}$$

Note that $1+\mathcal{O}^k = \mathcal{O}^{n-k}$ and that $\#\mathcal{O}^k = \binom{n}{k}$ for any $0 \leq k \leq n$. Hence, by virtue of Theorem 5, the orbits of $\mathbb{LOG}(L(\Gamma))$ on $\mathcal{B}(\Gamma)$ are

$$\delta(\mathcal{O}^k), \quad 0 \leq k \leq \lfloor n/2 \rfloor, \tag{32}$$

and

$$\begin{cases} \#\delta(\mathcal{O}^k) = \#\mathcal{O}^k = \dbinom{n}{k}, & \text{if } k < n/2; \\ \#\delta(\mathcal{O}^k) = \#\mathcal{O}^k/2 = \dbinom{n}{k}/2, & \text{if } k = n/2. \end{cases} \tag{33}$$

This is precisely the assertion of the theorem. □

Before addressing the case of $x + \mathcal{B}(\Gamma) \neq \mathcal{B}(\Gamma)$, we prepare an easy lemma.

Lemma 11. *Let Γ be a connected graph with a spanning tree T and let U be a nonempty subset of $E(\Gamma) \setminus E(T)$. Then, the action of $G_U(\Gamma)$ on $\mathbb{F}_2^{V(\Gamma)}$ has exactly two orbits which are $\mathcal{C}_0(\Gamma)$ and $\mathcal{C}_1(\Gamma)$.*

Proof. Owing to Remark 4, we need to prove that $G_U(\Gamma)$ acts transitively on $\mathcal{C}_0(\Gamma)$ and $\mathcal{C}_1(\Gamma)$, respectively. For each $e \in E(T)$, we have $e \notin U$ and so $\alpha_e^U = \alpha_e$. In addition, Lemma 6 guarantees that $\{\alpha_e^U = \alpha_e : e \in E(T)\}$ generates $S_{V(\Gamma)}$. Accordingly, we come to

Claim (*): Any two elements $x, y \in \mathbb{F}_2^{V(\Gamma)}$ with the same support size must lie in the same $G_U(\Gamma)$-orbit.

In order to prove the lemma, it suffices at present to show that each $x \in \mathbb{F}_2^{V(\Gamma)}$ lies in a $G_U(\Gamma)$-orbit containing some y with $\#\operatorname{supp}(y) \le 1$. We establish this fact by proving that for each $x \in \mathbb{F}_2^{V(\Gamma)}$ with $\#\operatorname{supp}(x) \ge 2$, there is $\beta \in G_U(\Gamma)$ such that $\#\operatorname{supp}(x\beta) = \#\operatorname{supp}(x) - 2$.

Assume that $x \in \mathbb{F}_2^{V(\Gamma)}$ satisfy $\#\operatorname{supp}(x) \ge 2$. Choose $e = ab \in U \ne \emptyset$. In light of the previous Claim (*), there is an $\alpha \in G_U(\Gamma)$ such that for $y = x\alpha$ we have $\#\operatorname{supp}(y) = \#\operatorname{supp}(x)$ and $y(a) = y(b) = 1$. But we then get that $\#\operatorname{supp}(x\beta) = \#\operatorname{supp}(x) - 2$, where $\beta = \alpha\alpha_e^U \in G_U(\Gamma)$. This completes the proof. □

Theorem 12. *Let Γ be a connected graph on n vertices. Let $x \in \mathbb{F}_2^{E(\Gamma)}$ be an element outside of $\mathcal{B}(\Gamma)$. If n is even, $x + \mathcal{B}(\Gamma)$ consists of two orbits of $\mathbb{LOG}(L(\Gamma))$, each of which has a size 2^{n-2}; If n is odd, $x + \mathcal{B}(\Gamma)$ itself is an orbit of $\mathbb{LOG}(L(\Gamma))$ of length 2^{n-1}. In both cases, the orbits of $\mathbb{LOG}(L(\Gamma))$ in $x + \mathcal{B}(\Gamma)$ coincide with the cosets of $\sigma^1(\Gamma)$ in $x + \mathcal{B}(\Gamma)$, namely the orbits of the sigma game on $L(\Gamma)$ contained in $x + \mathcal{B}(\Gamma)$.*

Proof. We pick a spanning tree T of Γ and choose a subset $U \subseteq E(\Gamma) \setminus E(T)$ such that $U^* + \mathcal{B}(\Gamma) = x + \mathcal{B}(\Gamma)$. As $x \notin \mathcal{B}(\Gamma)$, we conclude that U is nonempty. Lemma 11 now says that the action of $G_U(\Gamma)$ on $\mathbb{F}_2^{V(\Gamma)}$ has exactly two orbits, $\mathcal{C}_0(\Gamma)$ and $\mathcal{C}_1(\Gamma)$. Consequently, due to Eq. (12), Theorem 5 enables us to list the orbits of $\mathbb{LOG}(L(\Gamma))$ inside $x + \mathcal{B}(\Gamma)$ as

$$\begin{cases} \delta_U(\mathcal{C}_0(\Gamma)) = \delta_U(\mathcal{C}_1(\Gamma)), & \text{if } n \text{ is odd;} \\ \delta_U(\mathcal{C}_0(\Gamma)) \ne \delta_U(\mathcal{C}_1(\Gamma)), & \text{if } n \text{ is even.} \end{cases} \tag{34}$$

Comparing this with Eq. (14) shows that each orbit of $\mathbb{LOG}(L(\Gamma))$ in $x + \mathcal{B}(\Gamma)$ is a coset of $\sigma^1(\Gamma)$, as was to be shown. Finally, the sizes of the orbits can be read directly from Eq. (34) and thus we are done. □

Taking together Theorems 10 and 12, we now arrive at one of our main results.

Theorem 13. *Let Γ be a connected graph on n vertices and m edges.*

- *If n is odd, then $\mathbb{LOG}(L(\Gamma))$ has $2^{m-n+1} + \frac{n-1}{2}$ orbits, the length of which being $\underbrace{2^{n-1}, 2^{n-1}, \ldots, 2^{n-1}}_{2^{m-n+1}-1 \text{ times}}, \binom{n}{k}, 0 \le k \le (n-1)/2$, respectively.*
- *If n is even, then $\mathbb{LOG}(L(\Gamma))$ has $2^{m-n+2} + \frac{n-2}{2}$ orbits, the length of which being $\underbrace{2^{n-2}, 2^{n-2}, \ldots, 2^{n-2}}_{2^{m-n+2}-2 \text{ times}}, \binom{n}{k}, 0 \le k \le (n-2)/2$, and $\binom{n}{n/2}/2$, respectively.*

Let Γ be a connected graph. All orbits of $\mathbb{LOG}(L(\Gamma))$ on $\mathbb{F}_2^{E(\Gamma)}$ have been displayed in Eqs. (32) and (34). We proceed with applying the combination of Eqs. (5), (8)–(10) and (13) and then come to the next result.

Theorem 14. *Let Γ be a connected graph on n vertices and T a spanning tree of Γ. Let $b'(\Gamma)$ be given by*

$$\begin{cases} \max\{0, b_U(\Gamma) : \emptyset \ne U \subseteq E(\Gamma) \setminus E(T)\}, & \textit{if } n \textit{ is odd;} \\ \max\{0, b_U^o(\Gamma), b_U^e(\Gamma) : \emptyset \ne U \subseteq E(\Gamma) \setminus E(T)\}, & \textit{if } n \textit{ is even.} \end{cases} \tag{35}$$

Then, $b'(\Gamma)$ is independent of the choice of T and the minimum light number of $L(\Gamma)$ is $\max\{b'(\Gamma), b(\Gamma)\}$.

Corollary 15. *Let Γ be a tree. The minimum light number of $L(\Gamma)$ is equal to $b(\Gamma)$.*

Proof. For the tree Γ, we have $b'(\Gamma) = 0$ and so the result follows from Theorem 14. □

Note that none of $b_\emptyset(\Gamma)$, $b^o_\emptyset(\Gamma)$ and $b^e_\emptyset(\Gamma)$ can be bigger than $b(\Gamma)$. Therefore, a comparison of Eqs. (5), (17) and (35) reveals that

$$b'(\Gamma) \le \rho(\sigma^1(\Gamma)) \le \max\{b'(\Gamma), b(\Gamma)\}.$$

This in combination with Theorem 14 then leads to the following result which connects minimum light number with covering radius and edge isoperimetric number.

Theorem 16. *Let Γ be a connected graph. The minimum light number of $L(\Gamma)$ is equal to* $\max\{\rho(\sigma^1(\Gamma)), b(\Gamma)\}$.

We finish this paper with two easy observations about the lit-only sigma game on a path.

Corollary 17. *Conjecture 2 is true.*

Proof. It is evident that $P_{m-1} = L(\Gamma)$ for $\Gamma = P_m$. Up to isomorphism, let us suppose that $V(\Gamma) = \{w_i : 1 \le i \le m+1\}$, $E(\Gamma) = V(L(\Gamma)) = \{v_i = w_i w_{i+1} : 1 \le i \le m\}$ and $E(L(\Gamma)) = \{v_i v_{i+1} : 1 \le i \le m-1\}$. Thanks to Theorem 5, we know that the orbits of $\mathbb{LOG}(L(\Gamma))$ can be listed as in Eq. (32) for $n = m+1$ and $v_k^* = \delta(\sum_{i=1}^k w_i^*) \in \delta(\mathcal{O}^k)$, where $\mathcal{O}^k$, $0 \le k \le \lfloor n/2 \rfloor$ is defined in Eq. (31). Finally, an application of Eq. (33) then concludes the proof. □

Corollary 18. *Following the notation used in the proof of Corollary 17, we can never reach v_k^* from $x \in \mathbb{F}_2^{V(P_{m-1})}$ when playing the lit-only sigma game on P_{m-1} if $\#\operatorname{supp}(x) > 2k$.*

Proof. As we see in the proof of Corollary 17, v_k^* belongs to the orbit $\delta(\mathcal{O}^k)$. Since the maximum degree of P_{m-1} is at most two, we get that $\#\operatorname{supp}(\delta y) \le 2\#\operatorname{supp}(y) = 2k$ for each $y \in \mathcal{O}^k$. This is the proof. □

We remark that a quite explicit description of the orbits of the lit-only group of a path, including the distribution of the sizes of supports, has been obtained by Li and Wu [15]. We also mention that Chang and Chuah [5] obtained some results on the representatives of orbits of the lit-only group for several kinds of trees via a quite different approach.

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